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PREFACE

PIERRE NOYES, EDITOR

Quantum events are unique, discrete, irreversible, non-local and yet indivisible. Conventional quantum theory tries to embed them in a space-time continuum, — a step which in our view is the source of many conceptual difficulties such as the "collapse of the wave function", the EPR "paradox" and the infinities of second quantized field theory. What we will find in the theory developed here is that the quantization of action, events, masses and conserved quantum numbers and the limiting velocity of special relativity together with the Lorentz transformations have a common origin in the discrete substructure on which we build. We claim to achieve[1] a discrete reconciliation of quantum mechanics and relativity by going beyond the conceptual framework of Bohr and Einstein.

The ninth annual international meeting of the Alternative Natural Philosophy Association marks a turning point in the steps toward this goal which we attempt to record in this volume. While many significant results have been obtained since the pioneering work of Bastin and Kilmister appeared[2,3] and more than two decades have passed since the discovery of the combinatorial hierarchy by Amson, Bastin, Kilmister and Parker-Rhodes[4] this was the first time that a coherent presentation of the theory that has been evolving seemed possible.

The accomplishments now in hand show[1] that quantum mechanics and relativity, rather than being in conflict with each other, find a common conceptual basis in any finite, discrete, constructive calculus which has the basic structure of McGoveran's *ordering operator calculus*[5]. As he shows, both the limiting velocity of special relativity and the commutation relations of quantum mechanics, first encountered historically in physics, are general features of any finite, discrete, computable formalism.

To give precision to how this formal result allows us to identify the "universal constants" c and $\hbar$ by recourse to experiment, we need a *modeling methodology*. This starts from a rough agreement among the modelers as to what we hope to model called the *Epistemological-frame* (in our case particle physics), the working out of a computable representation that stands on its own feet — but is motivated by our informal intent — called the *Representational-frame* and rules of correspondence (*Procedural-frame*) that connect the formal representation to the "empirical facts" we are trying to model or might search for experimentally. This triadic structure can be iterated in either sense (ERP or EPR). If — as has always been the case historically in physics for all theories — the current iteration is unsatisfactory, we can tinker with any part of the structure and iterate again.

The current iteration, represented in this volume by papers whose content was discussed at ANPA 8 and ANPA 9, and elaborated in active correspondence during the remainder of 1987, allows us some very specific claims of interest to theoretical physicists. Many of these were achieved earlier in some sense, but as pieces of research rather than as parts of a coherent theory. The two most dramatic of these earlier results were the calculation of two of the scale constants of physics $\hbar c/e^2 \simeq 137, \hbar c/Gm_p^2 = (M_{Planck}/m_p)^2 \simeq 2^{127} + 136 \simeq 1.7 \times 10^{38}$ as the last two terms of a hierarchy $(2^2 - 1 = 3 \Rightarrow 2^3 - 1 = 7(+3 = 10) \Rightarrow 2^7 - 1 = 127(+10 = 137) \Rightarrow 2^{127} - 1)$ which *terminates* at the fourth level[4], and the Parker-Rhodes result[6] $m_p/m_e = 1836.151497....$ An heuristic argument of Dyson's[7] as to why the QED renormalized perturbation series diverges beyond 137 terms was shown by Noyes[8] to imply that 137 is the maximum number of charged particle pairs which can be *counted* by electromagnetic means within a volume whose radius is $\hbar/2mc$ and 1.7×10^{38} the maximum number of particles of protonic mass which can be counted by gravitational means within $\hbar/m_pc$. This showed that it made sense to give physical interpretation to these cardinals within the conventional structure, but left unsolved the problem of how to construct a *theory* in which these numerological results could be derived as *physical* predictions.

The research has proceeded at a faster pace since the formation of the *Alternative Natural Philosophy Association* in 1979. Each step[9–11] has led to the solution of specific problems, and broadened the community in which the work was found to be of interest. It was only in 1987 that firm foundations could be laid and shown to imply both the older and the newer results. It therefore seemed appropriate to publish the core of this work as it now stands, together with the more diffuse papers which were also presented at the meeting. The first four articles in this volume provide a reasonably systematic and complete presentation of the theory as it now stands. Since the first two papers (*Prephysics* and *Foundations for a Discrete Physics*) dealing with the philosophical and mathematical foundations of the subject may prove to be pretty heavy going for those whose interests lie primarily in physics, it is suggested that those readers start with the third paper and return to the foundations if the results seem to justify it. Prof. Kilmister, current president of the Alternative Natural Philosophy Association, comments on the conference and the papers presented here at the end of the volume.

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PREPHYSICS*

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"To be sure, it has been pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraic method of description of nature, that is the elimination of continuum functions from physics. Then, however, we must also give up, by principle, the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path." ... *Albert Einstein (1936)*

I

Ever since the ancient Greeks, speculations concerning man's place in the Universe has been an ongoing practice within science and philosophy. Whereas the antique and medieval scientists and philosophers, according to the common tradition in history and philosophy of science, interpreted Nature in subjective terms, it was the ingenious insight of Galileo to emphasize the *method* of physics (mathematization and experiment), which was to secure the objectivity of the practice of physics. This seemed to detach the measuring Subject from the unique and egocentric position it once enjoyed in scientific and philosophical thinking. It was Galileo who fathered the modern concept of mathematized natural science. He tried to achieve exactness and rational objectivity through the use of *mathematics*. According to Galileo, as he writes in the *Saggiatore* (1623), "(philosophy) is written in that great book which ever lies before our eyes, I mean the universe, but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. This book is written in the mathematical language" [1]. This bold statement has proved useful ever since it was written. Physicists started to formulate mathematically informally observed regular phenomena, like a falling stone or the flight of a cannonball. This method involved, inevitably, a high degree of scientific idealization. Physicists emphasized the construction and detailed study of scientific models. It was thought that all phenomena must be describable in terms of the mathematical method. The classical highlight was Newton's *Philosophiae Naturalis Principia Mathematica* (1687) [2].

The result was, as far as the practice of theoretical physics is concerned, the adoption of *calculus* (including the metaphysical idea of infinitesimals) as the mathematical tool of the theoretical physicist. The method was initially formulated by Newton and Leibnitz. Later it was perfected by Bolzano and Cauchy. Nature came to be regarded as being written in the language of mathematics and the whole Universe became understood as a mechanism, as a kind of a "universal clockwork," the blueprint of which is written in the language of calculus. In Kuhn's terminology: the classical calculus became, as far as mathematics is concerned, the

paradigm

of classical physics, which, in general terms, was characterized by the commitment "to the same rules and standards for scientific practice" [3]. The paradigm of Galilean physics is characterized by the adoption of mathematics and the experimental method.

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The Galilean method was assumed to have secured the *objectivity* of the practice of physics, since "that commitment and the apparent consensus it produces are prerequisites for normal science, i.e., for the genesis and continuation of a particular research tradition" [4]. For a long time this seemed to be the correct attitude to take; the Subject could safely perform the practice of physics by virtue of this paradigm in order to exhibit facts of the material Universe. This was the highlight of the era of the "classical" paradigm in physics regulated by the Galilean method, culminating with the formulation of Einstein's Special Theory and General Theory of Relativity.

However, in the beginning of this century, Planck formulated the idea of *quanta* in physics (1900). Within three decades, the problem of the Subject erupted once again. It was assumed that quantum mechanics forced the detached Subject back into physics; also, doubt concerning the space-time continuum, presupposed in the paradigm of classical physics, became—as a result of the introduction of the quantum theory—more manifest. Quanta introduced discreteness into physics and, as a result, thoughts concerning the necessity of a change of paradigm in physics, i.e., an essential discretization of physics, surfaced every now and then. A number of approaches to the problem of discrete physics were exhibited, e.g., in Bastin's book *Quantum Theory and Beyond* [5]. However, nothing substantial erupted which barred the creation of a novel paradigm.

To achieve such a change would have required a change of paradigm in physics. It amounts to a change in the paradigm (structure) of mathematics *used* in theoretical (mathematical) physics within the practice of physics. But, one may ask, what is really to be understood by a paradigm? In order to attempt to answer this question, one can begin by noting that usually, today, physics is divided into *experimental* physics and *theoretical* (and mathematical) physics. However, it has not been appreciated that there is a mathematical structure regulating the practice of theoretical physics. One can say that physicists (and mathematicians), usually, only have an implicit understanding of this structure. It is usually not an explicit part of a theoretical physicist's understanding. Nevertheless, a paradigm is always present as a tacit component in the competence exhibited by a theoretical physicist.

Now, when attempting to formulate a discrete physics, one is simultaneously dealing with an attempt to change paradigm in physics; thus, in order to accept a discrete physics, one must additionally accept the implied change of paradigm. This forces the proponent of discrete physics to be able, upon request, in principle, to exhibit the structure of the paradigm. One is to be cautious with this distinction, since the notion of a paradigm in physics amounts to a novel element in the practice of physics, in addition to the previous elements of theory and experiment. This has not been previously explicitly appreciated.

In order to investigate the notion of a paradigm in theoretical physics, one has to begin by grasping the insight that, indeed, there are objects (e.g., events), but that it is *acts* which are real, actual or concrete. Moreover, the acts are immediate acts and the objects are obtained as the *result* of acts. In the terminology of Austin: the immediate acts amounts to *performatives*. When an expression exhibits a performative, it does not *describe* my doing of what I should be said in so uttering to be doing, or state that I am doing it; it is to *do* it. The performative indicates that the issuing of an utterance (expression) is the performing of an action. In uttering an expression, one is *doing* something. We do something in saying and writing something when engaged in tasks of mathematics and theoretical physics: we *judge*. The activity of theoretical (mathematical) physics amounts to performing certain tasks, and to perform a task is to be understood in the sense of it being *immediately* performed. We are not to understand the words "perform" and "practice" in a representational way, as is usually done; to do so leads us astray as far as the point here is concerned. This is a crucial insight.

Thus, there seem to be three major (philosophical) problems connected with the current practice of physics:

1) Where is the place of the Subject in the practice of physics?

2) Ought the theoretical practice of physics to be grounded on a discrete paradigm of physics, instead of the prevailing contemporary space-time arena?

3) How is the notion of "physical reality" to be understood in relation to the practice of physics?

A philosophy of physics, if it aspires to be philosophically complete, as far as the meaning of the practice is concerned, must deal with all three problems. These three problems are not necessarily connected. One can attempt to deal with one, without attempting to deal with the others. However, for the practice of physics to be completely understood relative to the practice, itself, all three questions must be answered—taken together, they belong to a philosophical investigation of the practice understood as a whole; to understand physics as a practice presupposes a holistic conception of the practice. This point is not, yet, explicitly, required to be understood when attempting to formulate a discrete paradigm in physics. Here, the part dealing with experiments as immediate practice and their connection to transcendental reality is not dealt with at all. This part is presupposed to be in order. A physicist working within the paradigm of discrete physics can be equally in the dark concerning explicit understanding of transcendental reality as a physicist working within the continuum paradigm.

The aim, in this paper, is to make one aware that whereas the second point exhibits a genuine methodological task to be performed, the first and the third points only exhibit problems when being the target of a philosophical illusion, brought about by thinking along a representational way of grasping the problem. The first and the third points do not state genuine structural problems. Thus, by a paradigm in physics is to be understood a mathematical framework and the rules connecting this framework to the existing practice of physics [6].

In 1905, Poincaré wrote that mathematics is the mathematical physicist's "special language," indeed the "only language he can speak" [7]. He went on to say that the mathematical physicist uses mathematics not only for calculation but "above all, to reveal to him the hidden harmony of things" [8]. The mathematics referred to here, is, in contemporary practice of theoretical physics, usually taken to be "classical" mathematics (continuous mathematical constructions). With the arrival of the computer, a novel dimension in the history of physics has erupted. The time has come when we can seriously start thinking of replacing part of the work which physicists perform with ordinary mathematical equations (within the established paradigm) with discrete physics, which enables one to construct (computer) programs having (i.e., exhibiting) the behavior intended for the expert programs developed in the area of research called Artificial Intelligence. Thus part of what is currently called "theoretical physics" can, as a result of adopting the paradigm of discrete physics, be understood as belonging to what—in Dijkstra's terminology—is called "computing science" [9].* The switch of terminology from "computer science" to "computing science," reflects the insight of the presence of the transcendental Subject when programming is performed. As Polanyi has pointed out, the internal workings of the computer can, of course, be completely understood in terms of physical laws. What cannot be so explained is the computer's program. To explain the program requires reference to the *purpose* of the program [10].

The paradigm of discrete physics is also characterized by its adoption of *mathematics* (McGoveran's ordering operator calculus), while computing science essentially makes use of computer *programs*. These have usually been regarded as conceptually distinct practices; thus *classical* set theory and programming languages are conceptually distinct. Due to the work of Martin–Löf in grounding mathematics, we now know that this is not necessarily the case. This requires one to understand Martin–Löf's far-ranging insight that immediate mathematical practice exhibits a systematic distinction between *judgments* and *propositions*. This insight can be transferred to the practice of theoretical physics. However, it requires one to understand the connection between mathematics (constructive set theory) and computer programs, in order to understand the *purpose* of the mathematics used in discrete physics. In discrete physics, consequently, there is to be a connection between the notion of "mathematics" and the notion of "program," in agreement with the insight of Bishop [11].

The two methodologies of mathematics and programming have been regarded as conceptually distinct, as is easily seen in relevant literature. Furthermore, in current literature a sharp distinction is made between "programming"—which has not, for the most part, been understood as a mathematical practice—and "computer science"—which certainly has, as exhibited, e.g., in the work of mathematicians on graph theory, automata, combinatorics and formal languages. In fact, however, they are not conceptually distinct! Grasping this point requires a novel understanding of the method of mathematics—the method of Martin–Löf [12].

The method of Martin–Löf is not concerned with a theory of practical everyday use, but with a theory (paradigm) for understanding the practice of constructive mathematics. Alternatively, one can say that Martin–Löf's Intuitionistic Theory of Types (Sets) exhibits the *logical form* of the paradigm of mathematical language. As Martin–Löf said in a letter to Beeson: "I have been searching for a

* Of course, it is essential that the "computer" be realisable.

system which makes good sense, not only as an object of metamathematical study, but in its own right; one that stands on its own feet, so to speak" [13]. Achieving this, became, for Martin–Löf, the task of restoring the computational meaning of the well-known mathematical notions such as *function* and *proof*. As Martin–Löf points out, it was Brouwer, who realized the necessity of so doing: the true source of the uncomputable functions of classical mathematics is not the axiom of choice (which *is* valid intuitionistically) but the law of excluded middle and the law of indirect proof [14].

The intention of Martin–Löf's investigations is to make the Subject aware of the *common* structure of mathematics and programming languages. The genuine source of the difference between constructive mathematics and programming does *not* concern the primitive notions of either, since they are the *same*, but lies in the unreflective use of (1) program forms required in order to be read and executed by the computer, and (2) on the part of constructive mathematics in the fact that computational procedures (programs) are usually left *implicit* in the proofs (computations). Consequently, considerably further work is needed in order to exhibit them in a form fit for mechanical execution. Thus, Martin–Löf writes in his *Constructive Mathematics and Computer Programming* that "the whole conceptual apparatus of programming mirrors that of modern mathematics (set theory, that is, not geometry) and yet is supposed to be different from it. How come? The reason for this curious situation is, I think, that the mathematical notions have gradually received an interpretation, the interpretation which we refer to as classical, which makes them unusable for programming.... Now, it is the contention of the intuitionists (or constructivists, I shall use these terms synonymously) that the basic mathematical notions, above all the notion of function, ought to be interpreted in such a way that the cleavage between mathematics, classical mathematics, that is, and programming that we are witnessing at present disappears" [15].

These insights of Martin–Löf are reflected in the mathematics used in discrete physics. Thus one of the primary tasks of discrete physics (there are, indeed, many others) is to exhibit the common logical structure of theoretical physics, computer languages and constructive mathematics.

For example, the notion of a *function* is to be understood in the sense of a *method* to be applied in order to achieve a result within the paradigm of discrete physics. A function is *not* to be understood as a relation between arguments and value. A function is defined by providing *rules* (the method) for its calculation. As far as mathematical and computational practice is concerned, these rules amount to the 48 inferential rules of the Intuitionisic Theory of Types. Note that functions are not objects in the metamathematical sense of which it could be proved that they have the property of yielding unique values; rather, that functions yield unique values is to be *understood.*

The notion of *verification* (proof, computation) is to be understood in the same way as Martin–Löf. This amounts to understanding verification as a performative. One performs an immediate verification when one computes a *result* in theoretical physics. The aim when engaging in formulating the paradigm of discrete physics, in analogy with Martin–Löf's program, is not to formulate a language of theoretical physics for practical (explicit) everyday use, but to formulate a paradigm in physics in order to understand the practice of computation in physics in a more meaningful way.

In the paradigm of discrete physics, a "theory of physics" can be read as a *person program.* Thus, a theory is a piece of information (implicitly) giving instructions concerning what to do in order to attempt to falsify the theory (person program) in question. Consequently, a theory of physics and a person program amounts to the same. It is only a question of preference if one wants to adhere to a more object-oriented mode of language and talk of "theories," or, if one prefers a more subject-oriented mode of language to talk of "person programs." They amount to the same as far as content is concerned, i.e., they are synonymous ways of expressing the same point. Within the paradigm of discrete physics, a formulated theory *is* a person program (implicitly) giving instructions of its own validity when attempting to falsify a proposition by virtue of an experiment (performative).

Problem 1.

In order to show that the results of measurements, indeed, exhibit objective facts, the Subject has to engage in a philosophical (phenomenological) investigation of the practice of physics. It amounts to exhibiting the *practical* understanding *as it is given* in the immediate practice of physics. As far as theoretical physics is concerned, it amounts to understanding the practice of the theoretical devices used in theories of physics. Without a shared commitment to a set of symbolic generalizations, logic and mathematics could not routinely be applied in the community's work. In the terminology of McGoveran, to investigate this practical understanding amounts to investigating the "E–frame" (Epistemological framework) [16]. This amounts, in Husserl's terminology, to grounding (*begründen*) the practice of (theoretical) physics and is, essentially, a descriptive activity. This grounded practice is also the starting point when attempting to formulate a discrete paradigm in physics.

Contemporary practice of physics is usually regarded as consisting of, essentially, two subpractices (computer physics is usually not included in current practice): (1) experimental physics and (2) theoretical physics, where, according to Popper, "[theories] are nets cast to catch what we call 'the world': to rationalize, to explain, and to master it" [17]. What has not been generally understood, however, is that theories are always cast within some paradigm. Contemporary theories of physics are usually cast within the paradigm of continuum mathematics, where the paradigm "is what the members of a scientific community share *and,* conversely, a scientific community consists of men who share a paradigm" [18]. The current mathematical paradigm in physics is assumed to be based on the acceptance of the space-time continuum in the mathematics used (exhibited, e.g., as "renormalization" in quantum field theory or Wheeler's and Hawking's idea of "space-time foam" in quantum gravity).

The task of discrete physics is to change our understanding of the current paradigm. However, one cannot even *attempt* this if one is not already familiar with current practice of contemporary theoretical physics. One must have acquired the competence to engage in the practice of contemporary theoretical physics. In the terminology of Polanyi: what is required is "tacit knowledge" which is learned by doing science and not by acquiring rules for doing it [19]. These practices, as shared examples, must function as *data* for any attempt to engage in a paradigm shift in theoretical (mathematical) physics.

In order to engage in a paradigm shift, the Subject is to formulate the novel paradigm of theoretical physics. This amounts to engaging in what is to be called

Prephysics.

In McGoveran's terminology, it amounts to formulating the R–frame (Representational framework) and the P–frame (Procedural framework). To formulate these two frameworks amounts to formulating a paradigm in physics.

The R–frame "is an abstract formalism consisting of a set of symbols and a set of rules of manipulation" [20]. The logical form of the "rules of manipulation" amounts, when codified, to the 48 rules of Martin–Lŏf's Intuitionistic Theory of Types. To engage in formulating the R–frame is to engage in *syntax* in the terminology of prephysics. The activity of syntax is a speculative activity in the sense "that we really do not know what we are talking about," when engaging in this creative task. There are no rules regulating the activity of syntax. It is a speculative (and normative) activity. It is the absence of regulative rules which makes it possible to call syntax a creative activity.

What is still missing are explanations relating the observations (performed within the E–frame) and the symbols of the R–frame, which then, through recursion, serves to establish the relation between the E–frame and R–frame, until a sufficient level of agreement concerning accuracy is achieved or the paradigm fails (a la Kuhn) [21]. This explanation establishes the procedural framework, or the P–frame. In prephysics, the activity of formulating the P–frame is called *semantics.* It is a higher-order activity which, in a logical sense, can only be performed after the formulation of the R–frame is completed. In semantics we explain the E–frame by explaining the relation between it and the R–frame formulated in syntax. The explanation itself amounts to the rules regulating the connection between the old E–frame and the R–frame, thus supplying novel meaning to the E–frame.

The P–frame can also be understood as exhibiting, within the novel paradigm, a *translation manual* between the expressions occurring in the old E–frame, thus determining how far they can be given meaning within the novel paradigm. Alternatively, it can be understood as a *modeling* of the E–frame. In setting up the paradigm, we give, at the same time, a manual for translating between it and the ordinary forms of expressions used in practice (E–frame), and a model for these ordinary forms of expressions. The aim with semantics is to achieve a reflective equilibrium of understanding the practice of physics when performed according to the paradigm of discrete physics. Note that by semantics is not meant any representational, as is the case in the model-theoretical sense of semantics. The sense in which the word "semantics" is used in prephysics is not meant to be a branch of mathematics (like logical semantics, or its technical twin, model-theory); it is the activity of describing the relation between the E–frame and the R–frame.

In the research program of attempting to formulate a discrete physics, one is also concerned with establishing a novel paradigm in physics; or, as one could also formulate the point, we are concerned with establishing a novel paradigm of theoretical (mathematical) physics. Having successfully formulated the R–frame in syntax and described the connection between the E–frame and the R–frame by formulating the P–frame in semantics, one is to have achieved *theoretical understanding* of the practice of informal theoretical physics. Note that by the expression "theoretical" in the context of theoretical understanding is not meant "theoretical" as the expression is used in connection with, say, theoretical physics. In the sense the notion of "theoretical" is used here it purports, or attempts, to be a paradigm in the practice of theoretical physics. It cannot be conceptually separated from this practice because the paradigm determines the practice of theoretical physics to be what it is; to be able to formulate the point of the paradigm is to exhibit theoretical understanding of the practice of theoretical physics.

By engaging in prephysics one cannot exhibit the *sense* of the practice of physics, when the notion of sense (meaning) is understood as standing for what Frege called *Sinn*; that is, one cannot make the conceptual distinction between sense and reference in the modeling methodology of prephysics (as is assumed when semantics is understood in the model-theoretical sense). Prephysics establishes rules of meaning (definitions) in the sense of semantical descriptions (the P–frame), but presupposes rules of sense in order to be possible in the first instance at all. The rules of sense amount to the competence to use a natural language in order to grasp the point of the E–frame as a "universal medium of communication" (in Hintikka's terminology) in the first place.

Recall that we are dealing with the *practice* of physics requiring the presence of a transcendental Subject. All thinking presupposes the presence of a natural language as far as grasping the point with regard to the practice of theoretical physics is concerned. Indeed, this is what makes it possible to grasp the universality of physics in the first place. In this sense the transcendental Subject can be equated with natural language. This leads to the insight that *any* practice of physics (performed within any paradigm) exhibits a form of the Anthropic Principle to be called the

Transcendental Anthropic Principle

which, essentially, states that *natural language, being the Universal Medium of Communication, is necessary in order to bring the Scientific Universe into being-as-fact in the practice of physics.* This principle will be treated in more detail below.

Problem 2.

Since the classical period in physics, the idea has become familiar that a physical object is something real, existing outside of the thinking Subject, independent of whether or not the object has been subjected to observation. This has, in fact, on many occasions, been taken to be the criterion for scientific objectivity, since, it is claimed, one cannot attribute to a system at every instant its measurable properties. As a result of the introduction of quantum mechanics in the first half of this century, it seems at first sight that the concept of scientific objectivity has been strongly shaken. For example, one cannot even claim that a wave function has a well-defined meaning unless one explicitly refers to a definite measurement. Furthermore, it looks as if the result of a measurement is intimately connected to the *acts* (Bridgman: operations) of the Subject performing it, and thus, as if quantum mechanics drives one towards a complete subjectivism in the practice of physics.

One can formulate the problem like this: quantum mechanics is fundamentally about "observations." This is usually understood as implying a separation of the Universe into two parts, a part which is observed (*res extensa*) and a part which does the observing (*res cogitans*), if we use the terminology of Descartes. However, since Galilei and Newton it has been a basic requirement that physics ought to be objective. How is one to cope with this enigma? What is one to understand by the term "objective"? How is one to provide meaning to this term? The usual way out of the dilemma is to adhere to a *realistic* interpretation of theoretical physics by presupposing some kind of space-time continuum.

This cannot be done in discrete physics, which implies that one has to cope with the dilemma in some other way. Here we meet the first difficult insight:

> *To understand that physics primarily amounts to an immediate practice, and is only secondarily concerned with laws of nature and physical objects.*

To grasp that physics is essentially an immediate practice is more fundamental than to understand physics as concerned with certain laws and objects. Provided a measurement (experiment) of physics is made up of three discernible components: Object + Apparatus + Subject, as a combined and unique whole, then the philosophical problem becomes the task of grasping how a measurement provides objective knowledge of fact. By "objective" is to be understood the validity of a *result* of a measurement for any Subject participating in (performing) the practice. This amounts, essentially, to understanding a measurement in the sense of a *performative.*

To grasp the point that the immediate practice of measuring is a closed whole exhibiting *objective* knowledge of fact, can be regarded as the main puzzle in connection with measurements in "classical" physics, as well as in "classical" quantum mechanics. Both practices are "classical" relative to the mathematical paradigm applied. To correctly understand the objective semantical *force,* in Frege's terminology, of the immediate practice of measuring, amounts to grasping the objective self-evidence, or the *meaning,* of measurements in physics. And, as Wheeler emphasizes, "[no] feature in all physics voices more insistently the message 'meaning is central,' than the elementary quantum phenomenon" [22].

The philosophical task we are confronted with thus becomes to grasp the meaning of the *immediate* practice of performing computations and measurements in (discrete) physics. This must be somewhat qualified. It amounts to understanding these practices as exhibiting performatives. It is what the Subject actually *does,* i.e., the immediate acts (Bridgman: operations) of computing and measuring, that is real, actual or concrete. To be more precise, by an act is to be understood an act of *judging.* We shall return to this important insight below; it is enough for the moment to emphasize that the Subject is to grasp the point with regard to judgments; to be able to use public judgments in different informal practices of physics. That is, the Subject is to *understand* a certain practice in order to participate in that practice, to be able to exercise the faculty of judgment.

The Subject is to break into the circle of understanding by, in one way or another, achieving *practical competence* (the E–frame) to perform certain tasks, like computing and measuring. To exhibit practical competence to perform a certain task amounts to having practical understanding of the task; to be able to perform it. Here, it is simply a question of somehow achieving the practical skill to perform certain tasks. It is not, primarily, a question of describing verbally what is done. Practical understanding by the Subject is exhibited in having the competence to achieve results when engaging in the practice of performing computations in theoretical physics.

This competence is presupposed in order to be able to grasp the very point of engaging in prephysics. In other words, the primary task of practical competence (the E–frame) is to make the Subject grasp to what inductive reasoning amounts. Thus, prephysics makes us aware that *induction turns out to be the same concept as recursion.* They both amount to an immediate practice exhibiting practical competence. As an example of the presupposition of practical understanding (the E–frame), one can give the way a judgment of the form "$a \in N$" is introduced by Martin–Lõf: "a has value either 0 or $a_1\prime$, where $a_1\prime$ has value either 0 or $a_2\prime$, etc., until eventually, we reach an element $a_n\prime$ which has value 0" [23]. The point here is to emphasize that the task to be performed terminates after a finite integral number of steps and that this statement is not a metamathematical statement.

Here one clearly recognizes the necessity to have the competence of induction (recursion). It is crucial. As Poincaré said,

> "... induction, that is, demonstration by recurrence ... imposes itself necessarily because it is only the affirmation of a property of the mind itself" [24].

This was also Brouwer's position. In defending this position (of Poincaré and Brouwer), Weyl writes:

> "When Poincaré claimed that *mathematical induction* is for mathematical thought an ultimate basis that cannot be reduced to anything more primal, he had in mind precisely the processes, of composition and decomposition of numerals, that Hilbert himself employs in his contentual considerations and that are completely transparent to our perceptual intuition. For after all Hilbert, too, is not merely concerned with, say, 0' or 0"', but with any 0" $\cdots$', with an *arbitrary concretely given* numeral. One may here stress the 'concretely given'; on the one hand, it is just as essential that the contentual arguments in proof theory be carried out *in hypothetical*

generality, on *any* proof, on *any* numeral. This, of course, is not to be taken as an objection, for the procedure of the 'one after the other' can appeal to unshakable intuitionistic evidence; but, evident and primal though it be, we may nevertheless give it expression—not by formulating it as an 'axiom,' but simply by describing its concrete use—making its self evidence and primal quality explicit, and we are no doubt justified in seeing in it the characteristic mark of contentual *mathematical* thought" [25].

The important point to grasp is that induction essentially amounts to immediate practical recursive competence by the Subject. Practical understanding is exhibited in the competence to know *how* to do something practical (the E–frame), thus logically preceding theoretical understanding (this preceding is not a "preceding" in a temporal or empirical sense). As one cannot conceptually grasp *what* the Subject is performing logically distinct from *that* it is being performed, one cannot conceptually separate knowledge of the Universe (physics) from the method of knowing (implicitly or explicitly) *how* to achieve this knowledge, since the factual knowledge is established as the *result* of having performed a (repeatable) measurement. When immediate practice of physics is taken as fundamental, there is, in the end, no distinction between knowing *that* something is the case, and knowing *how* to reach this fact.

Problem 3.

The methodological task the Subject is confronted with in order to formulate and explain the point of discrete physics is to set up a *code* by engaging in *syntax* (the R–frame) and *semantics* (the P–frame). To engage in semantics is to engage in explaining the relation between the E–frame and the R–frame formulated in syntax; that is, semantics amounts to normatively prescribing the use of the expressions of the E–frame within the paradigm. Note that semantics comes last. We retain the idea with semantics coming last, to the extent that there are three discernible components of the paradigm corresponding to its three parts in logical semantics. We can differ between (1) object-valued and type (set)-valued functions (the E–frame), (2) objects and types (sets), i.e., symbols and rules of inference (the R–frame), formulated in syntax, and (3) the semantical part (the P–frame), which can be divided into a formal (stipulatory, mechanical) part and a nonformal (teleological, nonstipulatory) part.

The formal part consists of symbols, like, e.g., the natural numbers, or the symbols for length l, time t and mass m, which are already fully evaluated: if one evaluates the value of a formal symbol, one gets the value back. A symbol which is arbitrary formed need not, necessarily, have a value relative to the paradigm, but *if* it has a value then that value is necessarily canonical. This is why, for example, such symbols (those in the paradigm) amount to formal expressions. The nonformal part consists of the inferential rules *used* (implicitly) in the E–frame when performing practical tasks in discrete physics. The inferential (nonformal) part can be called the *teleological* part of semantics, because the Subject, in the practice performed within the paradigm of discrete physics, always tends to *use* (implicitly or explicitly) these rules in order to perform the computational task which the Subject set out to achieve. It is important to grasp that the Subject can apply these rules in computational tasks of theoretical physics without being able to *formulate* these rules in an explicit way. The logical form of the inferential rules, when formulated, exhibits itself in the inferential rules of Martin–Löf's Intuitionistic Theory of Types (Sets).* When performing theoretical tasks within the paradigm of discrete physics (also continuum physics), it is important to grasp that one is concerned with a single mechanism from which no one component can be removed without the others losing their nature. This is what makes the practice into a paradigm.

There is a precise rigid order when the Subject is to break into the circle of understanding discrete physics. To engage in syntax and semantics is a genuinely speculative activity. One could compare it with the moment when, after staring at a group of people playing a card game (the E–frame), with growing bewilderment and perplexity, something clicks, and all their operations with the cards fall into place. The Subject grasps what is done in these operations. Formulation of the R–frame and the P–frame does not amount to something that one can passively record from the E–frame. If it were just a question of passive recording, then the Subject would already know the method which is to be exhibited, since the Subject would *already* apply the method in order to record the agreed upon facts occurring in the E–frame.

* Note that this requires reading Martin–Löf's inference rules as pertaining only to finite domains.

Syntax and semantics is to provide a formal language and an explanation of this language which gives a *codification* of the informal concepts and rules used in the practice of theoretical physics. The formal language makes it possible to theoretically *exhibit* the meaning of these concepts and rules as they are used in the practice of physics. The expressions used in informal practice are translated in the language. Since the language is intended as a codification, one should not try to understand the point of the expressions and rules of the language through the translation. It is rather the other way around. The paradigm thus formulated provides the possibility of giving novel meaning (understanding) to the practice of theoretical physics. A practice of theoretical physics is never a conceptually "blind" practice as far as the task is concerned, since it is guided by the paradigm used. This, however, is not the case when engaging in syntax and semantics.

Here one can give an analogy with a machine (the E-frame) that has come down through several centuries. There are a number of people who can run this machine, some of them very skillfully. This would correspond to Kuhn's notion of "puzzle solving" or, alternatively, "research program," in the terminology of Lakatos. Lately, the machine has been put to use in unforeseen circumstances. Now it doesn't work properly; e.g., the Subject is faced with the conceptual separation of relativity theory and quantum theory. The result is that doubts arise whether some of the controls of the machine do anything essential, or whether they are indeed harmful or create havoc in the running of the machine in the new circumstances (e.g., space-time continuum), although they were harmless before; thus, it becomes urgent to understand the machine more profoundly, but *this* task is not just a descriptive undertaking. If the Subject is able to formulate in syntax any principles about the running of the machine, one may want to design new components which exploit these principles more effectively and improve the machine's performance. If we call the syntactical step from seeing just the physical operations (the E-frame) to grasping *what* is being done in the practice of physics *abstraction* (as Martin-Lõf does), then we can say that we know of no laws that regulate abstraction; thus, syntax and abstraction amount to the same activity.

Also semantics (the P-frame) is a speculative activity, the aim of which is to establish a reflective equilibrium between the regulative rules (the R-frame) and the practice of measuring in physics (the E-frame). This is performed by describing the point (semantical *force*) of the R-frame formulated in syntax. Such a description cannot be, in the last analysis, performed without the use of natural language. If the Subject is successful in semantics, a novel understanding of the practice of physics, i.e., a novel paradigm, is achieved. One must be aware, however—as Kuhn points out—that "[the] decision to reject one paradigm is always simultaneously the decision to accept another, and the judgment leading to that decision involves the comparison of both paradigms with nature *and* with each other" [26]. Note that one cannot *prove*, by virtue of semantics, that one paradigm is better than another. This can only be *understood*. There is no decision procedure by virtue of which this could be decided. Another way of stating this is to say that when a physicist is to choose between competing paradigms (of continuum and discrete physics), he behaves like a philosopher. In this sense, acceptance of a paradigm always amounts to a *normative* choice: to accept a novel paradigm within physics is to accept a *prescription* concerning practice of physics; thus one realizes that theoretical physics always incorporates a normative component exhibited by the paradigm adopted.

The important point to understand is that it is because the Subject previously has practical understanding and thus—employing the recursive (inductive) competence so achieved—that he can understand the point of prephysics in the first place, and, in addition, decide which of the competing paradigms is more meaningful. Here it is important to point out that semantics (the P-frame) is to bring about an understanding of the paradigm, but that there are certain limits to what verbal explanations can do when it comes to justifying the paradigm. As Martin-Lõf has pointed out, "In the end everybody must understand (the point of the paradigm) for himself" [27].

II

Problem 4.

Above we asked "Where is the place of the Subject in the practice of physics?" and "How is the notion of 'physical reality' to be understood in relation to the practice of physics?." We shall now attempt to answer these questions. According to Rorty, "[Discussions ...] in the philosophy of mind usually start off by assuming that everybody has always known how to divide the world into the mental and the physical—that this discussion is common-sensical and intuitive, even if that between two sorts of 'stuff', material and immaterial, is philosophical and baffling" [28]. This position exhibits a category mistake, which, according to Ryle, shows itself as the dogma of the "Ghost in the Machine" [29]. It maintains that there exist both bodies and minds; that there are mechanical causes of corporeal movements and mental causes of corporeal movements. In short, the doctrine assumes that there are physical processes and mental processes conjoined in the same category. This is a mistake. The idea of thinking as a process in the head, taking place in a completely enclosed space, easily provides something "occult." The judging (thinking) Subject is not anything over and above the judgments (thoughts), themselves. As Ryle points out, "(the) belief that there is a polar opposition between Mind and Matter is the belief that they are terms of the same logical type" [30].

This dualistic attitude, as a philosophical standpoint, has, implicitly, been transferred to all interpretations of the role of measurement when read in the light of quantum mechanics. However, it concerns *all* interpretations of measurement in physics; i.e., it concerns also measurements in "classical physics." Indeed, one of the characteristic features of current investigations into the "foundations" of physics is the attempt to provide intelligibility to, say, quantum mechanical structure, by attaching philosophical speculations of the role played by the psychological (empirical) Subject in physics (Wigner). This exhibits a mistake. The mistake concerns the way the notion of a "Subject" is understood, and is reflected in the way the *language* of physics is being understood. Actually, the problem is not a psychological problem, it is a problem of a *semantical* kind. So constructed, the meaning of physics amounts to a semantic thesis; a thesis about what, in general, renders a statement within the practice of physics true when it is true.

The crucial problem shows itself in a certain way of understanding the language of physics, based on the illusion that one, by using language, can provide an *interpretation* of physical reality. This way of thinking can be traced back to two conceptions of logic, which van Heijenoort has named "logic as calculus" and "logic as language" [31]. The conception of "logic as calculus" does not say that logic is like an uninterpreted calculus, but assumes that logic is reinterpretable like a calculus. The conception of "logic as language" amounts to the insight that one cannot get outside our logic, as it were, and its intended interpretation. It amounts to a doctrine of the *universality* (in the sense of *inescapability*) *of logic*. By "logic" one is to understand the point that the Subject, as it were, cannot get outside a practice (when performing tasks in physics) and its intended interpretation, i.e., nothing can be said outside *some* set of formal laws. Another way of formulating this point, is to say that the union of the laws that are possible is inescapable. Hintikka has generalized van Heijenoort's distinction into two basic ways of looking at one's language, which he call's "language as calculus" and "language as the universal medium," where "(as van Heijenoort noted) all logical semantics (model theory) is impossible if the view of language as the universal medium is correct" [32].

The standpoint of "language-as-calculus" leads to the belief that doing semantics (foundations) of physics amounts to providing the *correct* interpretation of, say, quantum theory; thus, philosophy, on this reading, amounts to a metainvestigation. This is, for a number of reasons, a mistaken attitude. By adopting the language-as-calculus way of thinking, the Subject is forced to accept the following theses:

1) Semantical relations are accessible.
2) The Subject can tell what it would be to have different semantical relations.
3) Model theory is possible.
4) Linguistic relativism is not a tenable doctrine.
5) The Subject can reach Reality as such because one can always subtract the influence of language.
6) The construction of metalanguage is possible.
7) Truth as correspondence is possible.

Acceptance of the language-as-calculus way of thinking amounts to a certain way of understanding, and this way of understanding is reflected in what the Subject expresses when attempting to understand physics. As a result of the adoption of the "language-as-calculus" way of thinking, there are a number of traditional philosophical pictures to which the Subject is habituated in the foundations of physics. First, perhaps most deeply rooted, is the philosophical "model of thought," which, in essence, finds its intellectual roots in Descartes' dichotomy between matter (*res extensa*) and mind (*res cogitans*). This "model of thought" can be visualized as in Fig. 1. In this philosophical "model," reality consists of all objects, and they are beheld by the Subject, Ego or Consciousness. Something like this picture occurs for example in perceptual psychology or, say, neurophysiology, where one analyzes the process of perception in terms of light waves, pressures, etc., which act upon the sense organs of the percipient and excite certain electrical and chemical phenomena in the nervous system. Here the Subject is understood in the form of an *empirical* Subject. When the Subject is understood in this way (as in neurophysiology), it is all right.

The analogue to this picture has also been used in *philosophy*. In this context, the model of thought represented by this picture has had a paralyzing influence, because of its emphasis on the *relational* character of philosophical (semantical) thought. One is forced to assume the real existence of relations in a mysterious "metaphysical" sense. This is the case in philosophy of mathematics, as well as in philosophy of physics. In the latter, it has erected the problem of the "detached observer." In the philosophical reading of this "model of thought" (language-as-calculus), the detached Subject receives sense impressions from the objects in reality, which are organized and sorted according to the categories of pure reason, canons of induction, etc., of the philosophical "tools" into iterative complexes through which the Subject can have knowledge of reality. As far as philosophy of physics is concerned, the objects of reality are either in a metaphysical reality in general, called "physical reality," perceived (somehow) by the philosophical mind, or the objects are part of the conceptual apparatus (mind) by which sense impressions are organized. The philosophical Subject is assumed to be in a logically separated "vacuum," exhibited as a relation between Mind and Matter.

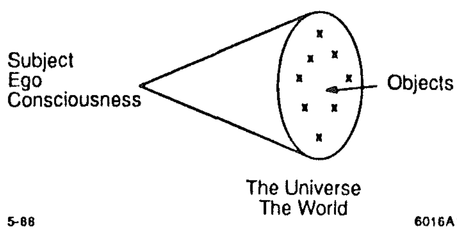

Figure 1

This picture, when formulated in the sense of "language-as-calculus" raises a number of problems in philosophical thinking.

It provides an uneasy oscillation between the philosophical doctrines of both verification-transcendent epistemological realism and the opposite, epistemological idealism.

This is the case because the status of the philosophical Subject is unclear. On the one hand, it is present in the process of recording facts of nature; on the other hand, its influence can be neglected since reality "is there" independently of any Subject. This way of thinking implies adoption of the model of thought (language-as-calculus) exhibited in Fig. 1. As examples of physicists who have been said to have embraced a verification-transcendent realist standpoint in this sense, one can mention Einstein and Schrödinger. Above, we tried to briefly describe the realist version of this "model of thought." One can also give an idealist emphasis of the "model." As examples of physicists who have taken this latter attitude, one can mention Wigner and von Neumann. In the idealist version, the emphasis of the picture is reversed from right to left; then, reality is not so much beheld by the Subject as it is constructed by him. In this case, reality is a product of the consciousness of the Subject, and depends on him. Both interpretations, realist as well as idealist, understood along the lines of "language-as-calculus," occur in physics; thus, e.g., Barrow and Tipler point out that "(the) Many–Worlds Interpretation is often classified as a 'realist' interpretation of quantum mechanics, as opposed to the idealist Copenhagen Interpretation, which brings the observer into physics in an essential way" [33].

The problem is that, however much one assigns priority to one side of the picture or the other, the "model of thinking" (language-as-calculus) remains essentially the *same*, and the philosophical problems inherently connected with it (the objectivity of the result achieved in the practice of measuring) remain unanswered. This leads to the next inherent problem of the "model."

The picture does not say anything concerning how the two sides of the picture are connected.

This model of thought does not say anything concerning, *how*, if one has access to reality only through one's impression, the connection (relation) is set up between those expressions and what they are expressions of, or, conversely, *what* principles regulate the construction of those objects, and out of what. Whatever side of the picture is emphasized, it remains silent on this crucial point. The relation remains mysterious. As far as this problem is concerned the picture is simply not intelligible to the intelligent Subject. The problems that the two sides in the "language-as-calculus" picture gives rise to, leads to the last inherent problem of the "model":

The philosophical Subject is separated from reality, as it were, by a pane of glass, to use Wheeler's metaphor.

The philosophical Subject is a spectator or observer, watching, perhaps, shadows on the wall of a cave, as Plato formulated the problem. The Subject has no contact with reality. One could, perhaps, say that the philosophical Subject watches an internal theatre, so one may ask, "Is the Subject *unreal?*"; but this is incompatible with what we learn from the quantum principle. As Wheeler puts the point, the quantum principle "demolishes the view we once had that the universe sits safely 'out there', that we can observe what goes on in it from behind a foot-thick slab of plate glass without ourselves being involved in what goes on" [34].

This situation, again, gives rise to the following question, "Is the model of thought asymmetric with regard to different Subjects?" In order to answer this question, the model of thought is, in traditional thinking, somewhat elaborated in order to moderate its subjective aspects, as in Fig. 2.

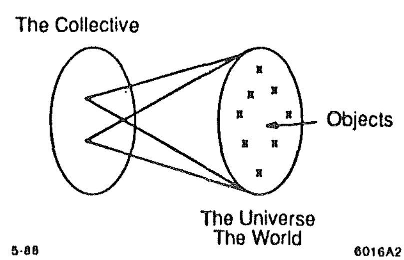

Figure 2

In such attempts, the empirical Subject is replaced by an intersubjectivity sustained between different empirical Subjects through their use of language; then, the single empirical Subject is replaced by a *collective* of empirical Subjects. To some degree, this picture, indeed, does manage to explain the objectivity of the Universe. Moreover, it achieves symmetry with respect to the observers. Despite this, there is still a crucial problem connected with this interpretation. This elaborated version of the "language-as-calculus" model of thought, still logically distinguishes, despite the intersubjective emphasis, the Subject and the Universe, whereas the Subject ought to be part of it. It still leaves the Subject *outside* the Universe.

Consequently, one again faces three (separable) parts, even if one attempts to "collectivize" the language: (1) the collective language, (2) the Universe and (3) the obscure connecting part, providing the (collective) interpretation of the Universe; also, in this case, the Subject (using a language) and the Universe are logically distinct; thus one can see that a "collectivization" of the Subject as a philosophical observer leads to the same *cul-de-sac* as in the case with the single observer. The empirical Subject ought to be a *participator* in the Universe. The quantum principle throws out the old concept of "observer" and replaces it, as far as the empirical Subject is concerned, with the new concept of "participator." That is to say, in Wheeler's formulation, "(in) some strange sense the quantum principle tells us that we are dealing with a participatory universe" [35]. The most serious defect of the "language-as-calculus" model of thought is the inability to exhibit the Subject as being a part of the Universe, and the necessity, following from this, of leaving the connection between the Subject and the Universe in obscurity.

To engage in the practice of physics in accordance with one's understanding, when based on the "language-as-calculus" model of thought, is to think in accordance with what one can call

Philosophical Separability.

The "language-as-calculus" model of thought is characterized, as we have emphasized, by three discernible parts: (1) Language and the Universe are always (logically) distinct systems. A corollary to this attitude is the doctrine of the distinction between Mind and Matter. (2) We set up the Language, say quantum mechanics (it could as well be the language of classical mechanics) to communicate among ourselves, and record (by experiments and observations) facts that we have discovered about the Universe. (3) We assign (in the fashion of logical/model-theoretic semantics) nonlinguistic items to the linguistic ones as their semantical "reference" (Frege), "denotation" (Russell) and "interpretation" (Davidson), in the hope of thereby showing their meaning and setting up the correct interpretation. This way of grasping the point of semantics is typical to the model of thought exhibiting philosophical separability. This is visualized in Fig. 3.

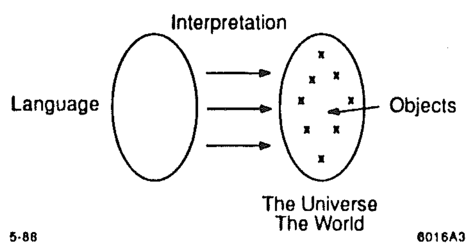

Figure 3

The outcome of the reflection on this model of thought (language as calculus) shows that there is something fundamentally wrong with this way of understanding physics. The problem focuses on the philosophical separability, which is an attitude based on illusion (as a semantical doctrine), because it makes *any* account of the connection between Language (the Subject) and the Universe obscure. According to this reading, we always interpret (a "veiled reality"?—in D'Espagnat's terminology), presupposing that we have access to anything real, actual or concrete, which we wished to make the denotation of in the Language. The problem arises because any means employed to identify that assumed real thing (of the Universe) would have an irradicable linguistic aspect (in the form of an interpretation). As examples of tacit adherence to philosophical separability, one can give the different "Quantum Realities" which are being provided as answers to the problem occurring when "(different) people looking at the same theory come up with profoundly different models of reality..." [36]. Here the assumed philosophical problem is regarded as the task of providing the (correct?) interpretation in addition (and *a posteriori*) to the quantum formalism, itself. This is not the way "theory" and "model" are understood in discrete physics. The result of the "language-as-calculus" way of thinking is that another linguistic system (the actual interpretation) interposes itself between the first linguistic system (say the quantum formalism) and its assumed field of denotation (the nonlinguistic reality). What provides the criterion that one of the formulated interpretations, indeed, is the correct one? Here, we see the problem inherently connected with this way of thinking: by virtue of *what* does one decide on the correct interpretation? Recall that we are here dealing with a relation between language and reality. Any attempt to formulate this correct interpretation requires a linguistic medium. By thinking along the lines of philosophical separability, one can *never* provide the adequacy criterion in order to decide which interpretation is the correct one. Let us look at this astonishing insight a little bit more closely.

One way of providing meaning to the various syntactical "entities" of a formal language is by modeling it in the way with which we are all familiar. The typical case would, of course, be the standard modeling of first order predicate logic. How does one proceed in this case?

- To begin with, one has a symbol, say A, for the type of individuals to which one assigns a set, which is referred to as the individual domain.
- Similarly, to each individual term, say t, one assigns an individual, say a; that is, an element of the individual domain.
- Furthermore, one assigns to each *formula* a proposition.
- Finally, one proves that if a formula is formally derivable, then the proposition which is assigned to it comes out true.

Intuitively interpreted, this means that one assigns to each formal derivation a proof of the proposition which is assigned to its end formula. This is a pattern which is followed in all kinds of modeling; most recently, in denotational semantics of programming languages, i.e., one assigns to the syntactical entities that one is dealing with certain mathematical objects and speaks of those objects as the *interpretations* of the syntactical entities. In model theory, one looks upon the interpretations in the object-oriented way in which one ordinarily deals with mathematical objects; i.e., one disregards language and handle the objects "directly," in the way one is accustomed to as a mathematician. This exhibits a mistake.

To begin with, every "object of knowledge" amounts to an *expression*. Indeed, a moment's reflection is enough to show that one is not at all dealing with these objects in a language-free way. How could one? After all, one is assigning a mathematical object to the syntactical entity by giving an *expression* for that object. One *always* uses an expression, a linguistic expression (which one, ultimately, in the last instance, must understand by virtue of one's "universal medium of communication") in order to express the object which is to serve as the interpretation of the syntactical entity. There are no "things-in-themselves" somewhere in a "linguistic vacuum."

Usually the meaning of the statement "the term t denotes the object a" is determined by three logically distinct things (in the terminology of logical semantics):

1) the meaning (use) of t as a term,
2) the meaning (use) of the denotation relation, and
3) the meaning (use) of the expression (interpretation) a.

Now, the expression a is, of course, *also* a term belonging to language, and it follows that in order to understand the above statement (the way it is used), the Subject must know the meaning of a statement of the form, "the expression a denotes the object b, " where b is another expression that denotes a. This last fact must be *presupposed* known when providing the actual explanation, and we are led to a regress concerning the correct interpretation. This can also be understood as exhibiting a critique against using Tarski's *correspondence theory* of truth in semantics: a true statement is a statement that is true to the facts. Here, one assumes *a priori* a certain relation between object-language and metalanguage and, again, a relation between metalanguage and metametalanguage, etc. This leads to a hierarchy of languages, the adequacy criterion of which cannot be stated. Indeed, we are led to a neverending regress concerning the meaning of the adequacy criterion. We never establish a paradigm. As Wheeler puts the point, "How can there be an end if we ask always for foundation of foundation of foundation?" [37]. We are led to an infinite regress. The situation is pictured in Fig. 4.

It becomes logically impossible to explain *how* the object a could be identified at all. The source of the paradoxical result is the philosophical separation of language (the Subject) and some assumed reality, as if the separation itself would be problem-free, like an inequality $A > B$ occurring as a relation between numbers. This is usually not understood. The problem is to understand how there can be a "semantics without semantics," if we use Hintikka's terminology. By the term "semantics" in the first part of the quotation, is meant semantics in the sense of prephysics (the P–frame).

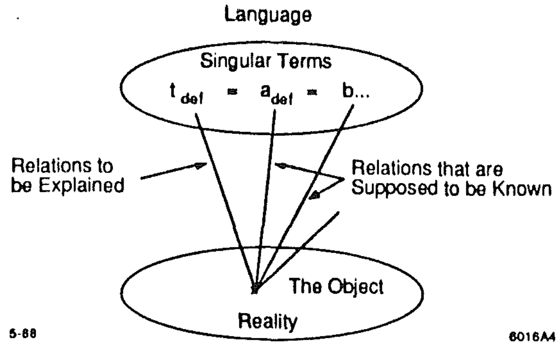

Figure 4

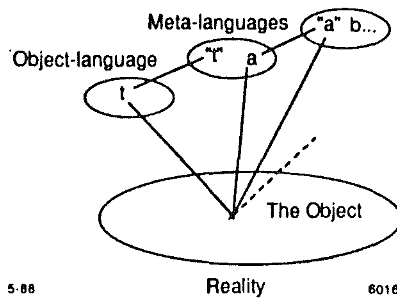

Figure 5

One might think that the problem could be avoided by introducing the distinction between object-language and metalanguage, or, what amounts to the same, the distinction between use and mention. Thus, the statement "the term **t** denotes the object **a**, " is a statement belonging to metalanguage and it is in that language that the expression **a** is *used,* while the term **t** is only *mentioned.* In order to be able to use the expression **a** in explaining the meaning of the term **t**, one must, of course, already understand the expression **a**. One must know *what* object it denotes. The result is that we have the same infinite regress, shown in Fig. 5. The distinction between object-language and metalanguage, exhibiting adoption of the "language-as-calculus" way of thinking, *itself* gives rise to the same difficulties by exhibiting adoption of philosophical separability; i.e., that one can separate a language from that (object or linguistic term) which the language treats, without being involved in the separation by having made an explicit formulation using the "universal medium of communication." This problem concerns the possibility of providing *any* interpretation. This point implies, as pointed out by M. and J. Hintikka, that "[the] impossibility of varying the interpretation of our language is an important additional reason why all model theory is impossible on the view of language as the universal medium. For a systematic variation of the representative relations between language (or at least its nonlogical vocabulary) and the world is a conceptual cornerstone of all logical semantics. Indeed, the development of logical semantics and its technical twin, model theory, has gone hand in hand with a gradual transition from the view of language as the universal medium to the view of language-as-calculus" [38]. Thus, precisely, as we pointed out in the beginning of this paper, the notion of logical semantics has made an "evolution" similar to that which the notion of function has made in mathematics; but one is not to be surprised: both logical semantics and "classical" mathematics rely essentially on philosophical separability, by treating the notion of function as a relation between arguments and value, a relation existing in a "linguistic vacuum."

One important insight to be gained by grasping the distinction between the semantical paradigms of "language-as-calculus" and "language as the universal medium" is that, in the end, philosophical problems of physics cannot be logically distinguished from philosophical problems concerning the language of physics as far as meaning is concerned. For example, the result of adopting the "language-as-calculus" way of thinking, allowing the logical distinction between object-language and metalanguage, is that we have a *diaspora* of interpretations concerning "quantum mechanical reality." This way of thinking exhibits itself in the way the problem concerning the "detached observer" in connection with the "foundations" of quantum mechanics is presented. The problem is that there is *no* universal adequacy criterion available for what is to be understood by a *correct* metatheoretical interpretation. Here, it is not a problem of providing the correct interpretation; it is the (logically impossible) task of discerning what is to count as an interpretation in the first place. This insight does not seem to have been generally appreciated in current literature on the topic.

The most important task facing philosophy of physics today seems to be to end this diaspora of "interpretations" and once more unite the understanding of the Subject concerning the practice of physics, by making the Subject grasp the practice of physics as a closed whole (a paradigm), embedded in the "life-world," i.e., in the universal medium of communication. Husserl emphasized the necessity of a paradigm, when he said "how could actual study and actual collaboration be possible, where there are so many philosophers and almost equally many philosophies? To be sure, we still have philosophical congresses. The philosophers meet but, unfortunately, not the philosophies. The philosophies lack the unity of a mental space in which they might exist for and act on one another" [39].

One simply has to accept that the Subject is being brought up in a life-world (*Lebenswelt*) by virtue of which scientific practices acquire their meaningfulness. The very point of the practice of performing experiments, for example, must incorporate practical understanding of the word "experiment" and what it implies. It must include understanding of experiments (measurements) as carrying the *semantical force of verification*. This seems to be Bohr's point, when he states that by the word *experiment* we can only mean a procedure regarding which we are "able to communicate to others what we have done and what we have learnt" [40].

To be able to engage in the practice of measuring in physics presupposes that the description of, say, an experimental setup and the result of the experiment, must be given "in plain language, suitably refined by the usual physical terminology," as Bohr formulates the point [41]. The deep insight here is Bohr's emphasis that, in order to engage in measurements in physics (in so far as understanding the point of a measurement), there is a presupposition that one is familiar with the use of a natural language as the universal medium of communication. This insight makes us understand that Bohr accepts (although never explicitly stated) the following theses:

1) Semantical relations are inaccessible.
2) The Subject cannot imagine different semantical relations.
3) Model theory is impossible.
4) Linguistic relativism is inevitable (the Subject is "trapped" in language).
5) The Subject cannot grasp Reality without linguistic interference.
6) The construction of a metalanguage is impossible.
7) Truth as correspondence is inexpressible.

The important point here is that semantical relations between language (the Subject) and reality are inexpressible (which is not to be confused with some kind of linguistic idealism). Recall that the empirical Subject is elevated to a participator in the Universe. In this sense, the Subject is always—as far as sense is concerned—embedded in a life-world (*Lebenswelt*).

Here, one is to look at the modeling in a different way—namely, think not only of what is to be interpreted as linguistic expressions, but think *also* of the interpretations which are assigned to them as linguistic expressions, expressing objects (of knowledge) in a linguistic way. Then, what appears to the model theorist as a modeling, appears—taking the attitude of semantics in the sense of "language as the universal medium"—simply as a *translation* into another language. A translation is always to be a translation into another language. Thus, we reach the insight that modeling and translation are the *same* thing within the semantic paradigm of "language as the universal medium," whether one takes an object-oriented attitude towards the interpretation or whether one looks at it linguistically. (Quine has emphasized this attitude already for a long time.) Whatever way one chooses to look at it, as modeling or as translation (within the semantical paradigm), this is certainly one way of giving meaning to the linguistic expressions of a formal language of theoretical physics. This point has been called by Hintikka the "paradox of formalization," in that language as the universal medium leads into formalism since, after excluding semantics, we retain only syntax. On the other hand, language-as-calculus also leads into formalism, since one is likely to mark those elements of language that can be reinterpreted.

One cannot conceptually separate factual knowledge and reality which the view of "language-as-calculus" assumes. This concerns *any* attempt by the Subject to formulate its understanding of the scientific physical Universe, and in addition, *any* interpretation of this understanding. This insight can be formulated as the *Transcendental Anthropic Principle* (TAP), which we formulated above. This principle differs from the Weak Anthropic Principle of Dicke (1957) [42], and the Strong Anthropic Principle, both in the form advocated by Carter (1974) [43] and Wheeler (1977) [44], in that these formulations all exhibit the Anthropic Principle as being concerned with *factual* knowledge and thus, essentially, being concerned with the empirical Subject.

The first to use a modern version of an Anthropic Principle seems to have been Whitrow [45], in a response to the question, "Why does space have three dimensions?" Although unable to explain why space actually has three dimensions, Whitrow argued that this feature of the Universe is not unrelated to the existence of the Subject as observer of it. Interestingly enough, the paradigm of discrete physics provides a proof that the measurable world with the richest dimensional structure consists of three dimensions, plus unobservable universal time and locally consequential time. This insight is treated in a rigorous way by McGoveran in *Foundations for a Discrete Physics* [46].

The Transcendental Anthropic Principle, essentially, tells us that it is impossible "to look at one's language from the outside and describe it, as one can do to other objects that can be specified, referred to, described, discussed and theoretized about in language," as M. and J. Hintikka formulates the point [47]. The Transcendental Anthropic Principle is another way (in a terminology perhaps more familiar to physicists) of expressing the necessity of understanding language as the universal medium of communication, when understanding that practice of physics is primary in relation to understanding physics as being concerned with objects and laws. The reason for the necessity of **TAP** is that one can use language to talk about something only if one can rely on given definite denotations (definitions), as we have emphasized above. That is, one must presuppose "a given network of meaning-relations obtaining between language and the world. Hence, one cannot meaningfully and significantly say in language what these meaning-relations are, for in any attempt to do so, one must already presuppose them" [48]. The Subject is truly a participator in the scientific Universe as far as meaning and understanding is concerned. This seems to have been emphasized by Bohr. In a discussion, as reported by Petersen, we find that Bohr is reported to have said that "...'reality' is a word in our language and that this word is no different from other words, in that we must learn to use it correctly..." [49].

Bohr is also reported by Petersen as having said that "(we) are suspended in language in such a way that we cannot say what is up and what is down" [50]. Compare this statement with the one made by Wittgenstein, when he states that "(human) beings are entangled all unknowingly in the net of language" [51]. What a striking resemblance between this statement and the one attributed to Bohr! One cannot express much clearer than this the adherence to the insight of the empirical Subject as being a participator in the scientific Universe, the insight of the *Transcendental Anthropic Principle*. This reminds one of the *hermeneutical circle*, which—in general terms—says that the Subject must always have understood in order to understand, and that the Subject nevertheless can improve this "preunderstanding" by methodological attempts to make the practice understood more meaningful, by engaging in syntax and semantics in the vocabulary of prephysics. This insight can be formulated by reflective transcendental statements like, "The Subject enters the realm of the life-world by grasping a natural language."

When the Subject learns language (the universal medium of communication), it can also perform conceptualizations; it expresses thoughts. One grasps that there are chairs, tables, trees, etc. (In Quine's terminology, "The Subject begins with 'ordinary things' and the totality of our so-called knowledge or beliefs, from the most casual matters of geography and history to the most profound laws of atomic physics or even pure mathematics and logic, is a man-made fabric which impinges on experience only along the edges.") The vocabulary is gradually expanded to include mathematical and scientific terminology. However, what is important to grasp is that immediate nature, consisting of chairs and trees (Quine's "ordinary things"), are *part* of the "background" of the Subjects preunderstanding of nature, relative to the more sophisticated language of physics. One could say that the life-world is rather like an "onion" of layers-within-layersof language, criss-crossing each other, to express the point in metaphorical terms. Expressions like "reality," "the world" or "the universe" belong to the "onion." With the words of Rorty, one can say that physics with its emphasis on objects (planets, elementary particles, quarks) and laws of nature (GTR, QED, QCD) functions like a "reflective mirror" of nature. The practice of physics becomes, essentially, like a mirror of participation in the life-world.

The upshot of this discussion ought to be the insight that there are no truths-as-facts outside logically possible experience. The scientific Universe is the totality of everything that can be the object of our experience, i.e., it is the totality of all possible experience. That is, with the words of Husserl, "[the] attempt to conceive the universe of true being as something lying outside the universe of possible consciousness, possible knowledge, possible evidence—the two being related to another merely externally by a rigid law—is nonsensical. They belong together essentially; and, as belonging together essentially, they are also concretely one, one in the only absolute concretion: transcendental subjectivity. If transcendental subjectivity is the universe of possible sense, then an outside is precisely—nonsense" [52].

Problem 5.

Prephysics amounts to a phenomenological investigation. To begin with, it amounts to grasping the insight that thought, language and Universe are one, "The transcendental ego inseparable from the processes making up his life," as Husserl puts it [53]. In the terminology of Parker-Rhodes, one can say that thought, language and Universe are *indistinguishable.* They are nonseparable ingredients of any scientific practice. Thought shows itself in the immediate acts of the Subject when attempting to solve scientific *tasks*. Thus one can say that only *acts* (Bridgman: operations) are real, actual or concrete. They exhibit performatives (= quantum principle of action). As far as the practice of physics is concerned, the most important acts are acts (operations) of *judging*. A judgment (operation) may be understood either as an act of judging (act of operating), or as *that* which is judged (the result of the operation). Likewise, an *assertion* (Ger. *Urteil*) may be understood either as an act of asserting or as that which is asserted. In its first sense, an operation is nothing but an act of knowing and, in its second sense, that which is known; that is, a piece or, more solemnly, an *object* of knowledge. The result is that judgments, operations and assertions amount to synonymous expressions, and one is to be preferred over the other only on stylistic criteria.

If one is not careful at this point, it is easy to be confused by the way the expression "operation" (judgment, assertion) is to be used in prephysics. It can, among other things, mean: (1) process of operating, (2) object obtained as the result of a process of operating and (3) operating-process as object, i.e., not understood in the sense of something "dynamic" [54]. Operations can be viewed as *processes* and differ from the resulting (constructed) object judged. The latter is a mathematical object (of knowledge) and can be *used* in the practice of physics; not so the former. In the sense one is to use the notion of "operation" in prephysics, it corresponds to form (2). This becomes evident, e.g., when contemplating on the task of engaging in syntax (formulating the R-frame). We are not interested in the "process" of what goes on in the mind of the Subject formulating the R-frame, but in the objects (of knowledge) obtained as the *result* of a "process" of operating (judging).

From the expression "operation," or "process," when understood in a representational way as, e.g., Whitehead does in *Process and Reality,* one might easily get the impression that there are actually two objects involved: the object formulated, or constructed, by virtue of the "process," and the "process" itself. This logical distinction is based on an illusion. The "process" and the object formulated by virtue of the "process" are *not* logically distinct things. They are two different ways of speaking of the same thing. To understand this, consider what must be done to construct each of them. To construct the object A, one must carry out (or perform) the construction of A. To construct the "operation" or "process" of A, one must do exactly the same. Both ways of formulating the task amount to the same: codify A. Each way of speaking has its advantages. The general insight gained is that one cannot engage in an "operation," or "process," without constructing A. Anything that needs to be proved is included. Thus, by an "operation" is to be understood a codified (in written or spoken form) object of knowledge.

Consequently, there are objects (in mathematics and physics), but only acts (operations) are actual, real or concrete. One can, perhaps, like Husserl, say that "[active] judging is not the only, but it is the original, form of judging. It is the sole form in which the supposed categorical objectivity, as such, becomes actually and properly generated. It is, in other words, the only form of judging in which the 'judgment' becomes, itself, as given originally" [55] (i.e., active judging is canonical). What the Subject has to understand, is that any assumed connection (relation) between Subject and object is *fused* in the acts of operating, performed by the Subject. One could say, as we already pointed out above, that the obscure (semantical) relation which comes last when thinking in accordance with philosophical separability (language-as-calculus), comes *first* when understanding measurement in physics according to prephysics. As far as existing "philosophical interpretations" are concerned, the reflecting Subject has to "bracket" (Husserl's terminology) the existing (language-as-calculus) convictions. Thus the Subject is faced with the task of grounding (*begründen*) the practice of measurement. This is performed by judging the *code* (R–frame and P–frame) regulating the practice of measuring. The Subject is never to deal with matters of fact, before questions of meaning have been settled, when formulating a novel paradigm.

However, an act of operating (judging, asserting) is easily read as a solipsistic act (without possibility of communication) and this, of course, is not what prephysical thinking adheres to as far as the practice of physics is concerned. This point is well formulated by Wheeler, when he states that "[what] is required in the analysis of genesis is not private judgment, but public judgment—which is to say science" [56]. But, one may ask, what is the difference between "subjective" operations (judgments) and "objective" operations (judgments)? Isn't the act of judging in both cases intrinsically connected to the judging Subject? This puzzle is dissolved by noting that by Subject is meant the *transcendental* Subject. It is this kind of transcendental Subject that Wittgenstein is addressing (despite his calling it the "metaphysical Subject"), when he writes that "(the) subject does not belong to the world; rather, it is a limit of the world" [57]. Thus the Subject engaging in prephysics, i.e., of judging the R–frame and the P–frame, is a transcendental Subject living in a life-world (*Lebenswelt*).

Here we meet an insight of crucial importance: *prephysical judgments can not be distinguished from pure realism.* The Subject is characterized by being a participator in the life-world and, thus, in the universe of intention. The Subject manifests this by exhibiting *thoughts* (*Gedanken*), and these thoughts always *mean* something to the Subject. In this sense, the Subject is always in the universe of intention, "The world and life are one" [58]. The outcome is, as Wittgenstein put it, that "... it can be seen that solipsism, when its implications are followed out strictly, coincides with pure realism. The self of solipsism shrinks to a point without extension, and there remains the reality coordinated with it" [59].

The transcendental (operating, judging) "I" (which can just as well be expressed by "you," "she," "he," etc.), coincides with the limit of the world. One easily gets the impression that there, in connection with the meaningful use of the language of physics, something psychological seems to happen, the closer inspection of which would be a purely empirical affair. This may, indeed, also be the case; but in this case, we are always dealing with the empirical Subject participating in the life-world. Now, one is not to confuse the philosophical, transcendental, Subject with the, empirical, psychological Subject; thus, the empirical Subject can be satisfied with merely noting that such and such things *must* happen (somehow) in order that meaningful use of the language of physics be possible. At that moment, the Subject steps beyond psychology and enters the sphere of transcendental reflection on the practice of physics; thus, the transcendental Subject is not totally unconnected to the empirical Subject. The transcendental Subject lays down conditions for the empirical Subject to fulfil and, as far as physics is concerned, it is the task of the transcendental Subject, by virtue of prephysics, to bring forth what these conditions are (the R–frame and the P–frame).

The code formulated can be explained in a more subjective form, i.e., a more idealistic form, emphasizing the presence of the empirical Subject (the "person program" terminology), or the explanation can take a more objective form, disregarding talk of the empirical Subject (the "theory of physics" terminology). Thus one can engage in prephysics by using a more subject-oriented way of formulation (Gefwert), as well as by using a more object-oriented way of formulation (McGoveran), by virtue of the semantical paradigm of "language as the universal medium." Both ways are imbedded in the life-world (*Lebenswelt*), being the transcendental ground for *any* understanding of the meaning of the practice of physics. The result is that whatever form one chooses to adopt, we seem to arrive at two kinds of truth in the practice of physics: (1) the transcendental notion of truth and (2) the notion of truth arrived at by virtue of the result of measurement (verification). One must be careful not to confuse these two notions of truth.

First, we have the notion of truth which is established as the *result* of a measurement (verification). This notion of truth can be called truth-as-fact. We have the analogous case in mathematical practice where a computed (proved) theorem is a proposition of the form "A is true," which is intrinsically linked with *immediate* provability (or verifiability). Wittgenstein expressed this when he said that "[the] stream of life, or the stream of the world, flows on and our propositions are so to speak verified by the present" [60]. Another way of stating this point is to say that prephysics emphasizes the importance of "do-it-yourself" (immediate) measurements in physics. In philosophy of mathematics, this way of understanding mathematical practice (analogous to prephysics) is best represented by the works of Martin–Löf.

Secondly, we have the notion of truth connected to the *validity* of a measurement in physics. This notion of truth can be called truth-as-validity. The notion of the validity of a measurement is the most fundamental notion of all, because to say that a measurement in (discrete) physics is valid, conclusive or correct, is nothing more than saying that the measurement *is* a real, or transcendentally true, measurement. It claims that a measurement is a valid measurement (a verification). Again we have the analogous case in mathematics, where we talk of a computation (proof) being valid. We can now grasp that the prephysical explanation of the notion of truth-as-fact (the analogous case in mathematics being Martin–Löf's explanation of the truth of a proposition—an expression of the form "A is true") is entirely compatible with pure realism.

Prephysics shows that the task exhibited by a question like "What objects does the world consist of?" is a question that it only makes sense to ask within the *practice* of physics regulated by a paradigm. As Wheeler says, "It tells what question it makes sense for the observer (participator) to ask" [61]. Thus, the aim with prephysics is to exhibit a *method* (the R–frame and the P–frame) in order for the Subject to be able to search and find propositions of the form "A is true," where "true" is to be understood in the sense of truth-as-fact. However, by the notion of transcendental realism is meant the philosophical insight that one can take the notion of reality for granted by virtue of the life-world. In this sense the Subject engaging in the practice of measuring in physics *already* does take the transcendental reality for granted. This means that the truth-as-fact point which one can exhibit, e.g., in the linguistic form of epistemological idealism (person program terminology), and which we, usually, exhibit in connection with physics in the linguistic form of epistemological realism (theory of physics terminology), can be understood as guiding the practice of measuring in physics, when physics is what Ryle has called "the game of exploring the world" [62]. Thus physics, understood as a "game of exploring the world," is entirely compatible with realism, if by realism is understood transcendental reality. This is already reflected in the Greek term *fysiká* ($\phi\upsilon\sigma\iota\kappa\alpha$) meaning "nature" and used by Aristotle to denote "natural science" (natural philosophy).

One is not rejecting the notion of "reality to be discovered" when engaging in the practice of (discrete) physics. As far as this point is concerned, there is no difference what paradigm of physics (continuum or discrete) the Subject adheres to. Whatever paradigm of physics the Subject uses, the transcendental reality is always presupposed. Maybe one can, like Prawitz, say that "(the transcendental) world is not there independently of us, but given that we are here, the world is also there waiting to be discovered" [63]. Indeed, to not adhere to this insight would amount to a genuinely irrational standpoint; to perform measurements without understanding the very *meaning* (point) of the practice of measuring in physics. Practice of physics would amount to an irrational practice for the Subject despite the fact that the Subject performs this very practice; a paradoxical situation. The Subject would be like a savage looking at an artifact (like a computer) not grasping what it is and what it is designed to do (despite being engaged in pushing certain buttons on the keyboard).

Another way of putting it, is to say that in such a case the Subject would not understand *what* to do when encountering such an artifact. Thus, the actual, or immediate, practice of measuring, just by being performed successfully, exhibits that transcendental reality is being adhered to. The actual practice of physics itself, just in the very performance of it (in whatever paradigm), *shows* that transcendental reality is being adhered to. Doubt at this level amounts to doubting the very meaning of measurements performed in physics. For a physicist it amounts to doubting the very meaning of the measurement he is currently performing. For a mathematician it would amount to doubting the very meaning of the computation (proof) he is currently performing. For the theoretical physicist it would amount to doubting the very meaning of the theory he is currently formulating or the calculation he is currently performing by virtue of the theory. For the ordinary human being it would amount to doubting the very meaning of the sentence he is currently uttering with the intention of conveying a certain point. One can say that practice is speaking for itself. In this sense the participating Subject is *always* a transcendental realist *au fond.*

III

Problem 6.

For the Subject to engage in the practice of physics presupposes practical competence to perform certain *tasks* in physics. One cannot exhibit any methodological rules for how to achieve practical competence. Practical competence does not *primarily* amount to verbal explanation (although one usually needs verbal explanations, i.e., semantics, in order to understand how to perform the task set out to achieve). To exhibit practical understanding is to exhibit *competence* to actually perform experiments and computations on request. This practical competence cannot be substituted for theoretical (descriptive) understanding of practice. The starting point must be the practical capacity to know how to perform measurements and computations in physics. One can list certain informal (heuristic) conditions that the Subject has to meet in order to achieve practical understanding (the E–frame):

(1) Agreement of cooperative communications
- commonly defined terms as fundamental
- fundamental versus derived terms
- agreement of pertinence

(2) Agreement of intent

(3) Agreement of observations

(4) Agreement of explicit assumptions

(5) The Razor
- agreement of minimal generality
- agreement of elegance
- agreement of parsimony

As we stated above, practical understanding of physics amounts to the practical competence to perform certain tasks in physics when requested to do so. These tasks are those which one normally would say correspond to the tasks of a trained "experimentalist" in the laboratory. However, practical understanding occurs also in what we call "theoretical physics" (including mathematics and computing/computer science). Here the practical understanding shows itself in the competence to formulate, explain and calculate, with theories of physics in the sense of a trained "theorist." The point of courses, examinations and laboratory training (including mathematics and computer/computing science), is precisely to convey the skill exhibiting itself as practical understanding of physics. In Kuhn's terminology, the practice of the "experimentalist" and the "theorist" belong to *normal science.*

As was realized a long time ago, there is no theoretical method by virtue of which the practical competence can be achieved. This is reflected in the term *heuristic (ars inveniendi),* by which is (and was in classical Greek) meant the methods and rules of discovery and invention. Important sources when investigating the heuristic method are provided by Euclid, Pappus (who has very interesting comments on the topic), Descartes and Leibnitz. In the last century the topic of heuristic was investigated by the philosopher Bernhard Bolzano in his *Theory of Science (Wissenshaftslehre)* of 1837. The modern investigations *par excellence* have been provided by Polya in his *Mathematics and Plausible Reasoning* and by Lakatos in *Proofs and Refutations.*

Now, it is clear that the informal conditions for achieving reflective equilibrium in the practice of physics can be understood as a certain kind of heuristic advice. To conclude: practical understanding (the E–frame) consists no more in the ability to state nor to describe verbally how a measurement, or the expressions of quantum mechanics are to be used, than the ability to drive a car consists in the capacity to describe how car driving is done. A similar example would be to stress that the Subject does not learn to talk by being told theoretically (physiologically) what happens when a person talks. To assume this (which seems to happen too often in philosophical or foundational discourse) is to become victim to what Ryle has called "the intellectualist legend," i.e., the illusion that intelligent performance involves explicit observance of rules. This point is reflected in the criticism of the possibility to formulate any "logic of induction" that prephysics exhibits. This insight was also reached by Einstein in his lecture *On The Method of Theoretical Physics,* where he states that "any attempts to derive logically the concepts and laws of mechanics from the ultimate data of experience is doomed to failure. There is no inductive method that can lead to the fundamental concepts or principles. The truly creative principle of theoretical physics is mathematical construction" [64].

Problem 7.

Wheeler has suggested that human communication is an essential part of the formulation of the laws of physics. This requirement amounts to that of point (1) of the modeling methodology of prephysics: agreement of cooperative communications. Both require adherence to the view of language as the universal medium of communication. We shall now investigate this specific question in more detail. Above we said that the practice of physics (the E–frame) presupposes natural language in the sense of a "universal medium" of communication, in Hintikka's vocabulary. Here we meet again an important point when understanding the practice of physics: the nonseparability of *rule* and *act*. This is exhibited in the very etymology of the word "practice."

Since the Subject participates in the Universe, every *meaningful* operation (judgment, assertion) exhibits a rule. For example, if we stick to physics, the practice of measuring a fact exhibits a number of rules (operations), making it possible to *repeat* the result of a measurement (operation). To make the point more precise: every scientific operation must be a *repeatable operation* if it exhibits a factual claim. This shows the criterion of objectivity relative to the practice of physics: the possibility of repetition. As Wittgenstein formulates this point, "The limit of language shows itself in the impossibility of describing the fact that corresponds to a sentence ... without repeating that very sentence" [65]. This concerns operations (speech-acts) of informal speech and writing (e.g., this article), as well as operations (performatives) in the practice of physics.

It is this point which makes the Universe, when exhibited in the practice of physics, a participatory Universe. This has not always been understood. The way laws of nature are being understood, e.g., exhibits this misconception clearly. Empirical "law-like" statements such as "The sun rises every morning" and "If a stone is dropped it falls to the ground" differs, so the story goes, from observational statements such as "The car is black," in that the latter statement, but *not* the former ones can be conclusively verified by observation. The former "law-like statements" can only be confirmed to a *high degree,* or can only be given a high degree of inductive support, according to conventional wisdom. They cannot be conclusively verified. They nevertheless express a definite empirical content, but this content can, in principle, only be known to an (unlimitedly) high degree. Another way of stating this conventional wisdom, is to say that laws of nature have *empirical* content which, on good grounds, we believe in (like Newton's laws, or, say, Einstein's General Theory of Relativity), but which in principle cannot be known with certainty. This inherited way of thinking is, it seems, very common. It is not always explicitly stated, but *shows* itself in what one is being taught in physics.

However, it contains a conceptual confusion. To say that a statement has a definite meaning or a definite content, which in principle cannot become complete knowledge, *is a paradoxical statement when the practice of physics is understood as being participation in the practice.* It is a paradoxical statement because to say that a statement has a definite meaning *is* to say that it expresses possible knowledge. The statement is meaningful since otherwise one could not grasp the very point of it. The Subject uses this statement in order to convey a certain *point.* When conventional wisdom states that the statement has a definite meaning which cannot become complete meaning, it assumes that the Subject can somehow "separate" itself from the universal medium of language.

The above way of thinking about laws of nature (language as calculus) is mistaken. Not only do we know many laws like the ones above, or more complicated ones, like Newton's first law of motion, "Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change state by forces impressed upon it," to a high degree—as conventional wisdom requires; the Subject actually knows these laws with *certainty.* As Wittgenstein put it, "All testing, all confirmation and disconfirmation of a hypothesis takes place already within a system. This system is not a more or less arbitrary and doubtful point of departure for all our arguments; no, it belongs to the essence of what we call an argument. The system is not so much the point of departure, as the element in which arguments have their life" [66].

The error is to a great extent based on the following misconception: a law of nature is assumed to have the character of a universal statement (or proposition). Usually by a law of nature is thus meant *empirical generalizations.* As examples of such laws, one can give the velocity of light or the gravitational constant. The meaning of a law-like statement is determined in terms of the meaning of its instances. The correctness of the law consists in the correctness of all its instances. This, however, is an illusion, because if it were correct, the Subject could not know the meaning of a single instance. A law of nature is not a universal statement (proposition), but a *law,* and a law is a rule or principle, something that the Subject follows; thus, a law of nature has to be something else than what one usually finds in books of physics (and philosophy). A law of nature does not express an empirical fact in the sense of something that the Subject can verify by observation or by engaging in an experiment. No, our basic laws of nature are what makes it *logically possible to make empirical observations at all.*

A law of nature belongs to the R–frame, formulated in syntax, and the P–frame, explained in semantics, in the terminology of prephysics. A law of nature (in contrast to, for example, a statistical law) is something that the Subject can only understand, and this knowledge of meaning is logically prior to the knowledge of an instance of it. Einstein seems to use the notion of "hypothesis" in this way when he writes that "[this] stipulation contains a further physical hypothesis,.... It has been assumed that these clocks *go at the same rate* if they are of identical construction" [67]. This statement cannot be conclusively verified within the practice of the Special Theory of Relativity. On the contrary, this statement makes it possible to formulate statements which can be verified in the practice of physics using the Special Theory of Relativity. Such a statement belongs to the E–frame in the method of prephysics; i.e., it is a statement concerning which one can only have practical understanding, providing one is to grasp the point of the Special Theory of Relativity.

To understand a law of nature in this sense amounts to being able to *use* it; thus, for example, expressions of natural language (when used) can be understood as exhibiting natural laws, in this sense, The rules of mathematics (the R–frame) and the rules of explanation (the P–frame) occurring in the practice of physics are also rules of nature in this sense. The laws of nature that we *do* follow in practice are genuine laws of nature. The Subject does not follow them *because* they are the genuine laws. That there are genuine laws of nature means that they are the ones that the Subject, usually implicitly, follows in practical computations and measurements. These laws of nature amount to *language-games* exhibiting the *ineffability of semantics* (in the language-as-calculus sense) in Hintikka's terminology.

By the notion of a law of nature, we thus mean the necessary rules that the Subject uses in practices of mathematics and physics: the paradigm. This is in agreement with Husserl's distinction between laws of nature and laws of *essence.* In prephysics we reverse this point. What we have called laws of nature are what Husserl calls laws of essence. It is these laws of nature (essence) which make it possible for the Subject to convey the possibilities of the objects of knowledge to combine with each other. These laws of nature (essence) are, and can be established as, self-evident. These laws of nature regulate what the Subject can or cannot practically understand. To engage in the practice of physics, by using these necessary laws of nature is *not,* as we said above, based on empirical generalizations. These laws are, as Wittgenstein put it, even more inexorable than the laws that empiricists (scientists) usually call laws of nature [68]. Another way of expressing the notion of a law of nature in the sense used here, is to call it a *law of thought* (law of practice). However, this does not mean that such a law is some kind of psychological law. As Wittgenstein said in one of his lectures in 1939, "The question is whether we should say we cannot think except according to them, that is, whether they are psychological laws—or, as Frege thought, laws of nature. He compared them with laws of natural science (physics), which we must obey in order to think correctly. I want to say they are neither" [69].

These laws are neither psychological laws nor are they laws of nature in the factual sense emphasized by empirical thought when read according to the "language-as-calculus" idea. A law of nature is, consequently, a law for what the Subject does when it engages in experiments and observations.

The basic system of rules (the paradigm) which the Subject (implicitly and explicitly) *uses* in the practice of physics *are* true laws of nature. They are constitutive of the meaning of the notion of empirical truth (truth-as-fact). From an empirical point of view, our basic laws of nature are absolute. The laws of nature are normative (and descriptive) rules regulating the practice of physics, making the practice what it is: a paradigm of physics.

What we have said here, of course, does sound strange for someone who is thinking as if the Subject were an observer of the Universe, being conceptually *outside* the Universe. This way of thinking is characteristic of the empirical way of thinking, and is rooted in the mechanistic philosophy of the seventeenth century. It is this view, which lies behind the conception of a law of nature, that we have criticized. This view suggests that there is some conceptually neutral way of observing objects and events in nature, independently of the laws of nature (the language-as-calculus view). It is not generally understood that the Subject is always conceptually *in* the Universe, and therefore always follows laws of nature when performing practices of physics. In general, however, one does not understand that this is the case. This attitude shows itself clearly, e.g., in talk of the "Big Bang" when the current "laws of nature" is said to have been formed (Wheeler: "mutability of laws of physics"). It is not generally grasped that talk of the "Big Bang" is a *metaphorical* way of expressing one's current expert knowledge of physics (analogous to the metaphorical way of talking of the Law of Excluded Middle in logic and mathematics). Thus one can say that a law of nature, as read in prephysics, does not amount to a *hypothesis* about what happens in some occult metaphysical reality, which is what the criticized view would amount to in the end. A paradigm is not a hypothesis in the sense familiar from theoretical physics.

In order to be able to participate in tasks of computing and measuring in physics, the Subject must be within a certain conceptual system or system of laws (the R–frame and the P–frame). To observe and describe the Universe in its variety, outside any conceptual system (paradigm) would be, as Einstein put it, like breathing in a vacuum. This is exhibited, for example, when Einstein showed that it is possible to use local, consequential time to *replace* the concept of Newton's absolute space and time. Recall that the concept of the homogeneity and isotropy of space used by Einstein (because of the need for boundary conditions in setting up a general relativistic cosmology), to analyze the meaning of distant simultaneity in the presence of a limiting signal velocity is, in fact, very close to Newton's absolute space and time. It was reflection on the *meaning* of the notion of time, space, simultaneity, etc., that led to the Special Theory of Relativity. What Einstein realized was that these notions have no absolute meaning independently of *what the Subject does* when observing and measuring in physics.

However, when it measures a fact within a conceptual system (paradigm), the Subject cannot relativize to *that* particular system without moving into another conceptual system. As Wittgenstein put it, "We are confronted here by a kind of theory of linguistic relativity. (And the analogy is not accidental)" [70]. Indeed, the analogy with Einstein's Special Theory of Relativity is not accidental. Recall that Einstein *motivated* his theory by discussing the ways in which certain propositions (ascriptions of simultaneity and time) can or cannot be verified. It is only possible to verify a proposition relative to a certain conceptual system (paradigm). The system (paradigm) cannot itself be verified. It is the conceptual framework relative to which verification is possible.

It is only from a logical point of view that certain laws of nature are relative (conventional). No paradigm is absolute from a *logical* point of view. In this sense the Universe is not deterministic. On the other hand, everything in nature must be understood in some conceptual system. In this sense the Universe is deterministic. This important point was emphasized by Poincaré and Einstein but has been misunderstood in the tradition exhibiting factual thinking (language-as-calculus).

In prephysics the conceptual system (paradigm) consists of the E–frame, the R–frame and the P–frame. The conceptual system (the paradigm) cannot be proven true (in the sense of truth-as-fact), it can only be understood. Wittgenstein formulated this insight by saying that "[the] thing that's so difficult to understand can be expressed like this. *As long as* we remain in the province of the true-false games a change in grammar can only lead us from *one* such game to another, and never from something true to something false" [71].

Problem 8.

The next point to be investigated is the requirement exhibited in point (2) of the modeling methodology: the requirement concerning agreement of intent. This amounts, when put in general terms, to the requirement concerning explicit specification regarding the *aim* of prephysics. In general terms, *the intent to engage in a prephysical activity can only be motivated by a requirement of making the practice of physics more meaningful* as a result of conceptual and practical (computational) problems existing in the current (continuum) paradigm of physics. When put into action, this is done in the form of heuristic advice.

Above, we gave certain heuristic conditions. We can now specify the requirement concerning agreement of intent when engaging in the "paradigm shift," in Kuhn's terminology as follows: one has to (1) *understand* the task (problem) to be met by syntax and semantics, (2) carry out the task of formulating the R–frame in syntax and (3) find the connection between the data (the E–frame) and the formulated R–frame in semantics (the P–frame). When the Subject has learned to put the P–frame into intelligent use, the reflective equilibrium is restored; we have achieved "normal science" in Kuhn's terminology. This closes the practice as far as meaning and understanding is concerned; the Subject practices "normal physics," i.e., we have agreement of intent when performing physics as a practice.

When the Subject has grasped how to engage in the practice of physics according to the novel paradigm, but *also* has explicit knowledge of the R–frame and the P–frame, he has what Gefwert calls *theoretical* understanding of the practice: an explicit method to use in order to find answers when measuring in physics. Thus, it is by virtue of theoretical understanding that the "counter paradigm" of Noyes is to be understood. In Noyes' formulation it reads, "Any *elementary event*, under circumstances which it is the task of the experimental physicist to investigate, can lead to the firing of a counter" [72]. However, there is an important point missing in Noyes' formulation.

As we said above, when the Subject has achieved theoretical understanding it has a method (explicit knowledge of the paradigm) for finding answers in the practice of physics. This is reflected in the etymology of the word *method*, which is derived from the Greek *meta* ($\mu\epsilon\tau\alpha$), meaning "after," and *odos* ($o\delta o\varsigma$), meaning "way." We have emphasized throughout that one is to grasp the practice of physics as being primary, in distinction to physics as being concerned with objects and laws. This leads to the requirement to distinguish, in Dummett's terminology, between *implicit* knowledge of meaning and *explicit* knowledge of meaning, of the counter paradigm [73]. The distinction between implicit, and explicit, understanding, or knowledge of meaning, is reflected in the "circumstances" which it is the task of the experimental physicist to investigate. These "circumstances" can be understood as referring (1) to the theories (person programs) used in everyday practice of theoretical physics, and (2) it can refer to explicit understanding of the paradigm itself: the ordering operator calculus and the (applied) Intuitionistic Theory of Types (Sets), coding the inference rules applied in the practice of theoretical physics within the paradigm. This distinction plays an important role in the modeling methodology itself, in that the practice presupposed (in prephysics exhibited as the E–frame) in the end *must* be explained as implicit knowledge of meaning of this practice. This is reflected in the necessity of the Transcendental Anthropic Principle.

The distinction between implicit and explicit knowledge of meaning is well formulated by Prawitz, when he states that "[knowledge] is explicit when the person can state what he knows, i.e., when he can assert some sentences that express the content of his knowledge; and then, of course, it is implied that he knows the meaning of the sentences that he asserts. To explain all knowledge of meaning as explicit knowledge would thus necessarily be circular, since any such explanation presupposes what it is to know the meaning of some sentences. Dummett's important conclusion is that knowledge of meaning has in the end to be explained as implicit knowledge, i.e., in terms of some practical ability, which of course must be some ability with respect to the use of language" [74]. This insight is also reflected in prephysics as exhibited by the modeling methodology and Noyes' counter paradigm.

To have explicit understanding (or knowledge of meaning) of the ordering operator calculus, amounts to having theoretical understanding of the practice of physics when coupled with the rules exhibited by the P–frame. The rules that determine the practice to what it is (the R–frame and the P–frame), constitute the theoretical, or objective, side of the Subject's actual knowledge of the practice of physics. When reflective equilibrium is achieved, i.e., when the method is being used (in an implicit way) in the immediate practice of physics (the E–frame), the paradigm constitutes the objective ingredient of the practice. Thus, one can say that the paradigm, as such and when applied, is naturally understood as the theory of knowledge of the practice of physics; that is, of the *demonstrative* knowledge of the practice of physics; prephysics amounts to exhibiting what Aristotle called *epistéme apodeiktiké* (επιστημη αποδεικτικη), for the practice of physics.

Problem 9.

We are now able to investigate the prephysical condition put forward in point (3): agreement of observation. As far as practices like mathematics and physics are concerned, the outcome of the previous discussion is the insight that the Subject does not investigate any assumed relation between, for example, the language of physics and reality, even if the Subject assumes being involved in such an investigation. This insight seems to have been understood by Bohr. According to Petersen, who was Bohr's long-time assistant, Bohr once declared, when asked whether the quantum mechanical algorithm could be considered as somehow mirroring an underlying quantum reality, "There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature *is*. Physics concerns what we can say about nature" [75].

This comment has sometimes been interpreted as expressing that Bohr is adhering to some kind of philosophical idealism (or instrumentalism), when understood in the sense of "language as calculus." This is a mistaken way of reading this quotation. What Bohr is saying is that physics as a practice is more fundamental than physics as concerned with factual laws and factual objects. Bohr adheres to the view of "language as the universal medium." This point has not been generally grasped. Physics as a practice aims at measuring facts. It aims at the notion of truth-as-fact; i.e., to say something "about" nature. To *say* (judge, assert) something about nature in this sense, is to judge some expression with a certain semantical *force*, in Frege's terminology. By performing this task, the Subject exhibits his participation in the practice, and thus implicit adoption of an R–frame and P–frame. By participating in the practice, the Subject also participates in transcendental reality. Adherence to transcendental reality is a presupposition in order for the practice of measuring to make sense.

What the Subject can say "about" nature is to be understood as expressing the insight that in the end, what is to be counted is what the Subject can *immediately* express by the measurement. This point is intrinsically connected with immediately *observing* the result of a measurement. To observe, in the practice of physics, is always to be within a conceptual framework (the R–frame and the P–frame). One can express this point by saying that observation is always *paradigm-laden*. Only a Subject "sees," not the eyes of the Subject. Cameras and eye balls do not see; they are conceptually blind. To "see" or "observe," is to be understood as exhibiting the semantical *force* of the R–frame and the P–frame; thus "observation" or "seeing" is to be grasped in the metaphorical sense, where the words "see" and "observe" are synonymous with "observing" (understanding). In the practice of physics, we always "see" in the sense of "seeing as" or "observing as." All seeing is "seeing as." If a Subject sees something at all, it must look like something. Another way of stating this is to say that "observations" are always understood within a practice of physics and thus, implicitly exhibiting an R–frame and a P–frame, when grasped in the sense of "language as the universal medium."

We shall now attempt to exhibit this insight by formulating what is meant by saying that acts, or events, are to be understood as being *immediate*. Assume that a Subject is looking at, or observing, two rods *a* and *b* lying in front of him. Assume, furthermore, that the two rods are placed at some distance from each other in such a way that the Subject is unable to see which of them is the longer one. Furthermore, assume that the Subject adopts the following working hypothesis, "rod *a* is longer than rod *b*. " Let us call this statement A. Look at A as a kind of primitive scientific theory. When this formulation has been performed (past tense!), the expression A is of the form "A is a proposition." The proposition formed exhibits a certain task: the task to *become* verified, i.e., to be established in the form "A is true (fact)." Here we distinguish *two* components in the expression A: *proposition* and *force*. The (semantical) force can exhibit a number of forms. One can express by $\vdash$A, !A and ?A, the assertion, the command and the question, respectively, the proposition-component of which is A; thus the proposition A, "rod *a* is longer than rod *b*" first has the force of a question: ?A (within the paradigm).

In order to answer the question, one has to engage in a measurement. Thus, the aim becomes to engage in falsifying or corroborating this statement, i.e., to affirm the proposition A. In order to perform this task, the Subject is to employ a certain *method*: the paradigm of the practice of discrete physics. Note that as the situation has been described, the statement A purports to express *what* to do in order to exhibit this statement as being a *fact* (truth-as-fact). In order to *find out*, in accordance with the task at hand, if the rod a is longer than rod b, the Subject has to engage in performing a measurement. The Subject can be understood as engaging in a task of searching and finding a fact, by virtue of the measurement. The Subject has to engage in a practice of searching and finding, verifying, or measuring, the factual truth of the proposition A. In order to achieve this, assume that the Subject brings the two rods a and b together, and that it looks like this:

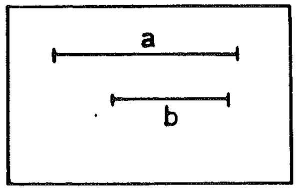

Figure 6

The result of the experiment performed exhibits the force $\vdash$A of the proposition. As a result of having performed this immediate experiment, the Subject is justified in asserting !A, i.e., that "the result of the experiment shows that rod a is longer than rod b." By bringing the two rods a and b together, the Subject exhibits (implicit) knowledge of *what* to do in order to solve the task A. One can also say that by bringing the two rods a and b together, the Subject exhibits practical competence (understanding) in order to solve the task at hand. When the Subject, then, as a result of having performed the measurement, asserts that rod a is longer than rod b, this shows that the Subject knows *that* rod a is longer than rod b. The general question of what it is to know the meaning of A, can now be given the informative answer, that it is to know what counts as a direct (immediate) corroboration of it.

The statement A above is of the kind that one *understands* rather than corroborates. Someone who was looking at the two rods lying close to each other, but who did not assent to the statement $\vdash$A, would *not* be in need of making another observation (experiment). He would rather need an explanation of the meaning of the experiment. One could say about such a person that he does not know what it *means*, in general, for a certain rod to be longer than another. One could characterize the situation by saying that (1) the Subject does not know what to look for, (2) the Subject is not able to recognize the result of the measurement, (3) the Subject does not know when the task has been solved, and (4) the Subject does not know when the question has been settled. To know the meaning (point) of the measurement in the example above, amounts to grasping the proposition A as a *problem* (Kolmogorov), *expectation* (Heyting), or *intention* (Husserl). When the Subject has brought a and b together, he has corroborated (verified) A to be a fact. The Subject knows *how* to verify A and he knows *that* this fact obtains. The Subject *sees that* A is a fact. Note that "seeing that" is always connected to a sentential clause. The Subject sees that A is a fact, where A *always* stands for a complete *sentence*: the vehicle of thought (cf., Frege's *Context Principle*: Never to ask for the meaning of a word in isolation, but only in the context of a proposition). In Bohr's terminology, the Subject grasps that A is a *phenomenon*.

This leads to the interesting insight that both *verification* and *corroboration/falsification* belong to immediate practice of physics. Relative to the *paradigm* (the R–frame and the P–frame), the Subject can be said to *verify* a proposition A to be a fact. Verification is thus connected to explicit knowledge of meaning. However, when performing a measurement *within* the paradigm, the Subject can—relative to that practice—be said to corroborate (or falsify) the proposition A. Thus, one can say that when the Subject corroborates (or falsifies) a proposition A, he has *implicit* knowledge of the paradigm used. Corroboration (or falsification) always takes place *within* the paradigm applied. Consequently, it is a question of relative to *what* [paradigm or theory (person program)] a fact A is grasped when being established, which determines whether it is verified or corroborated (or falsified). Note that in either case, the practice always *terminates* when the proposition A is found to be true (truth-as-fact). Another way of formulating this insight is, in Dummett's terminology, to say that whether one verifies or corroborates (falsifies) is dependent on whether the Subject has explicit knowledge of meaning (verification) or whether it has implicit knowledge of meaning (corroboration). In speech act philosophy (Austin's terminology) this is expressed by saying that implicit knowledge of meaning, when corroborating a fact, exhibits an *elocutionary force*; it is not an explicit part of what one corroborates, but is implicit in corroborating A to be a fact.

The notion of *verification* can be replaced by *solution, fulfillment, realization, winning strategy* or *measurement*. When the Subject knows that the statement A is a fact, he also understands the configuration of *a* and *b*. One cannot understand the proposition as being true (expressing a fact) without *having performed* the act (operation) of corroboration (verification). Thus, one can understand Einstein's point that a concept does not exist for the physicist until he has the possibility of discovering whether or not it is fulfilled in an actual case. This is also, precisely, what Bohr emphasizes when he stresses that no elementary phenomena (proposition) is a phenomena until it is an observed phenomena (fact). To understand A (to know how to verify it) and to know that it is true (truth-as-fact) amounts to the same thing; the meaning of the configuration is the meaning of the truth (as fact) of the proposition A. To know *how* A is true (a fact) and to know *that* A is true (a fact) amounts to the same immediate practice.

The meaning of the configuration (exhibiting practical understanding *how* to verify) and the meaning of A as true (expressing a fact) are logically nonseparable. Once a certain result A of a measurement has been observed (or found), i.e., A is a phenomenon (Bohr), then it does not take another measurement, or observation, to know *that* A is the result of the measurement. This is the case, because the result A is the result of the measurement, and to observe it *is* to observe it as such. The analogous case in mathematics amounts to grasping that one cannot prove that a proof is a proof; this can only be understood. When the Subject reaches the words Q.E.D. at the end of a proof of a theorem, one is supposed to have understood that it is a proof of the theorem in question. To identify the result is to understand it as a result (phenomenon) of the experiment. In this case the Subject is entitled to judge (assert) that A (1) is a *fact*, (2) is *true*, and (3) is a *phenomenon* (Bohr). The Subject is truly a participator in the practice of physics in the sense required by Wheeler.

IV

According to Hintikka "... as we cannot have any knowledge of things-in-themselves but only in a framework of knowledge-seeking activities, the references of our expressions cannot be given independently of those activities either" [76]. By a "knowledge-seeking activity" in physics is to be understood the immediate practice of measuring an object (of knowledge) to become a fact. This always takes place within a certain paradigm. The "knowledge-seeking activity" consists of two parts: (1) the activity of formulating a theory of physics (a person program) where the Subject is *searching* for knowledge of fact, and (2) the activity of *finding* a fact, i.e., the activity of engaging in measuring a proposition to be true (truth-as-fact). Thus, one can grasp that a practice of physics performed within a paradigm (either the continuum and discretum) consists of the immediate activities of searching and finding facts.

The discussion above leads inevitably to the insight that the primary task of prephysics is not affirmation or denial of the existence of some metaphysical objects, be they partially "veiled" or not (to use D'Espagnat's terminology) when understood in the sense of "language-as-calculus." The task of prephysics is to formulate and explain the *paradigm* of physics, consisting *both* of exhibiting the syntactical part of the paradigm as well as the semantical part of the paradigm. This is *not* to say that there are no objects; there certainly are, but the objects of physics are not metaphysical "veiled" objects (D'Espagnat) or some kind of "Ding-an-Sich" (Kant). The objects of physics are objects of knowledge, judged by the Subject to have meaning and understood by virtue of measurements. In other words, Physics is to be understood, primarily, in the sense of "language as the universal medium." However, this is not enough. Physics, when understood as an immediate practice of searching and finding facts should also be understood as exhibiting a *finite discrete* structure.

The reason for this is the general requirement that once a fact of physics is established, within the paradigm of discrete physics, one should always, on request, be able to exhibit an effective computable method (theory, person program) showing how to find the fact. This can be seen as a generalization of the constructivist tenet in mathematics, "There is an x such that $P(x)$," *means* we can explicitly produce an x such that $P(x)$. If the solution to the task (problem) at hand depends on some parameters, we must be able to produce the solution explicitly by some *algorithm* (rule) when given values of the parameters. That is, discrete physics requires that "for every x there is an y such that $P(x, y)$," where x is the measured *result* (truth-as-fact) and y is the explicit theory (person program) being the *method* for finding x. Thus, every theory of physics (person program), when formulated within the paradigm of discrete physics (the R-frame and the P-frame), implicitly exhibits a computer program (due to adoption of the notion of *function* occurring in constructive mathematics).

From what has been said so far, it is not difficult to grasp the relevance of constructive mathematics (applied in discrete physics) for computer programming. According to Nordström [77]:

1. The notions of *computation* and *method*, basic for constructive mathematics, is exactly the same as in Computer Science; it is a method which when applied to an argument of the right kind will output something of the right kind. The function concept in classical mathematics (a subset of a cartesian product with certain properties) is not what programmers use.*
2. From a constructive proof of a proposition it is possible to construct a program which computes relevant information from the proof. For instance, a proof of an existential proposition $\exists x.P(x)$ will yield a program which computes an object a which has the desired property $P(a)$.

This point can now be extended to the practice of theoretical physics when understood within the paradigm of discrete physics. Here, one is to apply constructive mathematics when formulating theories of physics (person programs). This amounts to understanding discrete theories of physics as, implicitly, exhibiting high-level programming languages. The practice of physics, if founded *(fundiert)* on the paradigm of discrete physics, is effectively computable. It does not employ any kind of *Principle of Omniscience,* in Bishop's terminology, which lies at the root of most of the unconstructivities of classical mathematics. As an example of the principle of omniscience, one can provide the following: if $\{n_k\}$ is a sequence of integers, then either $n_k = 0$ for some k or $n_k \neq 0$ for all k. This is called the *Limited Principle of Omniscience* (LPO), which states that:

$$\forall f \in \mathrm{N}^N[(\exists n f(n) \neq 0) \vee (\forall n f(n) = 0)] \ .$$

According to the absolute conception of truth (exhibited in the language-as-calculus semantics), this disjunction is true: the right side is true if every n has the quality f; if this is not the case, the left side of the disjunction is true. Assume that the right side is false. When the Subject starts the routine of searching and finding along the natural numbers, it sooner or later bumps into such an n for which f is not valid. The Subject has found a number which validates the existential proposition on the left-hand side of the disjunction above. One of the best known, so far undecidable, problems in mathematics is the *Riemann hypothesis* which can be formulated as follows: $c = 0.5000 \ldots$. Now, let $f(n)$, where the expression $c's\ n + 2'nd$ decimal is 0. The expression c is given in such a manner, that one can count its decimals indefinitely. Now, $\forall n f(n) = 0$ states that $c = 1/2$ and $\exists n f(n) \neq 0$ states that $c > 1/2$. According to the constructivist meaning of the logical operations LPO is not justified, even though we saw that it is classically true. Because we have not proved that $c = 1/2$ or $c > 1/2$, and, furthermore, since it is a consequence of LPO, the Subject has not proved LPO. Since we assume that we can always find undecidable (so far) mathematical problems of the form, "is every decimal 0 or is some decimal different from 0," we believe that LPO cannot be proven in a constructivist way. Classically, LPO is equivalent to the statement that $x > y \vee x = y \vee x < y$. The equivalence of the (classical) real numbers

* This last statement of Nordström's is debatable; consider a program function which encodes a look-up table or a sort-merge algorithm.

is, in its general form, impossible, provided one doesn't allow an infinite amount of evidence. From a constructivist point of view the equality of the (classical) real numbers is undecidable.†

This makes it clear that within the paradigm of discrete physics, one can allow computations involving the continuum, providing the Subject explicitly exhibits the crucial dependence on *LPO*. Recall that we are, primarily, dealing with the immediate *practice* of discrete physics (mathematics). This makes it possible to perform computations ("classical") within the paradigm of discrete physics (mathematics) without any loss of meaning and without any essential change in the method used. As Bishop puts it, "Classical mathematics would go on entirely as before except that every theorem would be written as an implication, either $LPO \rightarrow A$ or some extended version of an infinite computation implying A" [78].

Similarly, in discrete physics one can allow "classical" computations within the paradigm of discrete physics, provided one explicitly exhibits the dependence on *LPO*. Thus "classical" (continuum physics) can be regarded, when understood within the paradigm of discrete physics, as approximating discrete physics. However, the *canonical* formulation is to be performed by discrete physics adhering to constructive mathematics. This, among other things, shows the requirement of finite, and immediate, computability in discrete physics. As McGoveran puts it,

1) There is nothing in the knowable (or observable) Universe which cannot be described constructively.
2) There is nothing which can be described constructively which (that known as) the physical Universe can not produce (in a combinatorial sense).
3) There is nothing which can be observed or known which can not be described constructively.

This amounts, essentially, to the requirement of *finite reason* within the paradigm of discrete physics: the Subject is never in the immediate practice of physics to judge something which requires an infinite amount of evidence. Within the paradigm of discrete physics, consequently, there is no method requiring an infinite amount of evidence in order to solve of any problem A which option A or $\neg A$ we are justified in asserting prior to the actual verification of the task A. From this, one immediately notices the problem with the *Law of Excluded Middle*: its uninhibited use in immediate practice would lead to theories (programs) which one does not know how to execute (corroborate). Recall that a *law* is something that the Subject actually follows in immediate practice. There are no genuine tasks which require an infinite time (infinite amount of evidence) to be performed. Only when a computation (verification) is terminated can one claim a proposition to be factually true. There are no genuine laws prescribing an infinite amount of evidence. A law implicitly applied in immediate practice exhibits its finiteness in the very performance of the practice; there is nothing like an infinite practice. Every immediate practice (performative) is finite. Thus, a law only regulates a finite (immediate) routine. In order for this to be the case, the law must be meaningful, it must have a use within *both* the semantical as well as the syntactical part of the paradigm. As a law, "$A \vee \neg A$" clearly has no well-defined meaning within the paradigm of discrete physics in either case, when applied in an uninhibited way. It is never used (applied) in immediate (actual) practice. It is only used when *describing* a practice. The Law of Excluded Middle clearly has no use within the immediate *method* of discrete physics.*

There cannot be, within the paradigm of discrete physics, any law which is impossible to apply in order to solve a certain task. The uninhibited use of the *Law of Excluded Middle* is not valid within the paradigm of discrete physics (and can be seen as a kind of "metaphor" according to Dummett), whereas the axiom of choice *is* valid [79].† The interesting consequence is that the axiom of choice, essentially, belongs to the paradigm of discrete physics, not only to mathematics. This is one of the basic reasons why it is preferable that theories of physics ought to be founded on the paradigm of discrete physics

† Note that this argument against *LPO* can not be given in a finite and discrete system such as the Ordering Operator Calculus since it requires postulating an infinite amount of potential evidence in order to satisfy the assumption stated above.

* The Ordering Operator Calculus resolves this difficulty from the beginning since it is grounded upon finite tasks from the beginning. The Law of the Excluded Middle is therefore realised, and expressed, in an inhibited form only.

† Note, however, that the Ordering Operator Calculus need not and does not appeal to the axiom of choice, nor to prove it, since all finite orderings are well-orderings by definition. Recovering the continuum is not a goal of the Ordering Operator Calculus and, contrary to the position of Brower, it is a tenet of the theory that the continuum need not be recovered even in a constructive form.

which has as its structural core The Intuitionistic Theory of Types (Sets). Whatever is computable and possible to corroborate within the paradigm of discrete physics, the *canonical* formulation exhibits a finite routine (method) for achieving this task.

It becomes, then, natural to assume that it ought to be feasible to extract from theories of physics, formulated within the paradigm of discrete physics, "expert programs" making "computer measurements" possible [80]. For this, one can use programming languages like *Pascal, C, LISP* and *PROLOG*. Recall, however, that in Martin–Lőf's conception, mathematics *itself* is understood as exhibiting a high-level programming language. When extracting computer programs from constructive mathematics one can use Martin–Lőf's Intuitionistic Theory of Types (Sets) as a *programming logic*. As Nordström points out, "Type theory can be seen as a programming logic, a logic for the process when programmers write a program for a certain task and give arguments why the program is correct. It is an important open problem in Computer Science whether it is feasible to use the computer not only for editing, storing and executing programs but also to check that the programs are correct.... Type theory suggests one way of doing this" [81]. This possibility has been explored by Constable et al. with the Nuprl Proof Development System [82], which is a concrete implementation of Martin–Lőf's Type Theory as a programming logic. The system supports constructive type theory, whose primitive concepts can serve as building blocks for nearly any mathematical concept. This characteristic distinguishes Nuprl from most other proof-checking or theorem-generating systems. Nuprl runs in Zetalisp on Symbolics Lisp Machines and in Franz Lisp under Unix 4.2BSD.

Since discrete physics uses constructive mathematics (the ordering operator calculus), in the theories formulated within the paradigm, this means that the Type Theory also functions as a programming logic in discrete physics. One can say that it functions like a theory forming logic (programming logic) within the practice of discrete physics. This insight can now be extended to the practice of measuring a fact in physics. In discrete physics every theory (person program) exhibiting the canonical formulation, implicitly exhibits a logic for theory formation. In discrete physics (also in continuum physics) it is explicitly required that one can exhibit the theory (person program) to be tested. Thus, one can say that a theory of physics (person program) exhibiting the canonical form implicitly gives instructions for its own validity. Formulation of a theory in discrete physics always *intends* to achieve a reflective equilibrium between theory and the programming logic. It, furthermore, always attempts to achieve a reflective equilibrium between theory and fact measured within the paradigm—this, because the ultimate task of experimental physics is to exhibit (search for) the *factual structure* of the physical Universe. By factual structure is to be understood the judged, or corroborated (searched and found), facts of measurements.

In order to achieve this, it is not enough that the theory (person program) to be tested is formulated; one should also be able to exhibit a finite discrete method showing how to find the asserted fact. One way of formulating this is to say that every object of physics arrives equipped with its type-rule (Martin–Lőf), or, alternatively, to say that every proposition arrives equipped with its verification procedure (proof, computation). Verifications (computations) and corroborations (measurements) are built into the practice. This is an essential characteristic of McGoveran's Ordering Operator Calculus.

This leads to the insight that practice of physics can be understood according to the idea of

Phenomenon-As-Games .

In order to be able to grasp this point more easily, it is useful to compare this to the way Martin–Lőf explains the analogous case of mathematical practice: the "propositions-as-types" idea. First, the method of Martin–Lőf exhibits a number of *categories*. So also in prephysics. Recall that due to TAP one cannot analyze language (as the universal medium of communication) with the help of any category, since all categories only appear in language. The word "category" is used here in its older, philosophical sense, not in the modern sense of "category theory." To exhibit a category the Subject just has to tell "what a thing is." That is, as we said above, every object (of knowledge) arrives equipped with its type (set, category).

There are objects, since a judgment (operation, assertion) can be understood either as an *act* of judging, or, as an *object* of knowledge. However, due to philosophical nonseparability, these are not two separate entities: the act of judging and the object of judgment *fuse*. To judge an expression A to be a proposition, one must carry out the judging of A. Thus, if A is a proposition, we know (implicitly) what to do in order to tell what its canonical proof is; we know *how* to exhibit A. For example, in Martin–Löf's method, the set of integers **N** can be thought of as the proposition "there is a natural number." Now, any exhibited integer constitutes a direct (immediate, canonical) proof that there is a natural number. If the notion of "proposition" is understood in this general way, we can, for example, render the category "$a \in A$" as (1) "a is a member of the type (set) A," or (2) "a is a proof of the proposition A," as Beeson formulates this point [83]. Discrete physics is to be understood in a similar way, when read along the insight provided by the "phenomenon-as-games" idea.

If we understand the word "proposition" in this way, one ought not to have any difficulty with the equivalence of a phenomenon and a game (measurement). Following Bohr's terminology, one can say that a phenomenon and a game (measurement) exhibit *complementary* readings of a judgment. That is, this point exhibits the nonseparability (i.e., the unity of method and fact obtained) as it occurs in Bohr's characterization of the impossibility of any sharp separation between the behavior of atomic objects and the interacting with the measuring instruments which serve to define the conditions under which the phenomena appear. One of the insight's to be gained is that one can now realize that Bohr understood physics along the lines provided by the view of "language as the universal medium" and that a "complementary reading" belongs to the conditions that the transcendental Subject lays down for the empirical Subject to fulfil.

Theories (person programs) formulated within the paradigm of discrete physics can be given a number of complementary readings. Assume a to stand for a theory (person program) of physics and A to stand for a proposition. Then one can provide, at least, the following readings. The names occurring within parentheses express the corresponding readings in the practice of mathematics (presupposed known), within the paradigm, except for the last one. As examples one can provide the following:

1) a is a solution to the problem A (Kolmogorov).
2) a is a program for the specification of A (Martin–Löf).
3) a is a method of fulfilling (realizing) the intention (expectation, task) A (Heyting, Husserl).
4) a is a (winning) strategy for the game A (Hintikka, Ranta).
5) a is an instantiation (realization, instance) of the constructive (recursive) model A (Kleene, Rogers, McGoveran).
6) a is a measurement of the phenomenon A (Bohr, Wheeler, Gefwert).

The system is an inherently open system, in the sense that it is possible to extend it with new program forming (complementary) expressions and new type (set) forming operations.

We shall now investigate the last reading in somewhat more detail. As far as the practice of physics is concerned, one can give the formalism of theoretical physics, a genuinely novel and discrete reading, in accordance with the idea of a game (measurement) as exhibiting a phenomenon. Here one could use Ryle's terminology and say that one reads the practice of physics as exhibiting a genuine "game of exploring the world." The world or, more appropriate, nature is then understood as exhibiting the transcendental rules of the game (The Transcendental Anthropic Principle). The meaning of any practice of measuring in physics (implicitly incorporating the theory (person program) to be falsified or corroborated by virtue of the *result* of the measurement) can only be grasped provided the practice takes place in the life-world (*Lebenswelt*). Recall that the Subject is a participator in the factual investigation of nature performed in the demonstrative practice of physics. The notion of a "measurement" is to be understood as promoting the observer to participator status; i.e., it is "built into" the paradigm of (discrete) physics.

A phenomenon is only established as the result of engaging in a measurement verifying a proposition in accordance with certain rules. Thus, the practice of measuring a proposition to be true (truth-as-fact) can be read as a game corroborating (verifying) a proposition to be a phenomenon in Bohr's and Wheeler's sense. What is important is that a certain result is achieved (by virtue of some solution, program, method, winning strategy or measurement) and that the Subject understands it to be achieved. When the Subject has formulated (within the paradigm) a theory (person program) of physics and it is corroborated by virtue of the *result* of a measurement, it has formulated:

1) a solution to a certain problem A;
2) a method fulfilling an intention of corroborating A;

3) a (winning) strategy of searching and finding A;

4) a measurement establishing the factual structure of the phenomenon A.

As far as meaning is concerned, these linguistic forms exhibit redundant ways of expressing the same point: an effective strategy (method) implicitly applied in order to corroborate (verify) the existence of a proposition by "*immer ausfurbare Operationen,*" as Gödel formulates the point in his *Dialectica* paper [84]. This has not been generally understood.

In setting up the code (the R–frame and the P–frame) one is implicitly providing a translation (or modelling) between those frames and informal linguistic forms occurring in the E–frame. A certain judgment (category) may translate into several different forms of informal language (the E–frame). The translation manual (the P–frame) enables one to determine how far the ordinary forms of expressions (occurring within the continuum paradigm) can be formulated within the discrete paradigm setup intended as a code of the practice of physics without applying *LPO*. The translation manual (or modelling) enables one to grasp how far one's ordinary forms of expression may be given a meaningful explanation within the paradigm set up as a code of the practice of physics; that is, there are often many different ways in which a single judgment is expressed in physics. This may constitute a redundancy which the code eliminates.

It is important to realize that the translation manual is meant as a semantical explanation of how the practice is to be understood. This is precisely what one performs when engaging in semantics. To assume that the translation manual would provide the explanation of meaning, *presupposes* that the expressions occurring within the continuum paradigm are the ones determining how the expressions are to be understood. Not so. The aim with semantics is to convey how the expressions occurring in the E–frame *are to be* understood. Note that the practice of physics can only evade a verification transcendent semantics when understood within the semantical -paradigm of "language as the universal medium." This possibility is excluded within the semantical paradigm of "language as calculus" (model theory).

Discrete physics exhibits the basic constructivist tenet: when one asserts that an object of knowledge exists (having certain desired properties), one should be able to show how to find it by using a finite routine (theory of physics, person program) in such a way that a computer suitable programmed can verify (in normal science) the result. It may be, as emphasized by Truesdell in a lecture in Milan, that the computer may have an impact on mathematics comparable to that which the microscope had on biology and the telescope on astronomy. The understanding arising from the paradigm of discrete physics is that the computer, eventually, will be as influential in theoretical physics as it will be in mathematics. However, this requires that the semantical part as well as the syntactical part of the paradigm of physics is to change, so that the immediate practice of physics *can be made more meaningful* than it is when performed in virtue of the current paradigm accepting the continuum. This requires revolutionary changes, conceptually (unification of quantum theory and general relativity on a discrete basis) as well as in the mathematics used in theoretical physics. It is against this background, and only when it is achieved, that one is to understand Hawking's conjecture that "At present computers are a useful aid in research but they have to be directed by human minds. However, if one extrapolates their recent rapid rate of development, it would seem quite possible that they will take over altogether in theoretical physics." However, even within the paradigm of discrete physics, the computer will never take over from the judging Subject: only physicists judge whereas computers do not. Verification and corroboration exhibit judgments.

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FOUNDATIONS OF A DISCRETE PHYSICS*

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ABSTRACT

Starting from the principles of finiteness, discreteness, finite computability and absolute nonuniqueness, we develop the ordering operator calculus, a strictly constructive mathematical system having the empirical properties required by quantum mechanical and special relativistic phenomena. We show how to construct discrete distance functions, and both rectangular and spherical coordinate systems (with a discrete version of "π"). The richest discrete space constructible without a preferred axis and preserving translational and rotational invariance is shown to be a discrete 3–space with the usual symmetries. We introduce a local ordering parameter with local (proper) time-like properties and universal ordering parameters with global (cosmological) time-like properties. Constructed "attribute velocities" connect ensembles with attributes that are invariant as the appropriate time-like parameter increases. For each such attribute, we show how to construct attribute velocities which must satisfy the "relativistic Doppler shift" and the "relativistic velocity composition law," as well as the Lorentz transformations. By construction, these velocities have finite maximum and minimum values. In the space of all attributes, the minimum of these maximum velocities will predominate in all multiple attribute computations, and hence can be identified as a fundamental limiting velocity. General commutation relations are constructed which under the physical interpretation are shown to reduce to the usual quantum mechanical commutation relations.

1. INTRODUCTION

The purpose of this paper is to present a self-contained mathematical foundation for the modeling of diverse phenomena—in particular, physical phenomena—and to demonstrate its utility.

Twentieth century foundational mathematics is caught on the horns of several dilemmas. Perhaps the most difficult of these dilemmas is also the most ancient: the separation of description and process or, as more usually encountered, the separation of mind and body. This dilemma manifests itself in the split-mind with which the practitioner of mathematics must operate. On the one hand, we perform finite computations by prescribed methods; on the other, we must keep forever in mind that these are artificial limitations of space, time, energy and symbolism—as is evident in the ever present use of ellipses and the infinity symbol. The description ignores the process of describing.

Somehow the student of mathematics must simply accept the fact that we never quite complete (and in principle cannot complete) many tasks of either description or describing, but must extrapolate. Such acts of faith are deeply embedded in the foundations. Of course, one should not be too concerned that counterfactual paradoxes arise as a result of following the faith with fervor or that one can prove that a mathematical system, if moderately powerful, cannot be both consistent and complete [1]. One must simply accept. At once, the student must pretend that the system is faithful (generates trustworthy results) and unfaithful (is either inconsistent or incomplete).

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Twentieth century foundational physics is caught on the horns of a similar dilemma. The practitioner of laboratory physics appeals to the theoretician to completely describe his practice in an objective manner. Again, on the one hand, we perform finite measurements and computations by prescribed methods, while on the other hand we are asked to accept the fact that these are artificial limitations of space, time, energy and symbolism. Again the description ignores the process of describing. Dirac [2], seems to have been acutely aware of this separation of practice and formalism in dealing with the physical interpretation of discrete eigenvalues versus a range of eigenvalues:

"An eigenstate of ξ belonging to an eigenvalue ξ' lying in a range is a state which cannot strictly be realized in practice, since it would need an infinite amount of precision to get ξ to equal exactly ξ'.... Thus an eigenstate belonging to an eigenvalue in a range is a mathematical idealization of what can be attained in practice. All the same such eigenstates play a useful role in the theory and one could not very well do without them. Science contains many examples of theoretical concepts which are limits of things met with in practice and are useful for the precise formulation of laws of nature, although they are not realizable experimentally, and this is just one more of them. It may be that the infinite length of the ket vectors corresponding to these eigenstates is connected with their unrealizability, and that all realizable states correspond to ket vectors that can be normalized and that form a Hilbert space."

Neither the general nor the special theories of relativity readily admit of quantization. These theories are formulated within the space-time continuum using differential geometry. In conflict with this, quantum events are unique, discrete, irreversible, nonlocal, and yet indivisible. Conventional quantum theory tries to embed them in a space-time continuum, which is the source of many conceptual difficulties such as the "collapse of the wave function," the EPR "paradox" and the infinities of second quantized field theory. The properties of quantum events are more fundamental mathematically and conceptually than the properties of an abstract continuum.

One cannot construct a basis which is adequate for this thinking and for the description of phenomena with a language which is dependent on an embedding of discrete structures in a continuous one. We will develop terminology afresh, without the taint of a continuum (and infinities). Our point of view is more process-oriented than just descriptive: it must be possible to generate the structures and the properties which we explore.

L. E. J. Brouwer and others have attempted to constructivize mathematics since 1907 based upon severe and successful criticisms of classical mathematics. As noted by Bishop [3], "Many mathematicians familiar with Brouwer's objections to classical mathematics concede their validity but remain unconvinced that there is any satisfactory alternative." These are valid criticisms, but so are similar criticisms of various constructive attempts, which fail to recognize any of what we feel are some of the more essential aspects of the practice of mathematics. In particular, mathematics which is not process-oriented, context-sensitive, finite, discrete and constructive (primarily in Bishop's sense) [4] is of little use in practice, since the Universe in and about which mathematics is to be used is all of these things. The Universe is only knowable as a complete, consistent system: there exist no black holes arising from undecidability, halting problems, incompleteness or magic of any kind. It is not knowable or understandable in terms of its parts alone. We are strict, constructive, *systems* mechanists.

While we contend that the mathematical foundation presented here will indeed prove useful outside of physics (and we have reason to believe it will), the focus of this paper is restricted to demonstrating the utility of the mathematics for physics.

In order to construct a discrete basis for physics, we limit ourselves from the start to a finite number of symbols (e.g., 0, 1) and to an order parameter defined in terms of primitive recursion. In ordinary language, this allows us to count up to (or down from) some finite integer N which we specify in advance. No construction will be allowed to exceed this integer without additional articulation of the extant theory. This additional articulation will be consistent with and guided by our approach. These restrictions allow us to d-map our construction onto any "operational" description of physics in a sense even more strict than Bridgman's "pointer readings" and the finite specification of what operations are needed to make "pointer readings" are allowed only if we can reduce the operations to "counting." That this apparently impoverished starting point leads to interesting physics will be demonstrated in what follows. In particular, we achieve a fresh understanding of a number of the best established physical facts.

The context-sensitive process of ordering is fundamental; simple but subtle notions of ordering, carefully formalized, result in a rich mathematical structure. If one insists on finiteness, discreteness and a strong constructive approach, the power of the system is surprisingly undiminished from that of continuum mathematics [5]. For example, where others have claimed that a finite, discrete topology was indefinable, we assert that the proper notion of open set defined within the formalism is in fact more constructive than the usual definition from point set topology or Intuitionistic Zermelo–Frankel (IZF) set theory, and clearly avoids the paradoxes generated by the usual continuum-oriented definition of open neighborhood or open set.

In this paper, five principles will be introduced which should not strain the reader's credulity: finiteness, discreteness, finite computability, absolute nonuniqueness and strict constructionism. Then, after presenting eight key concepts (indistinguishables, *d*–sorts, ordering operators, *d*–sets, open *d*–sets, *d*–subsets, parameterization, dimension or basis and attributes) within the context of a larger development, the following consequences will be constructed: the 3+1 dimensional structure of space-time, a combinatoric construction of π, identification of the speed of light constant, the Lorentz Transformations, the relativistic Doppler shift, the relativistic composition law for *d*-velocities, the uncertainty principle, superluminal correlations without supraluminal communication, a combinatoric construction of the exponentiation operator, the commutation relations for linear and angular momentum, the de Broglie relations, the relativistic mass change, identification of Planck's constant and momentum conserving events.

1.1 PRINCIPLES

We will develop a theory which, both in terms of the constructs and operations defined on those constructs, possesses the properties expressed in the following five principles.

Principle I: The theory possesses the property of strict finiteness.

By finiteness, we mean that no infinities or infinitesimals are allowed in the theory. By infinities, we mean an x such that x is larger than any finite y in the system. By an infinitesimal, we mean an x such that x is smaller than any finite y in the system and is not identical to 0. In particular, no x in the system can be arbitrarily large or small. Furthermore, and in keeping with strict finiteness, we require finite definability of any derived (constructed) system, subsystem or attribute of a system.

Principle II: The theory possesses the property of discreteness.

By discreteness, we mean that the depth of partitioning by recursive descent (as by Dedekind cuts) or construction by recursive ascent (as in the construction of the transfinites) is bounded in advance from outside the theory. This absolute bound on the practice is a pragmatic constraint. Over the course of any effort, a particular bound will evolve by refinement.*

Principle III: The theory possesses the property of finite computability.

By finite computability, we mean that the theory is constructive in the following strict sense. It must always be possible to specify any procedure or argument used in the theory as an algorithm having a finite number of finitely definable steps and consuming a finite amount of memory. Such a theory is Turing computable, but only theories which are both Turing computable and which use a finite tape are finite computable. Those which use countably infinite tapes or countably infinite algorithms are excluded by this principle.

Principle IV: The theory possesses the property of absolute nonuniqueness.

* As an example, consider any practice which is realisable on a physical computer. The bound is fixed in advance by the amount of accessible storage. It is our point of view that altering this bound constitutes an alteration of the system (computer plus algorithm) which cannot be understood or modeled within the system. Thus, a system which allows for changes to this bound is ill-defined. If the behavior of a program written to run on a computer having a certain amount of memory is in any way dependent on the amount of memory available, then it is clear that changing the amount of memory available requires the programmer to reevaluate the program for unplanned behavior. If the programmer is wise, this is taken into account by coding "system parameters" into the program such that the system alterations will be "automatically" handled.

Simply put, we assume indistinguishability and uniformity unless we have constructively stated otherwise. By absolute nonuniqueness, we mean that no property which serves to single out or distinguish a construct within the theory from any other construct within the theory may be used in the construction in the absence of an explicitly stated computational mechanism. That is, we will invoke a finite algorithm within the theory whenever a property is to be used in a construction and will otherwise be required to deal with the absence of that property (i.e., by probabilistic means). Any two differently labeled, but otherwise indistinguishable, constructs must be treated as interchangeable in the absence of such an algorithm. Thus, the only *a priori* property that is acceptable is recognition of a lack of information as evidenced by indistinguishability.†

Principle V: The formalism used in the theory is strictly constructive.

Following Bishop [6], and in addition to the preceding Principles I–IV, we will argue by constructive means. As such, proof by contradiction will be considered to be justified, since we are restricted by Principle I to finite situations. The only way in which we may show that an object exists is to give a finite means for constructing it. Bishop would say "finding it," but we do not accept the idea of *a priori* existence of nonfundamental objects. Complex (derived) objects are constructed, not found.

A property P is called definable in the system, if, for every object x constructively shown to exist, x has a property P or it does not. This is different from saying that it has the property P or else it has the property "NOT P." If this cannot be said, then the property P is not constructively defined or even definable within the finite system. Within these constraints on the allowed subject matter, we will deny arguments by the principle of omniscience and of limited omniscience, except (again contrary to the position of Bishop) where the latter may be supported by a finite search. Because our theory is finitary, we embrace the Law of the Excluded Middle (as would Bishop).

We call this position **strict constructionist** because we understand it to be more restrictive than the constructive positions of both Bishop [7] and Beeson [8], which are among the more restrictive statements of the position, and clearly more so than Brouwer.

2. MATHEMATICAL FOUNDATIONS I

In this chapter and the next two, we develop a strict constructive mathematical system which we refer to as the **ordering operator calculus**. This system will be shown to have sufficient power to be a foundation for, or simply replace, significant aspects of conventional mathematics including set theory, lattice theory, differential topology, real and complex analysis and differential geometry.

2.1 Preliminary Concepts

Several concepts will be taken as fundamental in the development of our theory. These concepts are well-known to computer scientists and are rigorously defined by them. Nonetheless, we will provide definitions which limit the scope and applicability of the terms, since our usage will in general be more restrictive. It is especially important for the reader to keep in mind that we do *not* import the additional theoretical framework which is normally accepted within computer science and discrete mathematics.

Recursively Definable

By **recursively definable**, we shall mean simply that an abstract term is definable with a finite number of steps from simpler terms and values.

Computable

By **computable**, we shall mean that an effective procedure has been given by which an abstract construct can be constructed in a finite number of steps and with finite resources. We shall use the term *recursive* in a manner similar to that used in recursive function theory, in that it includes both recursive and iterative algorithms and is not restricted to mean a "recursive procedure call" in the computer programming sense.

† As we will see, this very general principle is at the heart of most invariance principles, including the assumption of equal *a priori* probabilities, isotropy, homogeneity and relativity.

Computational Complexity

By the **computational cost** $C(O)$ of an abstract, finite, discrete algorithm O, we shall mean a measure of the time cost and the space cost of the algorithm. Each of these is usually expressed as a procedure, which shows how to compute from the cardinality and/or ordinality of the domain upon which the algorithm operates (usually called the *size* of the problem), and yields a measure of the **computational time cost** $C_t(O)$ or the **computational space cost** $C_s(O)$ of the algorithm in time-like units (e.g., CPU cycles or algorithmic steps) or space-like units (e.g., bits), respectively. Note that, for us, these costs include the cost of running and storing the algorithm itself.

It is considered normal to express the **computational complexity** measure in terms of the dominant term of the appropriate polynomial, logarithmic, exponential or combinatorial expression; we will consider this to be shorthand for the exact expression. An algorithmic procedure $g(n)$ will be said to be of computational complexity

$$O[f(n)]$$

read "of order $f(n)$," if there exists a rational constant c such that

$$g(n) \leq cf(n)$$

for all n.* By the **total computational cost** of an abstract, finite, discrete algorithm O, we shall mean the result of a procedure which computes for each pair of inputs $C_t(O)$ and $C_s(O)$ a finite number $C(O)$ in a finite number of steps. Such a procedure (which in classical mathematics is representable by a polynomial expression) is said to represent a **computational metric**.

Representational Resources

By **representational resources** of an abstract, finite, discrete system, we shall mean the maximum of the spatial complexities of those algorithms which may be expressed within the system, without appeal to either spatial or time resources outside the system.

2.2 The Concept of Order Revisited

Ensembles

Consider a collection of mathematical (in the sense that physical properties are neither implied, nor are they denied) objects about which we have no knowledge, other than their quantity (cardinality), together with a collection of (mathematical) operators for selecting some of those objects. We call this collection of objects an **ensemble**, because it differs from the usual set-theoretic notion of a collection in ways which we now explain.

Ordering Operators

The notions of distinguishability and indistinguishability of such objects are relative. Without a stated computational mechanism, we are required to assume indistinguishability in keeping with Principle IV.† When asked if two objects are distinguishable, one must respond with a question, "distinguishable with what algorithm?". If presented with such an algorithm, we can think of that algorithm as inducing a property on the objects on which the algorithm operates; then the question of distinguishability becomes, "distinguishable with regard to such and such a property." Indeed, whether the objects are "truly" indistinguishables or not in the sense of Parker-Rhodes [9] is irrelevant: our inability to directly access the objects makes the properties of the computational mechanism used on the objects the essential knowledge in building our theory.

We choose a single means of establishing structure in our formalism, namely, the generalized concept of ordering relation called an **ordering operator**. These computational mechanisms are defined as having the following properties:

* It is usually permissible that a finite number of values of n violate the inequality. We do not allow this.

† This is not an ontological statement.

1) they are only defined on a finite ensemble (a domain);
2) the ensemble must have fixed cardinality N;
3) they take as single input a label;
4) each label carries an embedded unique inaccessible sequence number;
5) they operate on the ensemble or some portion of it;
6) they generate as output one or more labels;
6) the labels successively generated are not necessarily unique;
7) the labels so generated constitute a finite ensemble;
8) the mechanism has a stop rule;
9) the details of the mechanism, including the stop rule, are not inferable.

Note that without either the identification of the ensemble and the input or recognition of the output, there is no knowledge that the operator has been used. By recursively applying this mechanism, we generate an ordered sequence of labels. Clearly, the ordering operator counts as a generating function in the sense used by Kilmister [10], although it does not require the same mathematical foundation and has additional computational power. Since we lack knowledge about the nature of the indistinguishables, we need to specify a few more characteristics of the mechanism of ordering operators. Having done this, ordering operators then also serve an essential function in our axiomatic system as general rules of inference, since they determine precisely what can be constructively exhibited or evaluated.

By **indistinguishables**, we mean that, given the ordering operator mechanism, the objects in an ensemble come in two forms which we now define. By **identicals** we mean that there exists no algorithm constructed within the formalism which serves to distinguish two objects. Thus, identicals is* what one gets when an ordering operator operates twice on the "same" object. By **twins**, we mean that the algorithm used to manipulate the objects does not distinguish them, but that there exists some algorithm constructed within the formalism which does distinguish them. Thus, twins are what one obtains when the ordering operator operates on two objects, but does not distinguish between them in its output; that is, the objects seem to us to be the same within the context of the specific ordering operator. Thus, two objects are indistinguishables only for a specific alogrithm.

The output which results from using the ordering operators in either of these first two cases is two indistinguishable but sequence ordered object descriptions which we will call **labels** for short.‡ Thus, the ensemble of objects has no inherent ordinality as far as we are able to know.

Above, we said that the ordering operator operates on the ensemble. Specifically, we mean that the ordering operator picks a finite number of indistinguishables, given a label as input. If the operator picks more than one object, successive recursions of the operator via input of a label generate one sequence numbered label per object in the subcollection, until the subcollection has been exhausted. The sequence numbers "stick onto" the objects, and their significance can only be recognized by the ordering operator that generates them (it maintains the equivalent of a symbol table which allows it to look up the sequence number(s) associated with a label and vice versa); thus, other ordering operators simply ignore the sequence numbers if operating on the same ensemble. The subcollection is then returned to the ensemble. Further input of a label returns one to the initial situation.

Note that this mechanism allows the operator to generate both total and partial orderings of the labels. The ordering operator algorithm has a stop rule (it halts in a well-defined manner) and will not allow, without repetition, recursive generation of more than a fixed and finite number of labels. The process is defined with (a) the operator and (b) a unique starting label. For some label input, the number of labels output by recursively feeding the output label into the input (i.e., **recursive generation**) is a maximum. The maximal label output of the operator and the ordering operator, itself, are mutually defining. Thus, given a finite ordering of labels on a given ensemble, we define an operator, and vice versa.

* The grammatical "agreement" as used here is intentional.

‡ We suggest the use of **tags** where the term labels would be otherwise confusing as, for example, in Noyes [11] where label refers to a particular kind of label in our sense of the term.

Finally, the complexity of the operator mechanism (i.e., the algorithm) is too great to be represented by the labels alone. We would also have to know the intrinsic nature of the ensemble, but this can only be investigated with (other) ordering operators.

Suppose that a particular ordering operator O on a specific ensemble C (its "domain" in this instance) is given as input a specific label L_0. Let the resulting output of O be the label L_1. On input of L_1, O generates L_2, etc., up to some finite number of labels N. These labels need not be unique; however, each corresponds to some object in the ensemble C. Suppose that this correspondence is such that at least one label has been generated for each object in the ensemble C. If we keep a record of the labels so generated, we can be certain that rerunning the generation will produce an identical record, given the same ensemble C and the same initial label L_0. Indeed, if we begin with L_1, O produces an identical record with exception of the missing entry for L_0. If we begin with L_2, O produces an identical record with missing entries for L_0 and L_1.

If several successive entries in the record are equivalent except for the sequence in which they were generated, we cannot know whether the objects in the underlying ensemble to which these labels correspond are twins (indistinguishable but distinct) or identicals (indistinct). This can only be known by detailed knowledge of the objects in the ensemble and the algorithm by which the operator works. One might argue that in reproducing a record starting from equivalent records, that the recursion would not terminate since the labels are equivalent. However, note that conveying both the label and its record entry sequence is required for entry if the record is to be reproduced. The use of the notation L_n is not accidental. Two pieces of information are conveyed; the label and a number representing the sequence in which it was produced.

Define an operator $O^{\#}$ associated with O that behaves as follows: if the sequence number is left out, then $O^{\#}$ selects a default sequence number for the particular label consistent with the possible sequence numbers with which it might be produced. Thus, the operator may generate labels in two modes: with the sequence number or without it. When the sequence number is excluded, the recursive use of the ordering operator is similar to a sampling algorithm, subject to the constraint of an ordering relation. This ordering relation is not, in general, transitive. In this case, it is possible for $O^{\#}$ to generate L_M multiple times, given only label L for input. The output of $O^{\#}$ must then be ordered on the output sequence numbers to recover the ordering relation which $O^{\#}$ mutually defines. Unless we refer to sampling with repetition allowed, we will mean that $O^{\#}$ has as input and output both the sequence number and the label. However, we will ordinarily refer only to the input and output labels, the sequence being assumed and, since we then do not distinguish between O and $O^{\#}$, we will simply use O notationally.

Note that without the underlying objects, the algorithm for the ordering operator cannot in principle be defined, since the nature of the algorithm will depend upon the nature of the ensemble. Furthermore, depending upon which ensemble an ordering operator operates on, the statistical distribution of labels so generated (with or without repetition) is determined by the intrinsic character of the objects in the ensemble; namely, the cardinality of the ensemble and whether there exist indistinguishables or not, and how many. At best, having run through the operator once, we may use the output "sequence"♮ as a "look-up" table for additional runs, but only in the sense of checking off what has been generated so far and, in the case of a partial ordering, what is left.

2.3 Classifications of Fundamental Objects and Operators

The elementary unstructured object of our mathematics is taken to be a *d*–sort. We define a *d*–sort as any ensemble of n indistinguishable objects having cardinality n, of which it is NOT asserted that every pair of members is either identical or distinct [12]. A perfect *d*–sort is a *d*–sort for which every pair of members is either identical or twins (indistinguishable). We allow the members of a *d*–sort to

♮ We do not mean to imply that the output is "sequential" or totally ordered by the use of this term.

be labeled by an ordering operator, noting however that such internal labeling is inaccessible except via the ordering operator which performs the labeling.[◇]

In order to formally define the relationship between ordering operators and the objects of our formalism, we have need of some additional definitions.

An **ordering relation** $\leq$ is a binary relation (a relation on two arguments). From time to time, we will take the liberty of writing $y \geq x$ in place of $x \leq y$. We shall mean by the symbol for **equivalence** $=$, a binary relation such that the two arguments are either identicals or twins; that is, they are the members of a perfect d–sort. A **successor function** $'$ is an ordering relation such that P1 and P2 are satisfied:

P1: Given arbitrary objects a, b, b' and x in a d–sort S such that if $a \leq x \leq b$, $a \leq b$, $a \leq b'$, and either $a = x$ or $x = b$, then $b = b'$ (uniqueness).

P2: There exists an e in d–sort S such that $e \leq x$ for all x in S (infimum).

A **recursive enumeration** E is an ordering operator which provides or recovers a label for each member of a d–sort. It is, therefore, an effective procedure for listing the members of a d–sort, with repetition allowed. More formally, a recursive enumeration is a rule with successor function ($'$) such that, given a label for object x in a d–sort S as input, the recursive enumeration generates a label for object x' in S, not necessarily distinct from x.

Theorem 1: Neither the enumeration nor the successor function on a given d–sort are unique.

A **partial ordering relation** $\leq$ is a binary relation between two members of a d–sort S which, for all arguments x, y, or z in S, satisfies the following conditions:

P3: For all x, $x \leq x$ (reflexive).

P4: If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric).

P5: If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive).

A **parameterization** is a partial ordering relation induced on a d–sort by an ordering operator O such that, given a label x for a member x of S as argument, the parameterization generates a label x' for the successor of x, called x'. That is, the partial ordering relation satisfies P1 and P2, as well as P3, P4 and P5. We may refer to a ordering operator O as a parameterization if O is used to induce a parameterization.

A **total ordering relation** is a partial ordering relation which satisfies P6.

P6: Given arbitrary x and y in d–sort S, either $x \leq y$ or $y \leq x$.

We shall deem it convenient, at times, to speak of a particular type of recursive enumeration. In particular, we will want a recursive enumeration without repetition, and in which the binary relation is a partial ordering relation.

A **rule of correspondence** is a total ordering relation induced by an ordering operator O on a d–sort of cardinality 2.

A member x of a d–sort S of cardinality 2, with partial ordering, is called a **supremum** or **sup** if, for member x and arbitrary y in S, $y \leq x$ and it is not the case that $x = y$. Similarly, a member z of S is called an **infimum** or **inf** if, for members y and z, $z \leq y$ and it is not the case that $y = z$.

Theorem 2: Every total ordering relation induced on a finite d–sort defines a supremum and an infimum.

Theorem 3: A rule of correspondence defines a supremum and an infimum.

◇ Parker–Rhodes was insistent that indistinguishables could not be labeled at all. In this respect, we suspect that Parker–Rhodes would have likened d–sorts to a kind of multiset [13]. With a suitable mapping we could identify the notion of an ordering operator with Parker–Rhodes' functor in which case the ordering operator would produce ordinal sorts. However, this would relegate our theory to the domain of sort theory, an extra degree of ontological freedom and notational complexity which we cannot afford.

A member a of a d–sort S with ordering operator O inducing an ordering relation $\leq$ on S is an **upper bound** if there exists a member b such that for the d–sort S' consisting of a and b, with ordering operator O' inducing the ordering relation $\leq$, a is the sup of S'. Similarly, a member b of a d–sort S with ordering operator O inducing an ordering relation $\leq$ is a **lower bound** if there exists a member a such that for the d–sort S' consisting of a and b, with ordering operator O' inducing an ordering relation $\leq$, a is the inf of S'.

Note that we have used partial ordering in the definitions of sup and inf so that we may create d-chains of d–sorts with the same partial ordering. Defining the concepts of upper and lower bound in this way insures transitivity across d–sorts; then these concepts take on the usual lattice theoretic definitions and the uniqueness of the sup and inf (in a given d–sort of cardinality n with specified partial ordering) is assured [14].

2.4 CONSTRUCTED OBJECTS: FROM d–SORTS TO COORDINATES

A **d–set** is a d–sort with ordinality m imposed by one or more recursive enumerations. The ordinality m of the d–set is just the cardinality of the d–sort of labels given by the recursive enumeration. Note that there may be more than one such recursive enumeration associated with the d–sort.

We may now classify the generations of an ordering operator in terms of the cardinality and the ordinality of the labels it generates. When these are the same, the output is a d–set and the O is said to be a **total ordering operator**. When these are not the same, the output is a d–sort and O is then said to be a **partial ordering operator**. Like sets in set theory, d–sets have no members that cannot be counted uniquely, while d–sorts have members that can not be counted uniquely. Unlike sets, a d–set is only defined with respect to one or more *specific* ordering operators. It, like all other objects in our system, is *constructed.*

We say an enumeration E is **monotonic increasing** if it gives labels to elements in the order of the recursive enumeration of the d–set on which it is defined. Similarly, we say E is **monotonic decreasing** if it gives labels to elements in reverse of the order of the recursive enumeration of the d–set. An enumeration E will be said to be **nonmonotonic** if it cannot be shown to be either monotonic increasing or monotonic decreasing constructively.

We are now in a position to define an important concept of topology; the notion corresponding to an open set. Note that our definition in no way appeals to the notions of continuity (in the usual sense of the word) or infinitesimals.*

The **boundary** of a d–set S defined with ordering operators O_i generating ordering relations R_i, consists of those elements of the underlying d–sort which, when operated on by any of the O_i, generates a label which is either a sup or inf of the R_i.

A d–set S' is said to be an **open d–set** with respect to a d–set S defined with ordering operators O_i, if S' is just S without the elements which would generate the boundary of S. Thus, for the defining ordering operators of the d–set S, none of the defining ordering operators of the d–set S' on an element of the underlying d–sort of S' generates a label which is either a supremum or an infimum of the defining ordering operators for S. It will generally be the case that d–sets are formed from multiple ordering operators. The extension of the definition to more complex d–sets having multiple ordering operators is straightforward.

Clearly, from the definition of ordering operator (i.e., an ordering operator and its productions are mutually defining), S' is, itself, a closed d–set, but *for a different ensemble of ordering operators.* This makes clear the importance of the notion that a d–set is a d–set by virtue of the defining ordering operators.

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* In addition, the definition is purely constructive and recovers the "classical" definition of sets for sorts of sufficiently large cardinality n. We shall define what we mean by "sufficiently large" in a future paper in which we will discuss measurement—both abstract and physical—using the terminology developed here. Briefly, one measures the number of partitions n of a model by a d–map G to a d–sort of cardinality n'. Then, if $n > n'$ for all n' chosen, n is "sufficiently large," as no finiteness will be detectable independent of the measurement.

Note also that this eliminates the possibility of *d*–sets with deleted points. The transitivity of an ordering relation is, itself, defined constructively. Thus, the transitivity of any ordering relation is broken if a point is "deleted" and new ordering relations are induced, resulting in a new *d*–set. By defining open *d*–set as above, we have insured that there is no means of specifying a *classical boundary* for the *a* set independent of the construction of the set, as is done with a classical (infinite or continuum) set. Thus, every enumeration of the elements of a *d*–set would have to be an infinite enumeration, either by allowing (infinite and therefore not constructively definable) repetition (in the case of *d*–sorts) or without repetition (i.e., only if the *d*–sort is itself infinite and, in which case the *d*–set cannot be constructed as the ordering operator, is finitely definable only when all the members of the *d*–sort can be finitely specified). Thus, for *d*–sorts of sufficiently large cardinality, open *d*–sets are not distinguishable from the open sets of the classical definition.†

The fundamental concept of local topology is now within reach; i.e., an open neighborhood. A *d*-subsort (or *d*–subset) S' of a given *d*–sort (or *d*–set) S is itself a *d*–sort (*d*–set) which is defined with the same ordering operators, and for which x is an element of S' if and only if x is an element of S. We say that the S contains S'. Note that this precludes the possibility of "supersets" of *d*–sets being equivocated with *d*–subsets contained in the *d*–sets and, thus, the "set of all sets" of Russell's Paradox. For arbitrary elements of a *d*–sort S, an open neighborhood of x is any open *d*–sort containing an open *d*–sort containing x.

A closed *d*–sort (*d*–set) is a *d*–sort with defining ordering operators O_i, such that at least one element of the *d*–sort is either a supremum or an infimum for at least one of the O_i.

A *d*–map G on a *d*–sort S is an ensemble of rules of correspondence defined on S (i.e., inducing a *d*–sort of cardinality 2), such that there exists a *d*-subsort of S with members x_i all of which are the infimums of the rules of correspondence, and there exists a *d*-subsort of S with members all of which are the supremums of the rules of correspondence. We refer to one of these *d*-subsorts as the domain of the *d*–map G and to the other as the range of G. The range is said to be a *d*-subsort of some *d*–sort called the image of G. The domain and range have ordinality 1.

The union of two *d*–sets S and S' is a *d*–set S'', whose members consist of the members of either or both of S and S'. Similarly, the intersection of two *d*–sets S and S' is a *d*–set S'', whose members consist of the members of S which are also members of S'. The symmetric *d*–set difference of two *d*–sets S and S' is a *d*–set S'', whose members are either members of S but not of S', or members of S' but not of S.

A *d*–map M on a *d*–sort S is said to be one-to-many, if and only if the cardinality of the domain is less than the cardinality of the range. Such a *d*–map is called an operation. A *d*–map M on a *d*–sort S is said to be many-to-one, if and only if the cardinality of the domain is greater than the cardinality of the range and one-to-one, if and only if the cardinality of the domain is equal to the cardinality of the range. Such a *d*–map is called a function $f()$. A *d*–map M on a *d*–sort S is said to be onto, if and only if every element of S is either a supremum or an infimum of the rules of correspondence. If a *d*–map M on S is both many-to-one and onto, it is called an isomorphism. If M is said to be order-preserving or isotone if, given an ordering operator on two elements x_1 and x_2 in the domain of M, there exist corresponding elements y_1 and y_2 in the range of M, which are also valid arguments of the induced ordering relation in the following sense: given that x_1 corresponds to y_1 and x_2 corresponds to y_2, it follows that if $x_1 \geq x_2$ then $y_1 \geq y_2$ and similarly, if $y_1 \geq y_2$ then $x_1 \geq x_2$.

Theorem 4: An isomorphism is isotone.

Argument:

If we could not make this choice, there would exist some order on the *d*-subsorts, and the *d*–map would not exist. Note that such an ordering relation on the range or domain provide structure and thus increase the ordinality of the *d*-subsort.

QED

A bisection of a *d*–sort S is any one-to-one *d*–map defined on S and having both domain and range in S, in that a one-to-one *d*–map divides the *d*–sort into a range and a domain.

† This will be more obvious after we provide a constructive definition of a smooth recursive enumeration, below.

A partitioning is an ensemble of d–maps on a d–sort S such that no element is in the domain or range of any other d–map. Thus, a partitioning of the d–set S is a selection of disjoint d–subsets from the ensemble of d–subsets of S, which are disjoint union S (their union is equivalent to S). Note that a partitioning provides a natural means of "dividing" a d–set into parts, each d–sort distinguishable from the other.

We designate by $\{\}$ a d–set S and by $\{|\}$ the bisection of S. For convenience, we label the d–sorts thus created by a bisection of S, L and R. We call the label for $\{\}$ the **identity label**. The bisection $\{L|R\}$ is the bisection of S into d–sorts L and R. We have defined bisection in such a way that it is invertible. Thus, we may speak of the inverse process from time to time and call this **adjoining**. Note that for any d–set of ordinality 2, bisection yields 2 d–sorts of ordinality 1, namely L and R. Thus, adjoining L to R yields a d–set of ordinality 2. Similarly, adjoining L to S or S to R yields a d–set of ordinality 3. Note that adjoining L to itself (or R to itself) yields a d–sort of cardinality 1. We call this recursive adjoining, beginning with a single d–sort, the **von Neumann recursion**.

We leave the details of this recursion to the reader, but note that it differs from the original recursive process defined by von Neumann and the more recent explication by Conway [15], only in that we do not define a cardinal 0, nor do we require the (∞) real number line, since we are merely generating labels. Note that neither 0 nor ∞ are defined for us, since we cannot show how to construct (or even find in Bishop's terminology) either of them constructively and finitely. For us, the infimum is the successor of 0 and ∞ is the successor of the supremum of a recursive enumeration. Clearly, these are not constructable within the context of the specific recursive enumeration nor are they unique for the collection of recursive enumerations in the system, even though they are ordered with respect to its properly generated labels.

By a **number**, we mean a label given to an element x of a d–set via a monotonic recursive enumeration. We define a **primary enumeration** as follows: first establish a unique label to represent identicals (i.e., the identity element); then, a recursive enumeration which labels the initial element with that of the identity element, and which, on recursion, generates labels for which there exists an isomorphism with the recursively generated elements of the von Neumann recursion is called a primary enumeration. A primary enumeration generates the integers up to the cardinality of the d–sort.

We call a ternary relation $+$ **addition** if, operating on a d–set S of arithmetic elements (i.e., *numbers*), for arbitrarily chosen elements x, y, z, there exist elements of S x' and constant element e_1, such that the following relations hold:

$$x + (y + z) = (x + y) + z$$

$$x + e_1 = x$$

$$x + x' = e_1$$

$$x + y = y + x \quad .$$

We call a ternary relation $\times$ **multiplication** on a d–set S with addition if, for arbitrarily chosen elements of S, w, x, y and z, there exists elements of S w' and e_2, such that the following relations hold:

$$w \times (x + y) = (w \times x) + (w \times y)$$

$$(w + z) \times x = (w \times x) + (z \times x)$$

$$(w \times z) \times x = w \times (z \times x)$$

$$w \times x = x \times w$$

$$w \times e_2 = w$$

$$w \times w' = e_2 \quad .$$

Note that addition and multiplication are both intended to be relations manifested by the ordering operators and could be defined much like the reverse-polish notation calculator which has a single (arithmetic label) display and accepts one input (label) at a time. For operators, closure is not defined explicitly. The existence of a unique starting label and a stop rule guarantees that, for some input labels, the operator will simply stop and perhaps generate a special label. This is quite similar to an "overflow" or "underflow" condition in a physical calculator. In practice, arithmetic closure simply guarantees that a calculator is well-behaved and does not suddenly generate a symbol which is not a number. Our operators have this deterministic element built in, and so there is no need of a closure property.

A **reparameterization** is an ordering relation induced on a *d*–set by an ordering operator, such that, given a label for the element x of the *d*–set as an argument, the reparameterization gives a label for the successor of x; namely, x'. A reparameterization is not a primary enumeration.

A **segment** [x,y] for x and y in *d*–set P, is the *d*–set of all elements z which satisfy $x \leq z \leq y$. A partially ordered *d*–set is said to be **locally finite** if every segment is finite. Clearly, all partially ordered *d*–sets within the ordering operator calculus are locally finite.

Enumerations on *d*–sets may be divided into two classes. **Normal enumerations** are those for which, though not necessarily monotonic, there exists an isomorphism to the von Neumann recursion and which begin with the identity label. **Subclass enumerations** are those for which there exists an isomorphism to the von Neumann recursion, and which (up to redundancy) establish the identity label as the final label of the *d*–set.

Theorem 5: A subclass enumeration is a reparameterization.

A **fractional enumeration** is a subclass enumeration in which the labeling of each element is given in comparison to the final label of the *d*–set. The labeling thus generated follows a recursion relation induced by the inverse algorithm for multiplication. A monotonic subclass enumeration on a *d*–set of nonfinite cardinality would define the **real number line segment on [0,1]**. Since this is strictly not defineable within the formalism, we define the **discrete real number line segment** [0,1] to be a monotonic subclass enumeration on a *d*–set S of finite cardinality. We call the cardinality of S the **precision** of the segment. A reparameterization of the discrete real number line segment [0,1], with final label n and initial label m, defines the **discrete real number line segment** $[m,n]$.*,

Random versus Arbitrary

As noted in the introduction, where information regarding the construction of a property is not available, we shall be required to deal with the property by probabilistic means. In order to do this, we must introduce a concept of randomness which is constructive and finite. We are now in a position to do so.

Kolmogorov [17] and Chaiten [18] have defined the measure of randomness of a string in terms of the length of its shortest description, an inherent property of individual strings. Namely, if the space complexity of the algorithm is greater than the length of the string it produces, then the string is random. Unfortunately, this definition is not acceptable for three reasons: (a) it allows for infinitely long strings and infinitely complex algorithms, (b) it is nonconstructive (i.e., it does not tell how to construct a random string) and (c) the set of Kolmogorov-random strings is nonrecursive. A number of extensions have been considered, but none give an effective procedure for writing a pseudorandom generators.

Suppose that the algorithm for an particular computation O is not known. Select a computational metric. Let the computational cost $C(O)$ of representing the algorithm for O be greater than the **representational resources** n within the finite discrete system S to be constructed. Under certain conditions which we now determine, the algorithm may not be discovered or even constructed within S.

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* Note that if the reparameterisation is a one-to-many *d*–map with range of cardinality greater than the cardinality of the system N, and if monotonic decreasing, we obtain $(m,n]$, and if monotonic increasing, $[m,n)$, and the adjoin of these segments is (m,n). Furthermore, this provides a formal definition of the hierarchical nature of the real numbers. As pointed out elsewhere [16], the equivocation of this hierarchy of classes is the source of a number of apparent paradoxes.

Theorem 6:
An algorithm O with computational cost $C(O)$ is indistinguishable from a "true" random number generator within a discrete, finite system S with representational resources $R(S)$ whenever

$$R(S) < C(O) + log_2 C(O) \tag{1}$$

(where the operations $<$, $+$ and log_2 have their usual meanings). Call O an **arbitrary binary number generator.**

Argument:

Consider (1) a system composed of a Universal Turing machine with a finite memory, and (2) a binary number generator G. Such a system is incapable of deciding whether or not the number generator produces repeating binary strings of length n whenever the memory is smaller than an amount m equal to $n + \log_2 n$.

Suppose that the Turing machine takes as input a particular substring of length n output by G, and we wish it to determine whether or not the number generator G is producing this substring repeatedly. Select as a computational metric the computational space cost C_s, without regard to the computational time cost C_t. The Turing machine must consume an amount of memory equal to n in order to store the string; then, the computational space cost C_s for any computation on the substring, including direct comparison with a second input substring, is at least as great as C_s, for a count of the number of symbols n in the substring ($log_2 n$). Thus, $n + log_2 n$ sets a lower bound on the computational space cost $C_s(O)$ for any algorithm which may be selected to make the decision.

It follows that the system cannot decide whether or not the target string has been produced if it has memory less than $n + log_2 n$. But this means that the system cannot distinguish between number generators which produce repeating strings and random numbers. Clearly, the symbols in the repeating strings will occur with equal probability, as required for a random distribution. However, since the system cannot detect that a given string is repeating, it cannot detect that some string of cyclicity n is repeating. Thus, for systems with less than $n + log_2 n$ memory, a generator producing repeating strings of minimal cyclicity n is indistinguishable from a generator producing random numbers.

QED

This theorem means that we may actually construct ordering operators which are "perfect" pseudorandom generators, in our terms more properly called "arbitrariness generators." Thus, ordering operators can be constructed from existing ordering operators, and not all ordering operators need have *a priori* existence. According to the Theorem, such a situation will give rise to a nondeterminism born of computational complexity and representational impoverishment:† we cannot predict the output of the ordering operator, because we could not even express the complete algorithm if it were "known."

Given this situation, it is possible to understand the ordering operator foundations as arising from a complex, though finite system of space complexity $n + log_2 n$ greater than the space complexity n of the finite system in which we are working. The complexity of the *a priori* ordering operators is greater than the space complexity n of the known system. Note that this does not introduce an infinite regress, since we need postulate an extension of the known system only once to account for conditions of "randomness" and indeterminacy. The notion of *truly random* can have no meaning within the theory.

d–Spaces

A **d–set S' is said to be a permutation of the d–set S**, if the only difference between them is the partial ordering relation. Consider a d–set S partitioned into n mutually disjoint d–subsets.* These d–subsets need not be formed by equipartitioning of S, although this is what will usually be meant. In general, however, we will denote cardinality of each of the n partitions by $m_1, m_2, m_3, \ldots m_r, \ldots m_n$,

† The variables involved in an ordering operator's algorithm may rightly be called von Neumann hidden variables. This does not mean hidden variables in the usual "quantum mechanical" sense, since we do not have a Hilbert space. It is interesting to note that von Neumann had similar ideas when he referred to systems with "partial knowledge."

* We could just as well begin with n mutually disjoint d–sets S_n, and form a new d–set S which is the union of the disjoint d–sets; however, this would require care in specifying the ordering operator on which S is defined.

respectively. For each d-subset r, by definition, there exists an ordering operator which generates m_r distinct labels. Call an ordered d-set of the labels, one from each of the n partitions, an n-tuple or d-point. Form a d-set R from all the possible ordered n-tuples of labels. We call such a d-set R a d-space.

A d-space S on which addition holds for the d-points of S and for which there exist elements (defined via a primary enumeration) between which multiplication holds, is called a **vector d-space**. The d-points of a vector d-space are called **d-vectors**. By a **d-basis**, we mean an ensemble of n totally ordered d-sets.

A **d-curve** on a d-space is a d-set of d-points for which there exists at least one basis d-set which can be mapped 1-1 onto the d-curve, and for which there exists at least one total ordering on the d-basis, which is preserved by the d-map. A **smooth d-map** is defined here to mean that there exists a partial ordering over the domain and a partial ordering over the range of the d-map, such that the n^{th} enumeration in the partially-ordered domain maps to the n^{th} enumeration in the partially-ordered range (i.e., the d-map is isotone but not necessarily 1-1), and for every reparameterization of the domain there exists a reparameterization of the range which is isotone.

The **derivate** of a recursive enumeration $f(n)$ is the number x, where

$$P7\colon x = \frac{f(n) - f(n+h)}{h} \; .$$

Thus, x is just the divided difference $[n, n+h]$ of $f()$. Because the primary enumeration generates the integers, h is just 1 if $f(n)$ is a primary enumeration. Then x is also the **forward difference**. Indeed, most of the results of the calculus of finite differences may now be taken over intact [19].‡

A recursive enumeration $f(n)$ (or rule of correspondence or a d-map), is **locally differentiable** over some d-sort S if the corresponding primary enumeration exists; then there exists a number x such that $f(n) + x = f(n+1)$. Although there may exist a recursive enumeration on a d-sort, the primary enumeration need not exist if, for example, the d-sort is only partially ordered. A smooth d-map, which is locally differentiable for all the labels in the domain of the d-map, corresponds to the classical notion of a continuous function.

The series formed by summing a recursive enumeration $f(n)$ for successive values of n is said to converge if the recursive enumeration is monotonic decreasing, and **diverges** if the recursive enumeration is monotonic increasing.

If the domain of a function, with range defined on the discrete real number segment [m,n], depends on the parameterization chosen for the range, then we call it a **real-valued function**; otherwise we call it a **scalar function**.

For recursive functions with multiple arguments, we define the **partial derivate** $f_i'(x)$ as

$$f_i'(x_1, x_2, \ldots x_n) = f(x_1, \ldots, x_i + 1, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n) \quad .$$

With this definition, we recover the inverse function theorem for inverse suitably defined over d-sets. Further, the determinant (computed in the usual manner) will go to the infimum of the d-set, if the x_i are dependent. If the x_i are independent, then we are guaranteed that there exist linearly independent recursive functions

$$f_i(x_i)$$

such that some linear combination of the f_i yield f.

A **chart** is an 2-tuple consisting of a neighborhood N and a d-map from N to some d-space R^N, whose N disjoint d-subsets are each defined on a discrete real number line segment. If it is possible to construct a system of charts in such a way that each d-point of a d-space M is in at least one neighborhood, we call this system an **atlas**. A d-space M with an atlas A is called a **d-manifold**. Each d-map on a manifold M associates with each element of M an n-tuple of the space called the **coordinate** of the element under this d-map. A manifold can therefore also be understood as a d-set of d-points (N-tuples) where, for each d-point of the d-set there exists an open neighborhood which has a smooth one-to-one d-map onto an open d-set of R^N for some N.

‡ Note that for d-set with sufficiently large cardinality m, one may reparameterise P7 to read $P7'\colon f(n) + x\frac{1}{m} = f(n + \frac{1}{m})$, which reduces to the classical definition (L'Hopital's Rule) in the limit of large m, though without appeal to infinitesimals.

A coordinate d–basis x^i parameterized on the generations t of an ordering operator O^i of a d–space S, is a basis such that the d–sets of the basis have no element (indeed, no d–point) in common (they are mutually disjoint), other than a uniquely and arbitrarily identified d–point called the **origin**.

Each d–space is characterized by a unique number n, which is the maximum number of disjoint d–subsets of S of equal cardinality, such that the union of the d–points formed from these disjoint d–subsets is indistinguishable from the d–space S. This number is called the **d–dimension of S**. For a coordinate d–basis of d–dimension n, we may refer to one of the disjoint d–subsets as a *d–coordinate* of S.

Having defined a vector d–space, we can assume the usual definition for linear combination, linear independence, maximal linear independent set (d–set), basis, dimension, components, metric functions, inner product, etc. We may also define the usual continuum notions, as long as we adhere strictly to the conditions for d–sets. We can now include eigenvalues and eigenvectors as usual, except that a range of eigenvectors is not allowed. Keep in mind that each recursive function may have multiple arguments. Thus, the d–vector at d–point P of M is not just a real number, except in the case when the number of arguments is one, and even then it may have a sign. It is truly a discrete vector with n components.

For any coordinate system x^i in an open neighborhood of a d–point on a vector d–space, the coordinates define a coordinate d–basis x^i (since there are n linearly independent d–vectors in the tangent d–space, these being the vectors formed from first derivates at the d–point of the underlying d–space vectors). Good coordinates are those where the x^i are linearly independent—this is just the condition on them to provide a 1–1 d–map to some neighborhood of the d–point in M onto a region in R^N.

Notice that we have nowhere restricted the definitions of the elements of a given d–sort or d–set: the structure is always extrinsic, thus supplying a local topology. This also means that we can use our constructs as fundamental elements in the definitions we have just given, thus generating a further layer of recursion. In this way, we will be able to define hierarchical structures.

An **inner product** n() is a recursive function on a vector d–space V, which satisfies the following if x is a d–vector of V and a is any (discrete) real number, and $|a|$ is the value independent of sign:

P8: $n(ax) = |a| \times n(x)$.

An inner product n is a **distance function** or *norm* in a vector d–space V, if it satisfies the following, where x and y are vectors of V and 0 is the inf of the d–set of all such vectors:

P9: $n(x) \geq 0$ and $n(x) = 0$ iff $x = 0$.

P10: $n(x+y) \leq n(x) + n(y)$.

Note that the relations of $+$ and $\times$ need not be the usual addition and multiplication. For example, $\times$ can be multiplication modulo 2.

By treating d–sets in R^N, we are in no difficulty, as long as we remember that defining the members of the image d–set must be recursive. Our recursion serves to maintain the class of elements in the d–set (insuring the existence of what is usually called, and which we will continue to refer to as, an **equivalence class**). In practice, one may use the standard notation and properties of continuum mathematics as a kind of shorthand (effectively making a Dedekind cut to obtain the real number line segment). If this is done, we must remind ourselves that in so doing, we have changed class membership (e.g., 1/2 is not in the same class as 0 and 1): we have effected a reparameterization. Since our definition of d–set is dependent upon the ordering operator which generates it, this means we must reconstruct any relationships between the d–set and any other d–sets.

In a sense, the d–set of d–points S in R^N is the union f of the image d–sets for all classes in the domain. Thus, we may define a distance function on S. We may also define a distance function on a d–set (a non–Hausdorff space), but it will be "multivalued" in the sense that the ordering between elements need not produce a single chain; thus, there may be more than one "path" between elements. If we take the distance function such that the number of elements traversed is minimal, then at best we must assume that elements in the string defining the distance (i.e., the minimal simple chain between any two elements) may in fact be twins under the equivalence class defined by the distance function.[♮]

♮ Clearly, we do not care if the d–maps are strictly recursive; they may be analytic in the usual sense, as long as we keep in mind the constraints on the space.

A bilinear and symmetric inner product n satisfies P11:

P11: $n(x+y)^2 + n(x-y)^2 = 2[n(x)]^2 + 2[n(y)]^2$.

If we say that a curve passes through a d–point P of a manifold M, it follows from the definitions that M is a recursively enumerable d–set and the curve is then a monotonic recursive function f on M. Thus, we say that the d–points of M (or objects of M) form an equivalence class ordered under f.

A derivate of f at P is then the motion along f at P (how fast the ordering parameter is increasing and in what direction + or −). Furthermore, for monotonic f, there are $n!$ distinct orderings of M without redundancy, if M has n objects and a sufficiently large number of reparameterizations for a given f (i.e., adding $1, 2, \ldots n$ to f).

If a is an element of a d–set A and b is an element of d–set B, then a and b are comparable if and only if, for some ordering operator O, $a \leq b$ or $b \leq a$.

We say that a covers b when the segment $[a, b]$ has two elements.

Theorem 7: If M consists of objects which are not in the same equivalence class, then we order them such that there exists an f for each class in M; then we may cover M by a suitable choice of

$$x^i(P_i)$$

with $f(x_1, \ldots x_i, \ldots x_m)$ for m classes in M. Thus, we can establish a basis for M.

There are, of course, many such bases. A coordinate basis is, then, one for which M is covered and the f_i do not order the same P_i (i.e., they are maximal and (linearly) independent).

A d–fiber consists of the d–set of all the derivates for all the possible parameterizations at a d–point of the base manifold M. A projection d–map assigns each d–fiber to a d–point of M.

A product space $M \otimes N$ consists of all ordered pairs (a, b), with a in M and b in N.

A vector field is a rule which chooses precisely one tangent d–vector from the tangent d–space at each d–point and assigns this to the point. For every vector field, there exists a curve, just as every curve has a tangent vector at each point. A d–set of curves which cover a manifold is called a congruence.

It will be useful to step outside the theory from time to time in order to understand the relationship between the ordering operator calculus and other mathematical endeavors. Certain computational phenomena in the practice of standard mathematics as applied to laboratory physics may be accounted for in this way. For this purpose, we introduce two special terms. If a recursive function were to be defined on an infinite set, then it would be said to be analytic. The analytic interpolation of a recursive function defined on a d–set over the segment $[m, n]$, is just an analytic recursive function for which there exists a d–map between the reparameterization of some discrete real number line segment $[m, n]$ of the recursive function and some monotonic sequence belonging to the infinite set generated by the analytic recursive function.

Theorem 8: A monotonic recursive function on a finite d–set of cardinality n has at most $n - 1$ derivates.

From time to time, we may say that some aspect of our construction is global, by which we mean that it is characteristic of or applicable to the entire d–space. Similarly, we may say that some aspect of our construction is local, if it is characteristic of or applicable only to some proper d–subspace.

d-Vector Functions

A one-form is a recursive function which generates a (discrete) real number for each d–vector on M and follows the usual linearity. The formation of this number is called the contraction of the one-form on the d–vector. A metric is a linear, symmetric function of two d–vectors (the "dot" product).

The recursive enumeration for a general d–set provides a parameterization for recursive functions defined on the d–set. Clearly, the parameter takes values from 1 to n over a d–subset of cardinality n. Note that the function deals, in general, not with cardinality but with ordinality, and this is arbitrary under the permutation group. The input and output are only symbols. Consider finite d–sets only. Interpretation as having cardinality n induces (via the function) an ordering on the d–set; thus, some structure is supplied. The function is intrinsically a mechanics of typography—how we can combine and use symbols is a recursive function.

In general, the notion of a recursive enumeration of a d–subset of the recursive d–set goes over to a parameterization under a d–map: $J \to R^1$. The parameterizations must cover the d–set. If they do so independently without repetition, then we have a coordinate parameterisation.

Theorem 9: Exterior differentiation (defined as usual) commutes with any differentiable mapping of the manifold.

It is interesting to note that the cohomology groups depend only on the topological structure of M, and *not* on its differentiability. That is to say, cohomology theory passes over from standard differential geometry to the present theory almost intact, as there is no dependence on the definition of differentiability. This is particularly important for applications in physics, where Gauss' and Stokes' theorems are of such great use.

A Note on Computing Numeric Roots

It will often be the case that d-functions are needed, which make use of various roots, such as the square root. For us, not all numbers have a "square" root, meaning two equivalent roots, and similar comments hold regarding higher-ordered roots, such as the "cube" root. Where such references are made in the remainder of this paper, we refer to the so-called "symmetric" root. Symmetric roots are defined as being a rational root of the number plus or minus some other rational number. Thus, in general, any square may be expressed as

$$(a - \epsilon) \times (a + \epsilon) = a^2 - \epsilon^2$$

such that ϵ is a rational fraction up to the precision of the computation. Note that this definition literally inverts the Pythagorean construction of the irrationals, but in a manner which requires no irrationals. This is, of course, just the operational definition which is taken in finite computation such as that using Newton's Method. One performs a recursive computation until the error (our ϵ) is sufficiently small. The nonequivalence of the two *root*'s in the ordering operator calculus holds a special significance: it suggests that the d–space is intrinsically noncommutative and that a commutative d–space is meaningful only if constructed of perfect squares, perfect cubes, etc.

2.5 Constructed Objects: Partition Lattices

In this section, we provide the concepts necessary to make the appropriate connections to generating functions and the finite operator calculus, as well as incidence algebras and von Neumann's theory of games. These concepts will prove useful when we begin the process of interpretation of physical phenomena.

An **order ideal** in a partially ordered d–set P is a d–subset Z of P, which has the property that if x is an element of Z and $y \leq x$, then y is an element of Z.

The product $P \bigotimes Q$ of two ordered d–sets is (p, q), where p is an element of P and q is an element of Q endowed with order $(p,q) \geq (r,s)$ whenever $p \geq r$ and $q \geq s$. The **direct sum** or *disjoint union* $P \bigoplus Q$ consists of elements x and y with order $x \leq y$ if and only if

(i) x, y are elements of P and $x \leq y$ in P

or

(ii) x, y are elements of Q and $x \leq y$ in Q.

The blocks of a partition of a d–set S are the d–subsets of S making up the partition. A partition $\prod$ is a **refinement** of a partition $\sum$ if every block of $\prod$ is contained in a block of $\sum$. The inf or 0 of $\prod(S)$ is the partition whose blocks are the one element subsets of S, and the sup or 1 of $\prod(S)$ is the partition with one block. The **lattice of partitions** $\prod(S)$ of a d–set S is the d–set of partitions of S ordered by refinement.

Note that there is a natural correspondence between equivalence relations on a d–set S and partitions of S, since the equivalence classes of an equivalence relation form the blocks of a partition and, hence, there is an induced lattice structure on the family of equivalence relations of S.

2.6 CONSTRUCTED OBJECTS: COMBINATORIAL SYSTEMS

A **combinatorial system** consists of a unique initial label or *word* called the **axiom** of the system, and a finite d–set of strings called the **productions** of the system. Productions are the recursively generated words of a combinatorial system. The **alphabet** of the system is all the **symbols** or **letters** that occur in the axiom or productions of the system. A word of the system contains only the alphabet of the system.

Theorem 10: For every combinatorial system there exists a combinatorial system with precisely a two letter alphabet, whose decision problem is recursively solvable if and only if that for the first system is also recursively solvable.

Theorem 11: The d–set of integers generated by a combinatorial system is recursively enumerable.

Theorem 12: If the d–set of integers generated by the combinatorial system is not recursive, the decision problem for the combinatorial system is unsolvable.

Any n–form field divides all d–vector basis into two classes: those for which it is, on contraction, + and those −. This is called right- and left-handedness, respectively. If it is possible to be consistent in specifying handedness at each (not continuously here, since M is discrete) d–point P of the manifold, then M is said to be **orientable**. For every orientable manifold M, there exists an inverse to the derivate function called the **integral**.

3. MATHEMATICAL FOUNDATIONS II: ATTRIBUTE SPACE

By a **combinatorial attribute**, we mean a property of a d–sort that has been constructed by an ordering operator. In particular, for binary sequences or ensembles labeled by the generations of i ordering operators O_i, an attribute is any property which is recursively definable or computable on the ensembles of labels generated by the ordering operators. Let the sequence of labels output by each of the O_i be represented by a directed graph G_i (this is a Hasse diagram if there exist no cycles in the ordered labels produced by the ordering operator), consisting of labels as nodes and connecting arcs to represent the pairwise orderings between labels. Call the graph g_i which results from G_i by the removal of any number of arcs and/or nodes, a reduction of G_i. Consider the i collections of reductions R_i for each of the G_i. Form a new collection of reductions, consisting of no more than one reduction from each from of the R_i. If there exists an isomorphism between the all the pairs of reductions in such a collection, the reduction represents an attribute of the collection of ordering operators O_i.

Let the i ensembles of labels generated by the O_i be operated on by a new ordering operator O^*, so that the ensembles are (partially) ordered and labeled. Each generation of O^* can be classified according to whether or not there is a reduction corresponding to the underlying ensemble, which represents an instance of a particular attribute. We will call the instances of an attribute the **attribute states** of the attribute over the ordering operator O^*.

Since the output of any ordering operator O for each generation may be arbitrarily complex and have considerable internal structure (inaccessible unless constructed in the manner above), we will, henceforth, drop the notational distinction between O and O^*. The reader should, however, keep in mind the considerable structure which is implied when we refer to a combinatorial attribute or an attribute state.

A combinatorial attribute (or simply attribute where no confusion will result from the usage) is conceptually akin to a set-theoretic property, although mathematically distinct. First, attributes are constructive, whereas set-theoretic properties are not, being generally of an *a priori* nature and giving set theory that "tacked-on" look. Second, they are not "properties" of a set, but rather of a d–sort which has been constructed with an ordering operator. The definition of an attribute is thus much stronger than the definition of set-theoretic property, in that an attribute would certainly be a property but all set-theoretic properties are not attributes. Clearly, an ensemble has attributes as a set has properties, if one remembers that this similarity is metaphorical rather than mathematical.

As an example, consider the generation of the permutations of a discrete, finite, ordered collection S (noting that such a collection is a special case of a d–sort; in fact, it is a d–set). If the generator is specified via a recursive algorithm, has a unique starting ordering of the set and halts after generating all possible permutations generated in a specific order, the generator is a special (and useful) type of ordering operator. The notion of permutation so defined and used is an attribute of the ordered set

P of all permutations of the original ordered collection S (i.e., it is possible to recursively define the permutations of a particular finite ordered collection and to recursively give their complete denotation), and any specific permutation P_i of the ordered collection S is an attribute state. Thus, a permutation is an attribute with respect to a reference ensemble (the starting ordering of S) and the ordering operator which generates permutations. Similarly, any specific subensemble is an attribute, with respect to the subensemble and the **identity ordering operator**—the ordering operator which, given a label as input, returns it as output. In this sense, we may search a d-set for d-subsets which are equivalent; i.e., those which have the same (identity) attribute.)

3.1 Multiple Ordering Operators

Please note that more than one attribute (indeed, more than one ordering operator which generates permutations) may be defined on the ordered set. This is an essential characteristic of d-sorts which must not be overlooked. For example, given an ordered d-set of labels of cardinality N, there are $N!$ additional distinct permutations possible. There are then $(N!-1)!$ possible ways in which the d-set of all permutations can be generated and, therefore, $(N!-1)!$ permutation attributes definable starting from the ordered d-set of labels.

In any given construction, we must explicitly state what ordering operators generate the structure, as these provide the connectivity of the elements. We may then construct the ways in which two or more operators combine or interact with each other. Suppose a d-sort of labels L of cardinality N are independently generated by i ordering operators O_i. In order to treat the d-sorts generated by O_i as a single construction, it must be possible to demonstrate the constructive existence of a total ordering operator O' such that O' generates L. O' is said to be **decomposable** into the O_i if, to each generation of O', there corresponds one and only one O_i which generates the corresponding label. This is the first instance of the label being produced by this O_i, and all other O_i (except the O_i that generated the previous label of O' on its previous generation) generate the same label again. The O_i are said to be **serializable**.

Two or more ordering operators are said to be **intrinsically coupled**, if they generate at least one label or attribute state which is mutually indistinguishable (i.e., if the first operator cannot distinguish some label, (called the **coupling label**), generated by the second operator from a label which it generates and vice versa). Ordering operators which are serializable are **locally orthogonal**, since they are not intrinsically coupled. This does not mean that they are **globally orthogonal**, since they may be **extrinsically coupled** via a third ordering operator with which they are both intrinsically coupled.

The coupling of two ordering operators which are extrinsically coupled via a third ordering operator is said to be of **coupling degree** one; if via a third and a fourth, such that the output of the third is input to the fourth and the output of the fourth is indistinguishable from one of the labels of the first two, then the coupling is said to be of coupling degree two; the number of intervening ordering operators gives the degree of the coupling. The number of coupling labels gives the **coupling order**. Note that coupling is dependent on the specific ordering operators involved; two ordering operators may be coupled in multiple ways.

The coupling of two ordering operators O_1 and O_2 is characterized by a unique rational fraction called the **scale**, which is just the ratio of the cardinality N_1 of the labels, which may be produced by O_1 when coupled to O_2 to the cardinality N_2 of the labels, which may be produced by O_2 when coupled to O_1. The degree, order and scale, and the cardinalities and ordinalities of the ordering operators give all the information necessary to compute the probability (frequency) of one ordering operator interacting or **mixing** with another ordering operator. In a system of coupled ordering operators, the labels output by two ordering operators will be said to **superpose**.

We may relax the "set" restriction in our example: if a partial ordering is generated by an ordering operator, the collection (ensemble) is a d-sort with respect to that operator. In other words, distinguishability is meaningful only in terms of the ways (i.e., ordering operators) one has specified how to generate the ensemble. If the generator treats the order of "two" permutation states indifferently, then they are indistinguishable for that ordering operator, and we have no other means of determining distinguishability.*

It will be necessary to form quite complex attributes: attributes, then attributes of attributes, then attributes of attributes of attributes, etc. We will refer to these as **attributes of first order, second order, third order, etc.**, respectively. From the definition of combinatorial attribute given above, it can be seen that the construction of attributes is potentially recursive. One can form a collection of ordering operators O_i^*, each of which generate the attribute states of an attribute; perhaps being distinct (although they need not be) in the order of generation of the attribute states. From the collection of outputs of the O_i^*, one may construct a new attribute and define a new ordering operator O^{**}, which generates the attribute states of this attribute of second order. This method of constructing higher-ordered attributes may be recursively continued, up to the point at which the only reduction possible is the graph consisting of a single node.†

3.2 Establishing a Distance Function

Based upon the definitions of ordering operator, dimension, and coordinate d-basis, a one-dimensional d-space coordinate basis behaves as a totally ordered d-set. It is convenient to represent this d-set by a sequence of binary strings; i.e., a string in an alphabet containing two symbols, where the order of the symbols is dictated by the ordering operator. For example, given a string composed of n unique labels, one may use Huffman encoding [21] as a way of unambiguously giving a binary representation of the sequence of labels.

Now, define **attribute distance** for a specific attribute generated by an ordering operator O_i, as the measure dependent solely upon the number of distinguishable states s between two ensembles of labels which O_i may generate, normalized by the total number of states which O_i may generate, N_i. This is equivalent to the unique number of generations of O_i required to generate the first ensemble A from the second (called the ***reference ensemble***) B, and results in a **distance function** d() on the closed interval of rational fractions [0,1]:

$$d(A, B) = s/N \quad .$$

By a reparameterization on an attribute (from the definition of reparameterization), we mean a mapping of the labels for the attribute states generated by some ordering operator O, to the tags for the attribute states generated by a second ordering operator O'. Via a reparameterization, then, we may remap $d()$ into $d'()$, defined on the closed interval of rational fractions [-1,1]. Since the cardinality of O may be smaller than that of O', indistinguishable attribute states for O may be induced in the mapping in order to properly map all the tags of O'.‡ Note the similarity between the notion of attribute distance and statistical (read with the frequency interpretation) distance, as defined by Wootters [22]. as the "maximum number [N—added for clarification] of distinguishable orientations between" two measured

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* A further example may be helpful. In the d-space which is the positions on a chess board, the sequence of moves which any given chess piece takes during a game determines an ordering operator. The form or rule that specifies the legal moves that a piece can make specifies an attribute. If the ordering operator is parameterised on the attribute which defines a piece's legal moves, then each such legal position, generated in allowed sequence, is an attribute state.

† Note that the mathematical objects of our construction are each defined relative to one or more operations. This forces an intrinsic connection between the usual static form of mathematics and the dynamics found in physics. It also precludes arbitrary identification of constructed entities. If our analysis of paradox [20] is complete, then many (we hope all) of the paradoxes which arise in logic, set theory and philosophy are not possible here.

‡ In the chess example, the attribute distance in terms of the ordering operator which generates a specific piece's moves is just the number of moves that the piece has made, divided by the total number of moves it will make in the game. Then the distance is always some rational fraction of the total distance the piece will travel in the context of the game. Note, however, that having all the sequences of moves for all the pieces in a game does not allow us to reconstruct the game; we must know how the moves are interleaved. This can be accomplished by specifying the ordering operator in terms of the game clock; that is, for each move of the game, each pieces' ordering operator must generate some attribute state. For us, whenever a piece does not move, the attribute state generated is indistinguishable from the previous attribute state. Reparameterising the ordering operators in this way normalises all the attribute distances, thus providing a global distance function topology on the d-space of the chess board.

attribute values divided by the square root of N for normalization. This is just a measure of the distinguishability of two measurements, based on the number of values which an attribute being measured can take. Clearly, if the statistical distance between two values is zero, they are indistinguishable from an information theoretic point of view. Wootters presents strong evidence that "statistical distance equals actual physical distance." The specific relationship is derived for the case of photon polarization measurements. We will demonstrate that this relationship is even more general than (apparently) assumed by Wootters and that the relationship serves as the basis for an extension of relativity and explains much in quantum mechanics.)

Suppose that we reparameterize $d(A,B)$. Represent a generation of O_i which decreases $d(A,B)$ as a 0 and one which increases $d(A,B)$ as a 1. The total number of 1's is simply the Hamming distance, and is defined on the interval $[0,N]$. By subtracting the number of 0's and then dividing by N, the result is a Hamming measure on the interval [-1,1] (i.e., centered on 0, which has an ordinal but not a cardinal significance), and is independent of the number of generations of O_i. In general, we will find this to be a more useful form of the attribute distance function.

3.3 Synchronization

In order to perform operations on multiple ensembles with some ordering operator O, some means must be given for establishing a label in each as the common input to O. If the ensembles are not identical, then a d-map between the ensembles will be useful. In this section, we define a particularly useful mechanism for achieving such a mapping.

Pick three ensembles, A, B and C. Let the attribute distance between A and C be zero, but with $t_C > t_A$ by some ordering operator O local to A and C, with generations parameterized by t. Let there be an ordering operator O'' local to B. Furthermore, let the attribute distance between A and B be nonzero. We say that the ordering operators O and O'', with generations parameterized by t and t'', respectively, are synchronised if condition (1) holds, and A and B are said to be synchronous if the O and O'' are synchronized and conditions (2) and (3) hold:

1) $t_B - t_A = t''_C - t_B$;

2) if A is synchronous with B, then B is synchronous with A;

3) if A is synchronous with B and B is synchronous with C, then A is synchronous with C.

In other words, (1) states that the ordering operators are synchronized if there exists a binary symmetric relation between t and t'' over the specified attribute. By reason of the nonuniqueness principle (Principle IV), the property of synchrony between *ensembles* must also be reflexive and transitive as in (2) and (3), respectively. This simply means that it is possible to define a new distance function defined by T across the ensembles, which is consistent with the distance functions defined by t and t''.

Henceforth, we drop the notational difference between synchronous t and t'', since these may be replaced by a single universal ordering operator with parameter T.

3.4 The Dimensionality of D-Spaces

We are now in a position to construct a unique global property of d-spaces which have a distance function that is coordinate independent. We will begin by examining how such a coordinate independent distance function can be established, using the concepts we have defined and constructed. We will then investigate a global property of the resulting d-space.

Under the condition that the cardinality of the d-space or d-subspace precludes explicit representation of the algorithm for an ordering operator O, we are clearly faced with a severe lack of knowledge, the algorithm for O is arbitrary and the output may, therefore, may be treated as random for our purposes (Theorem 6).

Consider the labels produced by O to be represented by bit strings [i.e., strings of 1's and 0's] or, if repetition is being allowed and only two labels are allowed, to be arbitrarily treated as 1 and 0. The sequences of 1's and 0's thus produced meet the conditions of Bernoulli trials (see Figure 1):

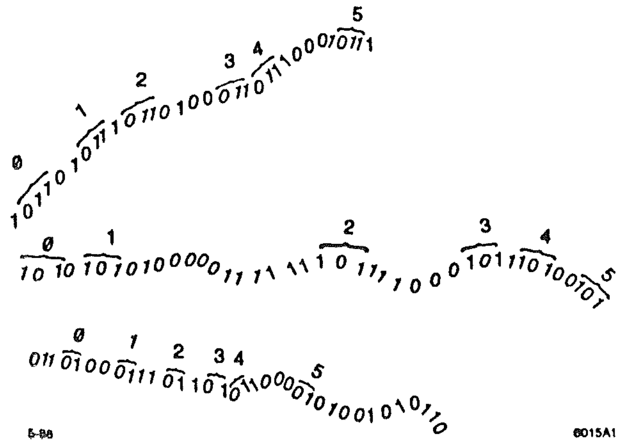

Fig. 1. Independent trials.

being unable to specify the algorithm used by O forces us to see successive productions of O as independent. Only after the fact, may we label the resulting output of O as representing some specific, previously known, ordering relation.

Now, let there be r such arbitrary binary number generators, $O_1, O_2, \ldots O_r$, with string (d–set) outputs $S_1(n), S_2(n), \ldots S_r(n)$ up to some maximum, R. We will refer to n as the ordinality of each string, parameterized by a counter we will call t. In the absence of other information about $O_1, O_2, \ldots O_r$, we assume them to be independent operators and, indeed, cannot discover otherwise, due to the computational complexity of the operators and our relatively impoverished (representationally) system of just two symbols.* Clearly, $S_1, S_2, \ldots S_r$ are synchronized on n via t, but this provides only trivial structural information about any relationships between the strings, given our criteria for "arbitrary" and the independence of $O_1, O_2, \ldots O_r$. $S_1, S_2, \ldots S_r$ constitute a coordinate d–basis, as long as we identify the initial outputs of $O_1, O_2, \ldots O_r$ as "identical."

Look for other means of synchronizing $S_1, S_2, \ldots S_r$; pick as an attribute any sequence of length M of binary symbols (i.e., a specific substring of $S_1, S_2, \ldots S_r$); interpret the first M symbols of $S_1, S_2, \ldots S_r$ as matching this substring; then examine the output of $O_1, O_2, \ldots O_r$ for further synchronized productions of this string. As long as the algorithm governing the ordering operator which generates the strings is of sufficiently great computational complexity, the occurrence of the substrings is arbitrary and simple statistics for concurrent, independent Bernoulli trials apply.

At $L = Mn$, (i.e., the position of the $n + 1^{th}$ string of length M), the probability that the number of occurrences of the substring is the same across multiple strings produced in this way is just

$$u_L = \frac{1}{2^{rL}} \sum_{k=0}^{L} \binom{L}{k}^r \tag{2}$$

where $r \in \{1, 2, \ldots R\}$.

Now, interpret the normalized number of occurrences of the substring as defining a discrete distance function across the R–dimensional d–space. That the normalized number of occurrences of the substring constitute an attribute distance, and, therefore, a distance function, is trivial; the identity attribute for

* Proving nonindependence would require a knowledge of the ordering operator's algorithm.

the given substring may be satisfied (the substring may occur) only finitely many times in a string of finite length. This defines the number of possible instances of the attribute. Take as a reference ensemble, the null string (the string of zero length, i.e., the empty string). The attribute distance follows from the definition, immediately.

Call a specific generation of an r-dimensional d-space with any distance function f, a reference frame. We see that Eq. (2) is just the probability that the distance function has a r-independent value, given a value of R and L (i.e., the distance function is length preserving).

Theorem 13: The upper bound on the global d-dimensionality of a d-space of cardinality N with a discrete, finite and homogeneous distance function is 3 for sufficiently large N.

Argument:

Note that for $R > 3$, the terms of Eq. (2) are monotonic decreasing [i.e., Eq. (2) converges]. That is, for sufficiently long strings, the probability of another synchronized occurrence of the specified substring must approach zero. For $R \leq 3$, however, the terms of Eq. (2) are typically monotonic increasing [i.e., Eq. (2) diverges]—there is always the probability of another occurrence of the specified substring. Hence, the possibility of an isotropic (Principle IV) distance function across more than three dimensions is unlikely, while it is certain if $R \leq 3$: [23]. Clearly, the case of $R = 3$ contains the greatest representational power.*

Formal Argument:

Two or more attributes are mutually independent if the generating or defining ordering operators on the space of ensembles are mutually disjoint, with the exception of a single element; in which case, they define the dimensions of the space. An attribute admits a distance function if and only if there exists a total ordering on the ensembles which possess the attribute. Two or more (R) such distance functions are symmetric (do not introduce inhomogeneities into the space—Principle IV), if and only if they can be synchronized (this is equivalent to demanding that there is exist an R-way matching criteria between the productions of the R ordering operators, such that a match is found arbitrarily often in sufficiently long productions).†

However, this is not possible if $R > 3$. Let the productions of the ordering operators be mapped to binary strings. These strings may be treated as the results of Bernoulli trials. The probability of a specific occurrence at the n^{th} trial is given by

$$u(n) = \frac{1}{2^{rn}} \left[\binom{n}{0}^r + \binom{n}{1}^r + \binom{n}{2}^r + \cdots + \binom{n}{n}^r \right] \tag{3}$$

for n trials, with $r = R$ equal to the number of dimensional metrics. The maximal term of the binomial distribution

$$\binom{n}{k} 2^{-n} \tag{4}$$

is of the order $\sqrt{2/\pi n})$ and $< n^{-1/2}$. Therefore,

$$u(n) < n^{-(r-1)/2} 2^{-n} \left[\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n} \right] = n^{-(r-1)/2} \tag{5}$$

and so the sum of $u(n)$ converges for $R > 3$.

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* The concentration of synchronisable events for "short" attribute distances, even with $R \gg 3$ will be related to the big bang and to quantum fluctuations.

† More importantly, note that these conditions are identical to those demanded by Einstein in deriving the Lorentz transformations. Namely, the demand that clocks be synchronisable is equivalent to demanding spatial homogeneity (i.e., that there is no preferred coordinate). In addition, the property of transitivity (from the definition of synchronisation) implies that there exists, at least mathematically, a "universal" clock. This is ironic, in as much as special relativity is usually understood to have removed the Newtonian concept of Universal Time. In fact, Einstein did not remove the concept, but rather showed that this global time need not be accessible, as long as synchronisation with transitivity was allowed. Under these conditions, local time is sufficient.

That it diverges otherwise is proved as follows:

Case 1: For $R = 2$, and from the normal approximation to the binomial distribution

$$u(n) = \binom{2n}{n} 2^{-2n} \simeq \frac{1}{(n\pi)^{1/2}}$$

and so $\sum u(n)$ diverges for $R = 2$. Note, however, that $u(n) \to 0$ as $n \to$ large N. Therefore, while synchronization is certain, it has a mean recurrence time on the order of $\sqrt{N}$, so that, in two dimensions, the synchronization is sparse.

Case 2: For $R = 3$, and from the normal approximation to the binomial distribution, for sufficiently large n and

$$\frac{1}{2} n - n^{1/2} \leq k \leq \frac{1}{2} n + n^{1/2} ,$$

we have

$$\binom{n}{k} 2^{-n} > cn^{-1/2} ,$$

where c is a (small) positive constant. Therefore,

$$u(n) > 2n^{1/2} (c^3 n^{-3/2}) = \frac{2c^3}{n} ,$$

and so $\sum u(n)$ diverges for $R = 3$.

Thus, as a recurrent event, any given sequence will be shared between more than three runs only a finite number of times, and, hence, is unlikely; whereas, between three or fewer, the same sequence will be shared in position arbitrarily often for sufficiently large strings.

It follows immediately, that we cannot define a metric-homogeneous discrete space with more than three spatial dimensions. Any other space must introduce either asymmetries or inhomogeneities over the metric.

QED

It might be argued [24] that in a d–space of finite cardinality N, the theorem no longer applies. However, consider what happens if the global distance function is defined with $R = 4$; then there exist local distance functions defined on the three–dimensional d–subspaces.

Suppose that some relationship is to be defined between the local distance function and the global distance function. For large, finite N, this becomes impossible. A comparison of $u(n)$ for each distance function shows that, as N becomes large, "meter marks" for the three–dimensional d–subspace become relatively more frequent, whereas those for the four–dimensional space become less frequent. Thus, the d–map becomes impossible, unless the one–dimensional d–subspace grows more rapidly than the three–dimensional d–subspace; i.e., unless one dimension is different from the remaining three. However, by hypothesis and in keeping with Principle IV, this is not possible, since it makes the four–dimensional d–space inhomogenous.

Furthermore, there would then be a three–dimensional d–subspace, composed of the one–dimensional d–subspace and any two other dimensions, which would generate as rapidly as the four–dimensional d–space, and the difficulty of defining a relationship between the distance function on this d–subspace and the global distance function would be undiminished. Notice that this difficulty becomes apparent for relatively small runs (as soon as $n^{1/2}$ is significant), since the ratio of expectations for synchronization between a d–space and its largest d–subspace is bounded by $n^{-1/2}$.

Constructing a Coordinate System

It is important to understand how one constructs a coordinate system using Theorem 13 and the definitions that preceded it. We make explicit use of the notion of independence in order to construct an orthogonal basis, since independence is the essential constructive notion underlying orthogonality when a geometry (i.e., some notion of "angle") does not, as yet, exist. Having taken this step, we are then *required* to construct a norm which vanishes when the two arguments are orthogonal. This is, of course, trivial if the usual operations of addition and multiplication are available; but care must be taken, since we deny the need for the usual properties of closure and commutativity.

Having once identified an attribute *independently* in each of three binary strings generated as in the discussion preceding Theorem 13, computed the distance from an *arbitrarily identified* origin using the appropriate one of three distance functions (each need only be defined on one of the strings), and, finally, established synchronization across the three strings, the only quantities of interest in performing d–vector computations in this three–dimensional d–space are the "meter marks" established by synchronization. This synchronization establishes a new distance function uniquely defined in the ordering operator sense, which is independent of which of the three strings are involved in the computation.

Thus, if we now treat the three strings as generating a coordinate d–basis x, y and z, we may say that a d–vector of a certain "magnitude" has a particular "direction." In the simplest case, the direction is either "parallel" or "antiparallel" to x, y or z. In such a case, the norm which we use must give the magnitude of the d–vector, when the arguments to the norm are the d–vector and the appropriate unit d–vector in the "same" direction and the infimum of the distance function, if either of the other two unit d–vectors are used. This proscription on the construction of a norm results in a unique norm only in that all such norms will be "orthogonality" preserving.

Similar comments hold with regard to the construction of a "vector product." Great care must be taken not to assume any intrinsic notion of direction and connectivity of the d–space, such as that which is often imposed by the Pythagorean theorem (which is valid only in a Euclidean or flat–space and generally not valid in a discrete, finite space). Furthermore, great confusion and apparent contradictions result if one insists on using the distance function defined, in order to construct "meter marks" on a particular string as though it were global (i.e., useful for all three strings or identifiable with the distance functions defined by the process of synchronization).

Note that if the d–vectors are represented by binary strings, it is necessary that the independent attributes be represented consistently; thus, the attributes must be independent under the operation of discrimination (exclusive or). If the d–vectors in a d–space are represented in a manner consistent with "meter marks," we then have a means of forming the vector product. In the canonical form, the independent attribute substrings for a three–dimensional d–space are just "001" (x), "010" (y) and "100" (z). These substrings form a complete representation and are independent under the operation of discrimination.

A d–vector represented by the binary string "001011110," then has an attribute distance in the x direction of 2, in the y direction of 2 and in the z direction of 1. Such a representation gives more information than just the direction and magnitudes—it contains a history of the generation of the d–vector. This explicit representation of the process nature of mathematical objects is an important characteristic of the ordering operator calculus. In order to use the usual notions of a vector space, including computation of components, this historical information must be obscured. Thus, one only considers the magnitudes and the directions, without the explicit binary representation of the d–vectors.

3.5 Characteristics of a Discrete Geometry

Having developed a d–space with a coordinate independent distance function, we may now explore certain other symmetry relations on the d–space. In particular, we will find it useful to understand the d–space equivalents of the familiar orthogonal and rotational symmetries.

It is a central point of this section that a measure of the discrete cardinality N and of the curvature of a discrete geometry in a d–space is given by the precision with which two ratios are identical in value: the ratio of the area of the maximally-sided symmetric polygon, which may be constructed in the d–space

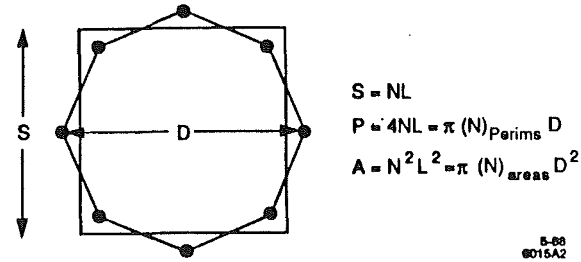

Fig. 2. Relation between $\pi(N)_{areas}$ and $\pi(N)_{parameters}$.

to the area of a square ($\pi(N)_{areas}$) in the d–space, and the ratio of the perimeter of that same polygon to the perimeter of the square ($\pi(N)_{perimeters}$); see Figure 2.

Indeed, the relationship between these values has global significance, and we shall have need of understanding that significance in later sections, as well as being able to explicitly use one or other of the ratios thus constructed.

In what follows we construct a square and a circle, and construct an algorithm for a rational fraction ratio which plays the role of π. We begin by constructing the equivalent of a square: an orthogonal, two–dimensional coordinate patch. The only elements allowed for construction are a finite (perhaps large) number of discrete elements (essentially indistinguishable mathematical objects), ordering operators, the ability to count and the ability to label the objects through an operator.

By nearest n neighbor of a label e in a sequence of generations of an ordering operator O, is meant any label n such that for labels a, b, e and n, $a : a = O(e)$ (read "label a such that a is generated on input of label e to ordering operator O') or $b : e = O(b)$; then for any ordering operator O' mutually disjoint from an ordering operator O, at least one of $n = O'(e)$, $n = O'(a)$, $n = O'(b)$, $e = O'(n)$, $a = O'(n)$ or $b = O'(n)$ holds. Clearly, a and b are nearest neighbors of e, as well. The process of identifying nearest neighbors simply defines a new binary ordering relation between a pair n and e, if there exists a third label a for which (possibly distinct) binary ordering relations exist between a, e and n.

Before proceeding with the constructions, a comment on notation: the ordering operators used will be total ordering operators. Those differing only in a subscript will denote a d–set of mutually disjoint ordering operators having domains of equal cardinality (and ordinality by definition). The symbols O and O' will be used for ordering operators whose output is to be taken as orthogonal: that is, the generations are mutually disjoint (distinguishable) except for a single generation of each which are indistinguishable. The generations of the O and O' will be notationally distinguished by x and y, respectively. A prefixed superscript of either 1 or -1 will denote the generations of the ordering operator as coming either before or after some specified and unique label, respectively. The subscripts associated with O and O' will be carried over to the respective generations.

A Discrete Coordinate Patch

Without reference to a particular geometry or distance function, a "square" can be defined as a closed d–set having the following properties:

a) two–dimensionality;

b) the edges or boundary consists of two d–sets of two mutually disjoint totally ordered d–subsets (four sides);

c) fixed center under interchange of the coordinate parameters;

d) it is possible to establish a distance function on the edges such that each of the totally ordered d–subsets is of equal length.

The criteria for two–dimensionality is satisfied by requiring two mutually disjoint ordering operators. The algorithm is as follows:

1) Select a label L_0; see Figure 3.

2) Establish a totally ordered ***d*-set chain** $^{-1}x_0$ of length n with L_0 as the supremum, using the ordering operator O_x; see Figure 4.

$$- L_0 -$$

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Fig. 3. A starting label L_0.

$$L_{-3} - - L_{-2} - - L_{-1} - - L_0 -$$

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Fig. 4. The subchain of length $n = 4$, $^{-1}x_0$ with L_0 as supremum.

3) Establish a chain $^{1}x_0$ of length n, with L_0 as the infimum, using the ordering operator O_x; see Figure 5.

$$^{-1}x_0 \qquad L_{-3} - - L_{-2} - - L_{-1} - - L_0 - - L_1 - - L_2 - - L_3 - \qquad {}^{1}x_0$$

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Fig. 5. The subchain of length $n = 4$, $^{1}x_0$, with L_0 as infimum added.

4) Call the union of $^{-1}x_0$ and $^{1}x_0$: x_0. Require that x_0 be totally ordered; see Figure 6.

$$x_0 \qquad L_{-3} - - L_{-2} - - L_{-1} - - L_0 - - L_1 - - L_2 - - L_3 -$$

5-88 6015A6

Fig. 6. The chain of length $n = 7$, x_0.

5) For each label L_i of x_0, establish chains $^{-1}y_i$ and $^{1}y_i$ of length n, under the ordering operator O'_y, with the selected label of x_0, as either the supremum and infimum of the chain. Require that the y_i are disjoint, as are the pairs $(^{-1}y_i, {}^{1}y_i)$. This is a unique labeling or total ordering requirement on the entire construction (i.e., there must exist an ordering operator O'' such that the labels of the entire construction are totally ordered; see Figure 7.

6) Require that the n^{th} label of the y_i form chains x_i, ordered by ordering operator O_{x_i}; see Figure 8.

7) The resulting object satisfies the requirements; it is the discretum version of a two–dimensional (square) coordinate patch. In particular, the two–dimensionality of the construction is satisfied by the definition of mutually disjoint ordering operators: at most, one label in a chain resulting from one operator will be found in a chain resulting from the other. For the given construction: at most, two operators can be used; a third would result in a partial ordering, instead of a total ordering, of the labels of the construction, and this would then represent an object which is not connected or result in an object for which "multiple" labels are doubly labeled. Thus, the ordering operators "parameterize" the object.

A Discrete Circular Patch

We can now proceed to construct an object which behaves as a discretum version of the 2–sphere. A 2–sphere (again, without reference to distance functions) has the following properties:

a) two–dimensionality;

b) every edge (boundary) label is indistinguishable from every other, under interchange of the corresponding ordering operators;

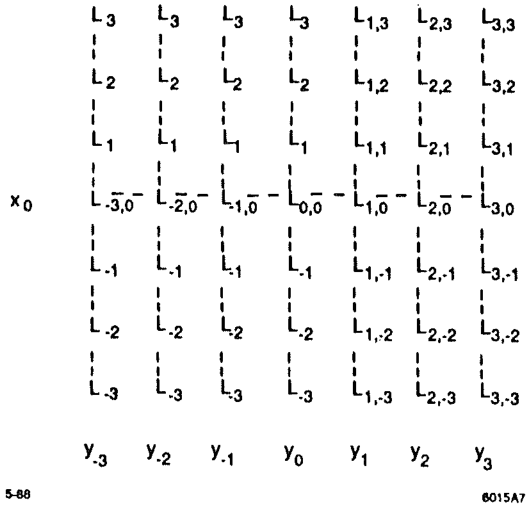

Fig. 7. The chains y_i of length $n = 7$ added.

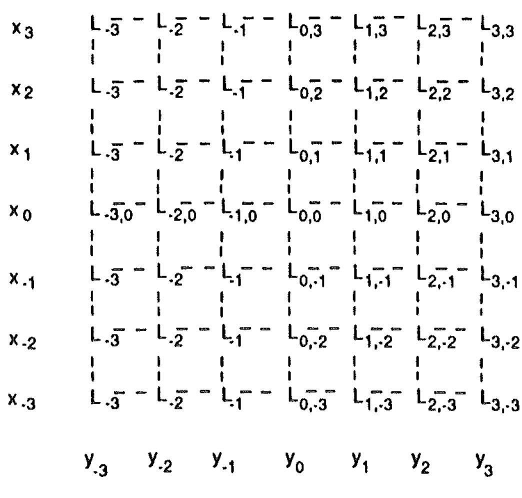

Fig. 8. The chains x_i for i.e., (-3, -2, -1, 0, 1, 2, 3) of length $n = 7$ added. Note that all labels are now subscripted twice, since they are identified as the production of two ordering operators.

c) existence of a unique label, which remains fixed in the construction, under interchange of any two ordering operators which generate it.

The constructive algorithm is as follows.

1) Select a (square) coordinate patch with center $L_{0,0}$ and all labels uniquely subscripted. Call this patch S_0; see Figure 9.

2) Constrain the possible ordering operators (as before) to those operators which produce chains of length n and which generate from $L_{0,0}$ a nearest neighbor of $L_{0,0}$, then a nearest neighbor of this label, and so on. We refer to the operators which generate these labels as radial permutations of the coordinate patch; see Figure 10.

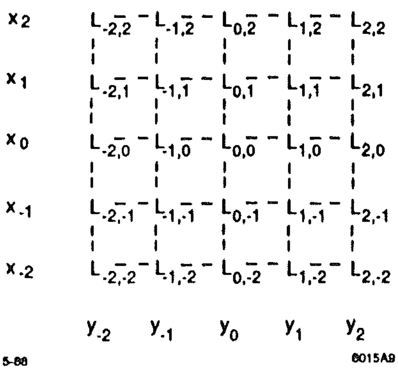

Fig. 9. Select a patch, s_0

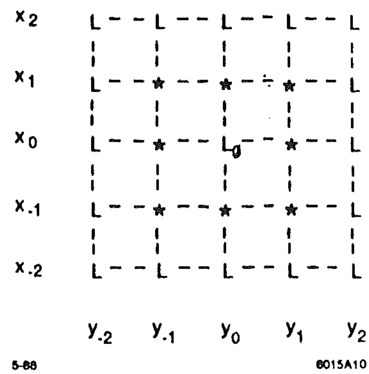

Fig. 10. The nearest neighbors of L_0 are shown as an asterisk (*).

3) Starting from $L_{0,0}$, construct a coordinate patch with a new pair of ordering operators which are radial permutations of the coordinate patch; see Figure 11.

4) Map the labels of this patch S_i to patch S_0, and eliminate any labels which do not have at least i subscripts; see Figure 12.

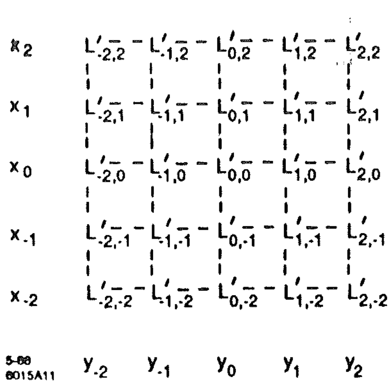

Fig. 11. A new patch, P_1.

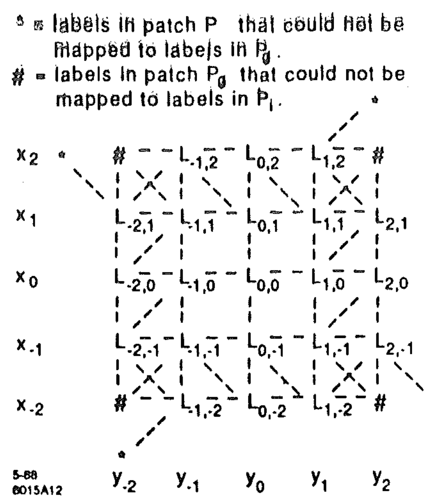

Fig. 12. Mapping the new patch to the old.

5) Repeat this process for all pairs of allowed radial permutations; see Figure 13.

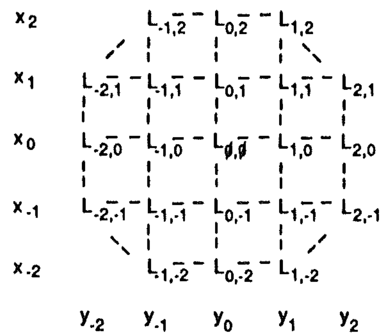

Fig. 13. Elements remaining after all allowed radial permutations.

The result is a discretum version of the circle, in that it has a fixed center ($L_{0,0}$) with radial symmetry (isomorphic to its radial permutations with identified center $L_{0,0}$). It has built-in bounds on "precision." The relation between the number of "sides" of the polygon formed by a set of cardinality n and the number of permutations is fixed: it gives a measure of the "size" of the circle.

Note that in Figures 3–13, the radial permutations which were not invoked would result in either the same labels of the construction being deleted, as here, or else would not maximally d–map the coordinant patch. The reader may readily demonstrate this. Also note that starting with a central label is a matter of technical convenience for the algorithms and may be circumvented.

$\pi(N)$

Given these two geometric objects, it is possible to define a ratio which plays the role of the ratio of the area of the circle to the area of the square patch from which it was formed. This number is obtained by counting the number of labels contained in the circle and the number of labels contained in the square and forming the ratio.

A second ratio is obtained from the ratio of the cardinality of the d–set of all radial permutations (obtainable by counting the labels on the perimeter of the circle) and the cardinality of the generations of one such radial permutation (e.g., the length n of the chain x_0).

In general, these ratios will be functions of the length n of the chain x_0. Furthermore, the values of the ratios will not, in general, be those obtained under Euclidean geometry. However, if one insists on isotropy, homogeneity and "density" (i.e., large n), it is easy to see that these values must be those obtained by the standard polygonal approximation to the circle. In particular, these ratios will be approximations to $\pi/4$ and π, with the appropriate precision. These constructions, and the results, are closely related to numerical and statistical "approximation" methods, as seen from within the traditional geometric paradigm. In fact, Archimedes came close to the construction used here (Measurement of the Circle). However, the definitions used here are completely constructive and general; matching the continuum definitions (which we prefer to think of as the "analytic interpolation") as desired.

CALCULATIONS

By Areas:

A (square patch) = 25

A (polygon) = 21

Ratio = π (areas)/4

= A (polygon)/A (square)

= 21/25 = 0.84

π (area) = 3.36

By Perimeters:

C (polygon) = 12

C (square patch) = 16

Ratio = π (lengths)/4

= C (polygon)/C (square patch)

= 12/16 = 0.75

π (perimeter) = 3.00

Indeed, if the cardinality of the d–space (N) is changing (evolving), then the two values of $\pi(N)$ will be changing, also. Furthermore, if the relevant discrete cardinality is related to a spatial volume, then, as this region becomes smaller, calculations involving $\pi(n)$ cannot be treated in a naive manner. Specifically, the multiple computational definitions of $\pi(n)$ must be disassociated, if the values are different (i.e., the ratio $\pi_{areas}/\pi_{circumference}$ will not be 1). The value of *pi* can no longer be taken as a constant, independent of spatial volume. Indeed, if the d–space is inhomogenous, the value will depend on the local inhomogeneities; it will have different values depending on the "density" of the local d–space. Even more important, if the d–space is discrete and finite, and if the values of $\pi(n)$ are not related to the local spatial volume via a cardinality of the local volume, it follows that the values of $\pi(n)$ used in calculations relate only to the cardinality of the d–space. In other words, $\pi(N)$ becomes a true global discrete topological constant, and local physical properties are then immediately dependent on the global properties.*

In the remainder of this paper we will use $\pi(N)$ to refer to the combinatoric computation of π, based on the ratios of perimeters for a d–space of cardinality N. We cannot use the ratio computed from area ratios, since we will not, in general, know the "curvature" of the d–space. Note that measuring the difference between the two ratios gives a means of locally measuring the flatness of the d–space. Similarly, the curvature can be measured by examining the ratio computed on the basis of "volumes."

Radian and Trigonometric Measures

Having constructed the largest coordinate patch and the corresponding inscribed "circle," we may now pick an orientation and specify a total ordering operator which generates the sequence of attribute states constituting the perimeter as labels. We then reparameterize the generations of this ordering operator into the interval of rational fractions $[0,\ 2\pi(N)]$. We call this parameterization the radian measure on a d–space of cardinality N. Similarly, we shall refer to the cardinality or length of the total ordering generated by one of the radial permutations used in constructed any circle, the radius r of the circle.

A radius r and a radian measure θ then correspond to that d–point which results from a translation in coordinate distance of attribute distance r from the origin, followed by θ generations of the reparameterized perimeter ordering operator. Since every pair r and θ correspond to a unique point on the perimeter, and $\pi(N)$ is constructed from the maximal coordinate d-patch, we may regard θ as a direction and define the trigonometric computations of θ in the usual manner using the norm function. In particular, take the cosine to be the unit normal projection on the x-axis and the sine to be the unit normal projection on the y-axis. Note that this does not assume the Pythagorean theorem, unless it is already entailed in the norm function.

* Applying this fact to physical phenomena, that π should then be of cosmological (global) significance is not surprising. Consider these results where the d–space is the physical Universe.

3.6 Properties of Evolving Systems: Attribute Velocities

Given a d–space, we require that there exist a total ordering operator on the space, so that a distance function (such as that produced by the Program Universe ordering operator) is possible. The universal ordering parameter T, on which the generation of this ordering operator is based, provides a local total ordering for the evolution of each ensemble, such that the local total orderings are isomorphic up to reparameterization. This in turn provides for synchrony.

We now define the **increment** I of an ensemble as the number of generations of some ordering operator† 0 needed to describe (establish local isomorphism with) the increases in attribute distance between an ensemble and a reference ensemble, with respect to T. This operator parameterizes the generation of the attribute states. Similarly, we define the **decrement** D of an ensemble as the number of generations t of the ordering operator O needed to describe the decreases in attribute distance between an ensemble and a reference ensemble, with respect to T. The **total size** S of an ensemble is defined as the arithmetic sum + of the I and D. Use $[I, D]$ to denote an ensemble with increment I and decrement D and total size $I + D$. Note that the total size S is not generally the same as the maximum cardinality N since total size refers to increments and decrements of the ordering operator, and not to the cardinality of the d–sort of labels produced by the operator.

Attribute velocity v is defined as the mathematical rate of change in attribute distance of an ensemble, with respect to generations t of an ordering operator O, computed as the difference between I and D, divided by the total size S:

$$v = \frac{I - D}{S} . \tag{6}$$

The relative attribute velocity v' is just v computed relative to a third ensemble (reference), having attribute velocity u. The relative attribute velocity may be regarded as a discrete map which transforms an ensemble $[I, D]$ into an ensemble $[I', D']$, where I' and D' depend only on I, D and u, and where v' depends only on u and v. This is just a change in the reference ensemble. The **increment quotient** is defined as the ratio of I' to I,

$$q = \frac{I'}{I} . \tag{7}$$

The **attribute speed** of an ensemble is the magnitude of the attribute velocity (note that direction is given by arithmetic sign or the degenerate cosine in the one–dimensional case, a discrete version of the $x^1 - x^2$ cosine in the two–dimensional case, and a discrete version of the $x^1 - x^2$ and $x^2 - x^3$ cosines in the three–dimensional case). Finally, we define independent ensembles as those having all states generated with respect to an ordering operator O, distinguishable. We will discuss the impact of indistinguishable states in a later section. Having defined these terms, we may now prove a series of theorems regarding the properties of such ensembles.

Theorem 14: The increment and decrement are additive for independent ensembles when aggregated; that is, the number of distinguishable states and the number of generations t relative to an ordering operator O required to describe I and D for independent ensembles is conserved.

$$[I, D] + [I', D'] = [I + I', D + D'] \quad . \tag{8}$$

Argument:

As long as two states of an attribute are distinguishable over t, we are certain that a generation of O is required for each. It follows that the total number of generations T for independent ensembles (those having all states distinguishable) is given by the arithmetic sum (total count) of the generations of O, for the increment and decrement of each. Indeed, the total size of the ensemble is just $S + S'$.

QED

† In general, this is not the same ordering operator which generated the ensemble.

Theorem 15: The attribute velocity v of an ensemble $[I,\ D]$ is a function of I and D, and nothing else.

Argument:

If ensembles A and B have the same attribute speed, then the aggregate ensemble $A + B$ must also have that attribute speed. Hence, v cannot depend on total size, but only on the ratio of I to D. Let $r = I/D$; then we can write

$$v = v(r)\ . \tag{9}$$

QED

Theorem 16: v is an increasing function of the ratio r.

Argument:

Trivially, the case from the definitions.

QED

Theorem 17: If the values of I and D are reversed, then v is reversed;

$$v([D,I]) = -v([I,D])\ . \tag{10}$$

Argument:

Inverting I and D is equivalent to counting distinguishable states from above, as compared to from below—i.e., if one counts from 0 to the maximum number of distinguishable states, one obtains the usual definition of I and D. If one counts from the maximum number of distinguishable states down to 0, consistency with the definition of additivity can be maintained if this is equivalent to a reparameterization resulting in a change of arithmetic sign.

QED

Theorem 18: If neither I nor D is 0,

$$v\left(\frac{1}{r}\right) = -v(r)\ . \tag{11}$$

Argument:

Trivially, from Theorem 4 and the supposition.

QED

Theorem 19: The attribute distance between any two ensembles has an upper and lower bound.

Argument:

Trivially, from the finitary principle (Principle I).

QED

Theorem 20: The lower bound of v CANNOT BE ZERO for independent (i.e., distinguishable) ensembles.

Argument:

If the lower bound of v were zero, the ensembles would be attribute indistinguishable and hence not independent.

QED

Theorem 21:
There is a limit to v as D approaches 0, which we can take as 1 by appropriate reparameterization; i.e., $v([I,0]) = 1$ and, hence, $v([0,D]) = -1$. We shall refer to this upper bound as v_{max}.

Theorem 22: Etters Velocity Relationship,

$$v(r) = \frac{(r-1)}{(r+1)} \tag{12}$$

holds for attribute velocities.

Proof:

Consider a d-space with distance function as previously defined. Now, examine the region between synchronization (metric marks or ticks). In this region, as we have shown, there exists a value for the isotropic distance function. Let I be the total number of 0's and D the total number of 1's generated up to n generations of the ordering operator which defines the distance function (called the metric ordering operator); then the total attribute "displacement" in $I+D$ generations is just $I-D$. This gives an Etters velocity relationship of $I-D/I+D$ or, if $r=I/D$,

$$v(r) = \frac{(r-1)}{(r+1)} .$$

QED

4. MATHEMATICAL FOUNDATIONS III: COORDINATE TRANSFORMATIONS

In order to explore the invariant properties of a system, we must have a means of expressing not only the coordinate bases defined in the previous chapter, but also transformations between coordinate bases. Of particular interest are those coordinate bases which define a reference frame. In the present chapter, we develop a series of theorems regarding transformations between reference frames.

Theorem 23: Suppose that synchronizable reference frames K, with coordinate bases x^i and k with coordinate bases y^i, i in $\{1,2,3\}$, are defined so that the origin of k has attribute velocity v in the direction x^1, with respect to the origin of K in the universal ordering parameter T; then the coordinates transform according to:

$$t' = \gamma \, \frac{t - vx^1}{v_{\max}^2} , \tag{13}$$

$$y^1 = \gamma \, (x^1 - vt) , \tag{14}$$

$$y^2 = x^2 , \tag{15}$$

and

$$y^3 = x^3 , \tag{16}$$

where

$$\gamma = \frac{1}{\left[\frac{1-v^2}{v_{\max}^2}\right]^{1/2}} .$$

Argument:

Select A, B and C synchronous with a distance function $d()$. Let $d(A,B)$ be the attribute distance between A and B and $d(B,C)$ be the attribute distance between B and C. Given $d(A,C)=0$, as above; then, by symmetry (Principle IV), we require that $d(A,B)=d(B,C)$, so that for maximum attribute velocity $v_{\max}$, we have

$$v_{\max} = \frac{2d(A,B)}{t(A)-t(C)} . \tag{17}$$

Since $d(A,C)=0$, note that A and C are indistinguishable, except by parameter t. Furthermore, with reference to a third ensemble with attribute velocity v,

$$t(B)-t(A) = \frac{d(A,B)}{|v_{\max}-v|} \qquad \text{and} \qquad t(C)-t(B) = \frac{d(A,B)}{|v_{\max}+v|} . \tag{18}$$

Now, suppose that we wish to compare the attribute distances d and d' and the operators t and t', with reference to third and fourth ensembles with attribute velocities 0 and v, respectively. Call these systems

K and k. Furthermore, assume that there exist at least two independent attribute distances (generated from mutually disjoint ordering operators, except for a single element) for K and k; call these x^i and y^i, respectively. We seek one-to-one transformations (discrete maps) between these operator values. Given (in the absence of specific cause—i.e., an ordering operator) homogeneity (Principle IV) of the system K and k in the parameters, these transformations must be linear and homogenous.

Let $x^{1'} = x^1 - vt$; then k has a system of values $x^{1'}$ independent of t. Define t' as a function of $x^{1'}$, x^2, t. Let $d'(A,B)$ be the attribute distance between A and B, and $d'(B,C)$ be the attribute distance between B and C. Given $d'(A,C) = 0$ as above; then, by symmetry, we require that $d'(A,B) = d'(B,C)$, and

$$t'(B) = \frac{1}{2}[t'(A) - t'(B)] \ ,$$

or

$$\frac{1}{2}\left[t'(0,t) + t'\left(0, t + \frac{x^{1'}}{v_{\max} - v} + \frac{x^{1'}}{v_{\max} + v}\right)\right] = t'\left(x^{1'}, t + \frac{x^{1'}}{v_{\max} - v}\right) \ . \tag{19}$$

Let $x^{1'}$ be chosen small, and use an appropriate reparameterization, so that we may use the calculus of finite differences in solving for the proper transformations. Then, taking the *finite derivates* (not the derivatives) [25],

$$\frac{1}{2}\left(\frac{1}{v_{\max} - v} + \frac{1}{v_{\max} + v}\right)\frac{dt'}{dt} = \frac{dt'}{dx^{1'}} + \frac{1}{v_{\max} + v}\left(\frac{dt'}{dt}\right) \ , \tag{20}$$

or

$$\frac{dt'}{dx^{1'}} + \frac{v}{[v_{\max}^2 - v^2]}\left(\frac{dt'}{dt}\right) = 0 \ , \tag{21}$$

and

$$\frac{dt'}{dx^2} = 0 \ . \tag{22}$$

Since t' is linear, and we can assume $t' = 0$ when $t = 0$, the solution is just

$$t' = a\left(t - \frac{v}{v_{\max}^2 - v^2}\, x^{1'}\right) \ , \tag{23}$$

where $a = f(v)$, unknown for now.

Let $v_{\max}$ be represented by the same fixed value for both K, and k by a suitable reparameterization in each reference frame. Let attribute information transfer with attribute velocity $v_{\max}$ over a positive attribute distance y^1,

$$y^1 = v_{\max} \times t' \ , \tag{24}$$

and

$$y^1 = a v_{\max}\left(t - \frac{v}{v_{\max}^2 - v^2}\, x^{1'}\right) \ ; \tag{25}$$

then, with reference to the frame K, an ensemble expressed in the system k has attribute velocity

$v_{\max} - v$, or

$$\frac{x^{1'}}{v_{\max} - v} = t \; . \tag{26}$$

So

$$y^1 = a \; \frac{v^2_{\max}}{v^2_{\max} - v^2} \; x^{1'} \; , \tag{27}$$

and

$$y^2 = v_{\max} t' = a v_{\max} \left(t - \frac{v}{v^2_{\max} - v^2} \; x^{1'} \right) \; , \tag{28}$$

where

$$t = \frac{x_2}{[v^2_{\max} - v^2]^{1/2}}, \qquad x^{1'} = 0 \; ; \tag{29}$$

thus,

$$y^2 = a \left(\frac{v_{\max}}{[v^2_{\max} - v^2]^{1/2}} \right) \; x^2 \; . \tag{30}$$

By substitution for $x^{1'}$, we obtain

$$t' = f(v) \; \gamma \left(t - \frac{v x^1}{v^2_{\max}} \right) \; , \tag{31}$$

and

$$y^1 = f(v) \; \gamma (x^1 - vt) \; , \tag{32}$$

$$y^1 = f(v) \; x^2 \; , \tag{33}$$

where

$$\gamma = \frac{1}{\frac{1-v^2}{v^2_{\max}}^{1/2}} \; . \tag{34}$$

To find $f(v)$, introduce K' with coordinates x^1, $x^{2'}$ and t' in parallel translation relative to x, such that the origin of k moves with attribute velocity $-v$. Assume the origins coincident. Applying the transformations we obtain

$$t' = f(-v) \; \gamma(-v) \left(t' + \frac{v y^1}{v^2_{\max}} \right) = f(v) \; f(-v) t \; , \tag{35}$$

$$x^{1'} = f(-v) \gamma(-v) (y^1 + vt') = f(v) f(-v) x^1 \; , \tag{36}$$

$$x^{2'} = f(-v) y^2 = f(v) f(-v) x^2 \; . \tag{37}$$

Since the transforms from K' to K are independent of t, it follows that K and K' are relatively at rest. Therefore,

$$f(v) \; f(-v) = 1 \; . \tag{38}$$

Now, let there be an attribute distance of value ℓ, given independent of x^1 and $x^{1'}$; call this x^2 and $x^{2'}$, in k and K, respectively; then ℓ in k, with reference to K, is just

$$x^2 = \frac{1}{f(v)} \, . \tag{39}$$

Since, from symmetry, attribute distance can depend only on v, and not on direction or the sense of attribute speed, it follows that the interchange of v and $-v$ does not change ℓ. Hence,

$$\frac{1}{f(v)} = \frac{1}{f(-v)} \qquad \text{or} \qquad f(v) = f(-v) \, . \tag{40}$$

Thus, from Eqs. (37) and (39), it follows that $f(v) = 1$. Therefore, we have

$$t' = \gamma\left(t - \frac{vx^1}{v^2_{\max}}\right) ,$$

$$y^1 = \gamma(x^1 - vt) ,$$

$$y^2 = x^2 ,$$

and

$$y^3 = x^3 ,$$

where $\gamma = 1/[1 - v^2/v^2_{\max}]^{1/2}$; these being Eqs. (13), (14), (15) and (16), respectively.*

QED

Theorem 24: If $u = 0$, then $I' = I$ and $D' = D$; that is, an ensemble with zero attribute velocity induces the identity transformation.

Argument:

Trivially, from the definition of attribute distance, an ensemble with zero attribute velocity, with respect to some reference ensemble, is indistinguishable from the reference ensemble.

QED

Theorem 25: If $u = v$ and attribute speed < 1, then $I' = D'$; i.e., if ensemble A (which we may interpret as an observer) has the same attribute velocity as ensemble B, their relative attribute velocity is 0.

Argument:

Trivially, from the definitions of attribute distance and velocity.

QED

Theorem 26: If $I = D$,

$$v' = \frac{(I' - D')}{(I' + D')} = -u \, ; \tag{41}$$

i.e., with respect to a reference ensemble A with nonzero attribute velocity, an ensemble B with zero attribute velocity is an ensemble with the same attribute speed, but with opposite sign (direction).

* Note that, although our derivation is finite and discrete, we have deliberately followed the derivation of the Lorentz transformation developed by Einstein. We wish to emphasize that, contrary to common belief, the derivation of these transformations are not dependent upon the continuum. Where Einstein used derivatives, we use finite derivates, Eqs. (20) and (21). Where he allowed for a continuum of coordinates and velocities, we are restricted to the rational fractions which suffice per Pauli and Brodsky.

Theorem 27: If the reference attribute speed is less than 1, a reference ensemble A with attribute $-u$ induces the inverse of the transformation induced by changing to a reference ensemble B with attribute velocity u.

Corrolary 27A:

Reversing ensemble A attribute velocity sign (direction) inverts the transformation induced on the attribute velocity of ensemble B.

Theorem 28: The relative attribute velocity in the frame of ensemble A is bounded from below by the speed of ensemble B.

Theorem 29: The limiting attribute velocities for an ensemble are invariant under the transformation induced by nonzero attribute velocity; i.e., $[I,0]' = [I'',0]$ for some number I''.

Argument:

If the sign of the relative attribute velocity is positive, this follows from lower bound. If negative, the inverse transformation corresponds to positive relative attribute velocity, so that D must remain invariant.

QED

Theorem 30: The increment quotient q is a function only of u.

Argument:

$[I,0]' = [I'',0]$, where I'' depends only on I and u. However, by Theorem 14, $[I',D] = [I,0]'+[0,D]'$, hence $I' = I''$; thus $q = I'/I$ depends only on I and u. However, q cannot depend on I, since otherwise $[2I,2D]'$ would have a different attribute velocity than $[I,D]$.

QED

Theorem 31: The inverse transformation induced by an ensemble with attribute velocity $-u$ has an increment quotient of $1/q$.

Argument:

The inverse transformation is I/I'.

QED

Theorem 32: The decrement quotient is the inverse of the increment quotient:

$$\frac{D'}{D} = \frac{1}{q} \,. \tag{42}$$

Argument:

First, reverse I and D to get $-v$, then take inverse transformation associated with $-u$, which multiplies the increment (which is now D) by $1/q$ to get $-v'$, then reversing I and D again to get v'; Thus, D' results from multiplying D by $1/q$, and it follows that $D'/D = 1/q$.

QED

Theorem 33:

$$q = \frac{(1-u)}{\gamma} \,,$$

where

$$\gamma^2 = 1 - (u^2) \,. \tag{43}$$

Argument:

By the definition of the decrement quotient [Eq. (42)], $D' = D/q$, and from the increment quotient [Eq. (7)], $I' = qI$, so that from the definition of $v = (I' - D')/(I' + D')$ [Eq. (41)], we can write $v = (qI - D/q)/(qI + D/q)$. Since q is a function only of u, we can choose any values of D and I that lead to an equation in q and u, and its solution will define the general functional dependency. Assume $I = D$ so $v = 0$ and $v' = -u$; then, from Eq. (41),

$$-u = \frac{(qI - I/q)}{(qI + I/q)} = \frac{(q^2) - 1}{(q^2) + 1} .$$

Solving for q results in the relationship to be proved.

QED

Theorem 34: Relative to the zero velocity frame v = 0, the size change δm of an ensemble with attribute velocity v' is

$$\delta m = \frac{S}{\gamma} . \tag{44}$$

Argument:

Multiplying in the first part of (43) by $(1 + u)$ gives $1/q = (1 + u)/\gamma$ and $D = I$x for an ensemble with zero attribute velocity, this follows immediately.

QED

Theorem 35: Attribute velocities combine according to

$$v' = \frac{v - u}{1 - vu} . \tag{45}$$

Argument:

By definition,

$$v = \frac{r - 1}{r + 1}, \qquad r' = \frac{I'}{D'} = \frac{r(1 - u)}{(1 + u)} ,$$

and

$$v' = \frac{r' - 1}{r' + 1} .$$

Then, by substitution and recollection of terms, we have

$$v' = \frac{v - u}{1 - vu} . \tag{46}$$

QED

Theorem 36: For an ordering operator O of cardinality N and for each run of cardinality k, the minimal attribute distance increment $\imath$ is

$$\imath(O) = \frac{1}{k!} . \tag{47}$$

Argument:

Consider a sequence of productions from an unspecified ordering operator of cardinality N to be used as a coordinate basis. We can compute the minimal attribute distance increment which can be generated in a given run of cardinality k of the operator, straightforwardly: it is the ratio of the number of (order) distinguishable states C (i.e., combinations—by excluding order, we take only those states that are distinguishable under a particular ordering operator) to the number of states P (i.e., permutations—by including order, we include all states, even those which are not distinguishable under a particular ordering operator).

$$C(k;N) = \frac{N!}{k!(N-k)!} , \tag{48}$$

$$P(k;N) = \frac{N!}{(N-k)!} , \tag{49}$$

$$i(O) = \frac{C(k;N)}{P(k;N)} = \left[\frac{N!}{k!(N-k)!}\right] \times \left[\frac{(N-k)!}{N!}\right] = \frac{1}{k!} .$$

QED

In general, C gives all the possible attribute states that could produce a sequence of state ensembles of the proper cardinality k, while P gives the number of ensembles of cardinality k possible in the same total space of cardinality N. This is, of course, subject to the constraint $k < N$.

Theorem 37: The total attribute distance $d(k;I;N)$ for an ensemble of cardinality k implied by I increments of i in a total space of cardinality N is

$$d(k;I;N) = \frac{I^k}{k!} . \tag{50}$$

Argument:

Suppose that we want to generate I increments in the attribute distance; then we want to turn the crank of the ordering operator which produces each attribute state I times. In the absence of further knowledge about the specifics of the ordering operator generation, we cannot enforce sequence so that the increments are disjoint; this is equivalent to sampling k objects from a population of I objects, with repetition allowed. Call this $R(k;I)$; then, in general, for an ordering operator to generate an attribute distance d equal to I increments from a run of cardinality k on a space of cardinality N, we have:

$$d(k;I;N) = R(k;I) \times \frac{C(k;N)}{P(k;N)} = \frac{I^k}{k!} ,$$

where

$$R(k;I) = I^k .$$

QED

Theorem 38: The sum of all values of Eq. (50) from $k = 0$ to $k = K$ approaches e^I (for any expression of I) as K becomes large. We call this the combinatoric definition of $e(K)$.

$$e^I \approx \sum_{k=0}^{K} \frac{I^k}{k!} = e^I(K) . \tag{51}$$

Argument:

From the identity of definition of terms of the power series for e^I and the combinatoric definition of $I^k/k!$, the result follows for all discrete, finite values of k, N and I.

QED

Theorem 39: The attribute distance, given a distance function g transformed by reparameterization from a distance function f, is just:

$$d[k;I;g(M)] = \sum_{k=0}^{K} d[k;I;f(N)] \times D(f;k;N) , \tag{52}$$

where $D(f;k;N)$ are the k^{th} derivates of f.

Argument:

Consider a reparameterization of $d(k;I;N)$ from a distance function f on a d–space of cardinality N to a distance function g on a d–space of cardinality M, where the attribute is first order for both f and g. This is given by multiplying the attribute distance increment for $D(k;I;N)$ by a conversion factor (rational fraction), D. Since the attribute distance increment is inversely proportional to N, we have:

$$d[k;I;g(M)] = d[k;I;f(N)] \times D \approx d[k;I;f(N)] \times (N/M) \ . \tag{53}$$

Now, examine a general distance function $f(I;N)$ defined on a d–space S. By Principles I (finiteness), II (discreteness) and III (finite computability), $f(I;N)$ may be expressed as some ordering operator O, which generates attribute states of an attribute of some order.* Call this order K. To express the generation of O in terms of the underlying discretum of cardinality N, we must take into account the possible contributions from all orders k from 0 to K. In general, D is not constant, but is dependent on f, N and k. Thus, for a general distance function $f(I;N)$, we have:

$$d[k;I;g(M)] = \sum_{k=0}^{K} d[k;I;f(N)] \times D(f;k;N) \ .$$

Note that the $D(f;k;N)$ may be solved for by the method of difference quotients [26]. These are the k^{th} derivates of f. The series is always finite (and, hence, there is no question of "divergence" for a given evaluation of the series) since N is fixed. For sufficiently large N, the series Eq. (52) approaches the Taylor series with arbitrary precision.

QED

The Lagrange form of the remainder is of particular interest here, since it gives a measure of the deviation from the discrete form by the analytic form of the truncated Taylor series Eq. (52).

$$R_n(x) = \frac{f^{n+1}(\epsilon)}{(n+1)!}\,(x-a)^{n+1} \ , \tag{54}$$

where

$$x < \epsilon < a \ .$$

For sufficiently complex attributes and large N, this approaches the usual form of the exponential operator, as normally used to describe transport along a parameter. The sum may be understood as the contributions to distinguishability by successively more complex aspects of the attribute, weighted by the probability that a particular sequence that can generate the required distance is the correct one.

Theorem 40: The incremental transport x_0 along a basis x^i at x parameterized on t is just

$$d[k;I;f(x+x_0)] = \sum_{k=0}^{K} \left(\frac{(x_0)^k}{k!}\right) \times D(f;k;t) \ , \tag{55}$$

where $D(f;k;t)$ are the k^{th} derivate operators on x with respect to t.

* Recall that an attribute of an attribute is called an attribute of second order, an attribute of an attribute of an attribute is called an attribute of third order, etc.

Argument:

We wish to compute the incremental transport δx along a given coordinate basis x in terms of the above formulation. This is equivalent to a reparameterization from f to g, in which f and g are related as follows:

$$g(x) = f(x + x_0) \ , \tag{56}$$

with x_0 being the minimum attribute distance increment.

Since we do not know the particular ordering operator, but only the ultimate cardinality of the ensemble and the cardinality of the space, we must use the general form of reparameterization, Eq. (52). The result follows from substitution of Eq. (56) in Eq. (52).

QED

If the ordering operator produces a sequence which is of first order (linear in the ordering parameter), then the rate of change of attribute distance with respect to the ordering parameter is constant. This is, of course, just the first discrete derivative (derivate). If the ordering operator produces a sequence which is of second order, then the rate of change of attribute distance with respect to the ordering parameter is a first order function of the ordering parameter, i.e., the second derivate. Similar arguments hold for ordering operators of higher order.† In order to compute the transport along x^i from x to $x + x_0$, we must take into account the contributions of each order up to the order of the operator.

Theorem 41: Given reference frames F and F', coordinate transformations between unsynchronized events satisfy Eqs. (13), (14), (15), (16) of Theorem 23, statistically.

Argument:

Consider two reference frames, F and F', given by two sets of independent generations S_1, S_2 and S_3, and S_1', S_2' and S_3'. Again, we initially synchronize each set of three and let them go independently (Theorem 13). We count the occurrence of an attribute state which may be used as a metric mark in one of the generations as a 1, and any other attribute state as a 0, for purposes of analyzing the statistics.

Now, however, we have two ordering operators which we label O and O', global to F and F', respectively. In the absence of further information regarding the ordering operator, we will assume a normal distribution (Principle IV) of distinguishable states about a metric mark in either F or F' (i.e., generated by independent O and O' as per Theorem 13).*

Consider a discrete mapping from F to F'

From the combinatoric definition of the base of the natural logarithm and the definition of the normal distribution, a sample size of two standard deviations around an attribute state, taken as the mean or center of the distribution, will consist of all the distinguishable states around the mean [28]; and, therefore, a metric mark, with a probability equal to the ratio of distinguishable states to all states, summed over all possible attribute states that might be selected in F' as a metric mark. However, this is just $1/e(N)$; thus, for a well-defined "metric mark" in F, a arbitrary transformation to F' results in a measure in F' which deviates from a metric mark by $\pm\sigma$. For a normal distribution, 2σ is just the transport for a minimum attribute distance increment. Computing the population variance σ^2 is then, for population of size N

$$\sigma^2 = \sum \frac{(U - \mu)^2}{N} \ , \tag{57}$$

where μ is the average of U. Suppose U is just the attribute distance in F; then the "mean attribute distance" μ is just the attribute velocity multiplied by the number of generations over which the attribute velocity has evolved. This is equivalent to giving the number of increments minus the number

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† This analysis is consistent with the requirement that the k^{th} derivate may be obtained from confluent divided differences of k arguments. The k arguments are order independent and, hence, are "sampled from a population of cardinality I with repetition allowed," as previously noted.

* In the absence of large N, we could as easily use the binomial distribution justified by the combinatorics to reflect finite N, and use the appropriate Yates adjustment in which $y_0 - \frac{1}{2}$ is substituted for y_0 in the computation of the probability $Pr(y > y_0)$, so that the unit normal variable probability $Pr(z > z_0)$ is just $z_0 = (y_0 - \frac{1}{2} - Np)/\sqrt{Npq}$, where p is the probability of a 1 and q is the probability of a 0. However, we assume here that the normal approximation is adequate in the light of the usual criteria that $N > 5$ and the absolute value of $|(1/\sqrt{N})(\sqrt{q/p} - \sqrt{p/q}|$ is less than 0.3 [27].

of decrements in terms of a global ordering operator spanning both F and F'. In other words, in the frame of the minimum of the maximum attribute velocities, the maximum number of generations for the ordering operator producing the attribute will be N, the cardinality of the universe. The normalized variable x^* corresponding to x with mean 0 and variance 1 is just

$$x^* = \frac{(x-\mu)}{\sigma} . \tag{58}$$

Here, $x-\mu$ is just the difference in the global frame between the increment and the decrement. Sigma is then the probability of obtaining an attribute increment corresponding to the ordering operator, which produces metric marks in F' relative to the ordering operator, which produces metric marks in F. Thus, Eq. (58) is equivalent to going to dimensionless (i.e., frame independent) quantities [29].

Note that for a discrete function on finite domain, this x^* is always bounded and finite; i.e., sigma is never 0 whenever $x-\mu$ is not 0. In addition, since the fluctuations in x are bounded and finite, it makes no sense to speak of specifying x beyond that discrete step length which results in the smallest fluctuation.

Now, let μ be a d-velocity v multiplied by the number of generations t over which it is measured, and let x^1 be the attribute distance in F, and y the attribute distance in F'; then, from Eq. (58),

$$y^1 = (x^1 - vt)\,\gamma , \tag{59}$$

so that

$$\sigma = 1/\gamma \qquad \text{and} \qquad \mu = vt ; \tag{60}$$

then y^1 is interpretable as the normalized variable associated with x^1. Clearly, as long as β is defined as v/v_{max} as in the derivation of Theorem 23, we have recovered the coordinate transformation in the absence of synchronization. Therefore, the coordinate transformations, Eqs. (13), (14), (15) and (16), are applicable at all rational scales, for all frames and for all attributes.

QED

It is important to understand that the mean attribute distance increment computed by going to dimensionless coordinates and transformed from a metric mark in F arbitrarily to F' (i.e., in the absence of synchronization between F and F'), is, thus, identical to the minimal attribute distance increment, transformed under synchronized frames for metric marks. This result may also be taken as proof by construction that the combination of the minimum attribute distance increment and the coordinate transformation of Theorem 23 has bounded (i.e., over the range of meaningful rational fractions which may be defined by reparameterization on the d-space) scale invariant significance.‡

Theorem 42: Let $P = Prob\,(I \to I+1)$ and $Q = Prob\,(D \to D+1)$ for $N \to N+1$. The uncertainty associated with a coordinate transformation satisfying Theorem 41 between meter marks is given by:

$$1 - (P-Q)^2(\Delta x)^2 = 4PQ(\Delta x)^2 > 1 . \tag{61}$$

‡ This analysis shows why the random walk derivation of the Lorentz transformation, as presented by Stein, works [30].

Argument:

Now, since the variance is given by

$$1-(P-Q)^2 = 4PQ \ , \tag{62}$$

and with

$$P = \tfrac{1}{2}\,(1+\beta) \ , \tag{63}$$

$$Q = \tfrac{1}{2}\,(1-\beta) \ , \tag{64}$$

the probabilities of I and D, respectively, for N generations, we obtain

$$\sigma = (NPQ)^{1/2} = \left[\frac{N}{4(1-\beta^2)}\right]^{1/2} \ , \tag{65}$$

so that

$$\sigma L = L\left[\frac{N}{4(1-\beta^2)}\right]^{1/2} = \left(\frac{L\gamma}{2}\right)\ N^{1/2} \ , \tag{66}$$

where L represents the discrete increment for the variable and

$$(P-Q)L = \beta L \ . \tag{67}$$

Thus, we arrive at an interpretation of the coordinate transform between reference frames and between metric marks. Note that, because N is finite, the variance is finite, i.e., bounded. This provides normalization of the transform, as well as a "maximal velocity." We have simply applied a consistency requirement to all allowed (i.e., rational) velocity frame transformations, namely bounded scale invariance.

Furthermore, because σ is bounded from below by one generation, it follows that the minimum deviation is always 1 between metric marks. Fluctuations between metric marks are thus bounded above and below. Letting Δx represent the discrete increment in x, the bound from below gives the uncertainty in the region directly from the variance:♮

$$1-(P-Q)^2(\Delta x)^2 = 4PQ(\Delta x)^2 > 1 \ .$$

QED

4.1 Multiply Connected Attribute Spaces

We now show how a d–space can be multiply connected, and derive some consequences of this multiple-connection. Unlike other notions of nonlocality, a multiply-connected d–space has a sequence of maximal attribute velocities.

Theorem 43: In a multiple attribute d–space, the sequence of maximal attribute velocities V_i has at least one value which is a least upper bound $V_{\max}$ and at least one value which is a greatest lower bound $V_{\min}$.

♮ As we will see in the physical interpretation, this fact implies that we do not require the concept of the wave function. Our "collapse" is nothing more than the attainment of more information about the specific ordering operator involved in the "evolution" of the discrete system. The uncertainty is nothing more than a quantification of the amount of detail expressible, given the selected basis having rational fraction values; i.e., as "meter marks."

Argument:

Trivially, from the definition of maximum attribute velocity, indistinguishable, attribute state and Principle I.

QED

Theorem 44: A multiple attribute d–space has relationships between attribute distance functions satisfying Eqs. (13), (14), (15) and (16), which display nonlocal correlations (i.e., require more generations than allowed by the ordering operator for v_{max}) and indeterminate relation (i.e., cannot be expressed as a function of N and the attribute states alone) to at least one of the attributes.

Argument:

Consider a discrete d–space U of cardinality N, with attributes E and P such that the number of attribute states of E is much greater than the number of attribute states of P. Further, consider d–subspaces L, R and S of U.

For a particular attribute A, we will represent the attribute distance from one d–subspace X to another d–subspace Y by $d(A : XY)$. By $V(A)$, we will mean the maximum of an attribute velocity $v(A)$ in the attribute A. By $C(X : EP)$, we will mean the minimum computational power necessary to represent the relationship between the attributes E and P in the d–subspace (or d–space) X.

Let the combined cardinalities of L, R and S be represented by M, and suppose that the number of attribute states of E in U is greater than $M + \log_2 M$ (Theorem 13); then there exist sequences of attribute states of E algorithmically producible within U, which cannot be differentiated from randomly distributed sequences of attribute states from within L, R or S, or any combination of L, R and S. Now consider the further relationship between E and P within U. Suppose that E is related to P via a function F which, by virtue of the fact that the number of attribute states of E is much greater than P, is a many-to-one d–map. It follows that the relationship F cannot be known within L, R or S, even when well-defined on U. Clearly, such a system is capable of exhibiting local "random" behavior.

Furthermore, it is clear that there must exist correlations (or anticorrelations) of P in $L + S$ and P in $R + S$, since this relationship is completely determined by F and incompletely expressible to either $L + S$ or $R + S$.

L, R and/or S are not large enough to discern the algorithmic relationship between P and E. By hypothesis, the maximum attribute velocity of P, $V(P)$, is greater than the maximum attribute velocity of E, $V(E)$. It follows that the correlation of P between L and R in U is limited by the velocity $V(P)$ rather than $V(E)$, and is thus nonlocal. Within the context of describing the system via the attribute E with maximum attribute velocity $V(E)$, these correlations appear instantaneous, based upon measurements of $d(E : SR)$ and $d(E : SL)$.

QED

Theorem 45: The attribute having the infimum of d–set of maximal attribute velocities for a maximal attribute velocity also has the smallest of the corresponding minimal attribute distance increments.

Theorem 46: The attribute having the infimum of d–set of maximal attribute velocities for a maximal attribute velocity corresponds to the attribute having the largest number of possible attribute states.

Theorem 47: The maximal range of attribute velocities over which relationships may be specified between arbitrarily selected attributes defined on some d–space is bounded from below by 0 and from above by the infimum of the d–set of maximal attribute velocities $V_{\min}$.

Argument:

Trivially, from the fact, if the zero attribute velocities are identified equal, then d–maps between attribute velocities can only be 1–1 over the interval $[0, V_{\min}]$.

QED

4.2 A Combinatoric Construction of Commutation Relations

The commutation relations as normally understood in quantum mechanics actually involve two quite distinct principles. The first is the principle that noncoordinate bases do not commute. Given a coordinate system x^i, one can adopt the derivate operator d/dx^i as a basis for the vector field. However, any linearly independent set of vector fields can serve as a basis, and one can easily show that not all of them are derivable from coordinate systems. This is because the operators d/dx^i and d/dx^j commute for all i, j, while two arbitrary vector fields do not commute.

The Exponentiation of the Derivate Operator d/dp

Theorem 48: The transport $p_0 + \epsilon$ along $x^i(p)$ may be given as

$$e(N)^{\epsilon d/dp} x^i \Big|_{p_0} . \tag{68}$$

Argument:

Let $D = d/dp$ evaluated at some point p_0 on a particular coordinate parameterization. Suppose the coordinate values $x^i(p)$ of points along the integral "curves" of a "vector field" d/dp are discrete functions of p; then the coordinates of two points with parameters p_0 and $p_0 + \epsilon$ are related by Eq. (52):

$$x^i(p_0 + \epsilon) = x^i(p_0) + \epsilon \left(\frac{dx^i}{dp} \right) \Big|_{p_0} + \left(\frac{1}{K!} \right) \epsilon^K \left(\frac{d^K x^i}{dp^K} \right) \Big|_{p_0}$$

$$= e(N)^{\epsilon \left(\frac{d}{dp}\right)} x^i \Big|_{p_0}$$

where $e(N)$ is just the power series expansion of e truncated at the N^{th} term by the definition of $e(N)$.
QED

Discrete Geometric Interpretation of Generalized Commutation

We will use the shorthand notation for Eq. (52) developed in the previous theorem in the derivation of the discrete commutation relations which follows*

Theorem 49: The order dependence $x(B) - x(A)$ of the derivate operators d/dp, d/dq is given by

$$x(B) - X(A) = \left[\frac{d}{dp}, \frac{d}{dq} \right] + \bigcirc \left(\epsilon^3 \frac{d^2}{dp^2} \frac{d^2}{dq^2} \right) . \tag{69}$$

Argument:

Notice that by definition of a coordinate basis (orthonormality), x^1 is constant along the lines of x^2, which are the integral curves of the derivate operator d/dx^2. That is why the derivate operators d/dx^1 and d/dx^2 commute: each is a derivate along a line on which the other is fixed.

Consider a basis d/dp combinatorially produced by Bernoulli trials vis-a-vis an ordering operator. Consider a second basis d/dq similarly, but independently produced. Now consider a transformation from one basis to the other; i.e., we seek a transformation which takes us a distance ϵ from a point P to a point R in x^i, using λ for transport; see Figure 14.

The two arbitrary vector fields V and W are defined by $V = d/dp$ and $W = d/dq$. Even the fact that the parameterizations look like that of a coordinate system is an artifact of 2–space; in 3–space it may happen that curve 2 intersects curves a and b, but that curve 1 only intersects curve a; see Figure 15.

* Adapted from B. Schuts. [31].

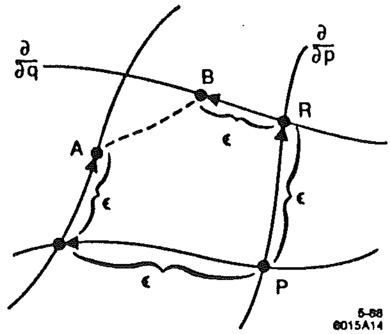

Fig. 14. Transport from a point P to a point R, using d.

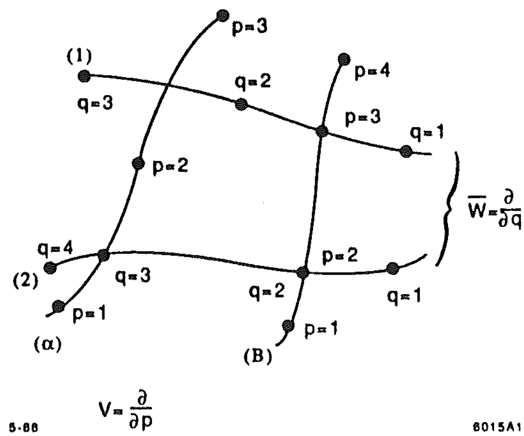

Fig. 15. Relation between parameterization and transport (see text).

We obtain a picture of the vector $[V,W]$ in the following manner. Consider a starting point P, moving $dp = e$ along the V curve through P, and then moving $dq = e$ along the W curve. One winds up at A. Starting again at P and going first along the W curve, and then along the V curve, takes one to B. The vector stretching from A to B is $\epsilon^2[V,W]$, to lowest order in ϵ; see Figure 14.

The transport along x from P to R in discrete step lengths is just:

$$x(R) = e(N)^{[\epsilon d/dp]}x \quad \text{at} \quad P\,. \tag{70}$$

Now assume that we have similar relationships for d/dq. For a point A in x, ϵ distance away from P along d/dp and ϵ distance further along d/dq, the transformation is just the product of the two operators (i.e., transform along d/dp, then along d/dq).

$$x(A) = e(N)^{[\epsilon d/dp]} \times e(N)^{[\epsilon d/dq]}\, x \quad \text{at} \quad P\,. \tag{71}$$

Similarly, we may travel from P to a point B, which is located by just changing the order of the transforms. We then obtain

$$x(B) = e(N)^{|\epsilon d/dq|} \times e(N)^{|\epsilon d/dp|}\, x \quad \text{at} \quad P \,. \tag{72}$$

Now find the distance from B to A:

$$\begin{aligned} x(B) - x(A) = |e(N)^{|\epsilon d/dp|} &\times e(N)^{|\epsilon d/dq|} \\ &- e(N)^{|\epsilon d/dq|} \times e(N)^{|\epsilon d/dp|}\, x \quad \text{at} \quad P \,. \end{aligned} \tag{73}$$

Now we undo our shorthand notation for Eq. (52), in order to multiply out the terms explicitly, and explicitly ignore higher-ordered terms which result. Expanding, we have the right-hand side of Eq. (73) as:

$$\left[1 + \frac{\epsilon d}{dp} + \tfrac{1}{2}\,\frac{\epsilon^2 d^2}{dp^2} + \mathrm{O}(\epsilon^3) \quad , \qquad 1 + \frac{\epsilon d}{dq} + \tfrac{1}{2}\,\frac{\epsilon^2 d^2}{dq^2} + \mathrm{O}(\epsilon^3)\right] . \tag{74}$$

This is just

$$= \epsilon^2 \left[\frac{d}{dq}, \frac{d}{dq}\right] + O(\epsilon) \,. \tag{75}$$

Thus, for two discrete operators ("vector fields") d/dp, d/dq which are not part of the coordinate d-basis x, the commutator is just the open part of an incomplete parallelogram, whose other sides are equal parameter increments along the integral curves of the vector fields. Note that the parallelogram is complete if and only if d/dp, d/dq are one to one with the coordinate d-basis; see Figure 16.

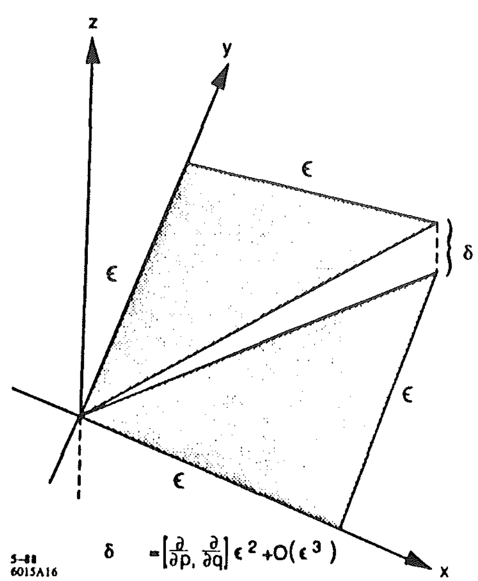

Fig. 16. Incomplete closure for parameters that are not part of a coordinate d-basis.

QED

It is important to understand how the operators which generate discrete distance functions might not be a part of the coordinate d–basis. Earlier, we noted that two ensembles A and B with increment and decrement I, D and I', D', respectively, were said to be independent if and only if all the defining states for A and B were distinguishable.

Theorem 50: For any two bases P and Q, the commutator of P and Q vanishes if and only if P and Q are independent; i.e., if and only if P and Q are coordinate bases.

Suppose that not all the defining states for A and B are distinguishable; then for some generation of the ordering operator, a redundant attribute state (instance) is generated. As a result, the additive law for attribute distance must fail; i.e., the sum of the total sizes for A and B does not equal $S' + S$. The sign of the deviation depends upon whether the deviation from $S' + S$ is accounted for by a deviation from $D + D'$ or by a deviation from $I + I'$ in the summation. Although the deviation can be treated as an attribute distance in its own right (indeed the inverse function of the additive law encourages us to do this), the ordering operator required to generate this deviation is clearly not independent of the generation of the two ensembles (consisting of a mixture of distinguishable and indistinguishable states), and is absolutely independent of the representation of both ensembles as being strictly independent (i.e., incorporating only distinguishable states); thus, it may be counted as a basis which behaves locally as an independent dimension.

If any distinguishable states are shared between the two coordinate parameters (i.e., one parameter is a function of the other), the product of the transports becomes order dependent: the computation of attribute distance for the first basis transport consumes the state and, thus, alters the ratio of distinguishable to total states for the second basis transport. Since the derivates for the basis are not in general the same, this results in a nonvanishing commutator. On the other hand, if the bases are independent, the commutator will clearly vanish.

Theorem 51: The commutator is bounded above and below.

Argument:

In a finite system, the commutator can clearly be no larger than the absolute maximum attribute distance representable in the dependent basis, where we assume that a dependent basis provides less information than the independent basis. Hence, the commutator is bounded. If the dependent basis has cyclicity ξ with respect to the independent basis, mapping each successive ξ distinguishable attributes of the independent basis to the same ξ attributes of the dependent basis, then the commutator is bounded by ξL (and in fact is equal to ξL), where L is the "conversion length" between bases. Based upon arguments previously given regarding dimensionality, it is clear that fluctuations of the commutator less than ξL are not consistently representable within the n–space (i.e., they occur between meter marks).

QED

Theorem 52: If $P = P(Q)$ is a first order derivate, then Eq. (75) is exact without higher-ordered terms.

Argument:

Since higher-ordered terms in Eq. (75) depend on higher-order derivates not vanishing, the theorem follows immediately.

QED

Theorem 53: For bases P and Q, if P is cyclic in Q (an angle variable), then

$$[P, Q] = \pm \frac{i\,\mathrm{Constant}}{2\pi(N)}\,, \tag{76}$$

where $\pi(N)$ is just the discrete computation of π by the combinatoric method in a d–space of cardinality N, as given above.

Argument:

If the indistinguishable attribute states involved combine to behave as distinguishable attributes in the proper manner, this independent dimension will behave mathematically just as though it were imaginary. Suppose, as in Theorem 52, that one of the two bases P is a function of the other:

$$P = P(Q) \ . \tag{77}$$

Furthermore, suppose that $P(Q)$ describes either a closed "orbit" or a periodic function of Q. If one of the bases is cyclic, its "conjugate" basis is constant. The corresponding orbit in the QP discrete 2–space is then just a "horizontal straight line." Following Goldstein [32], the "motion" may then be considered as the limiting case of a rotation type of periodicity, in which Q may be assigned an arbitrarily long period (subject to N, of course). This is just a change of coordinates from the real coordinate P to an imaginary coordinate J in a complex discrete 2–space, following the usual practice of using complex plane to represent such a change of coordinates; see Figure 17.

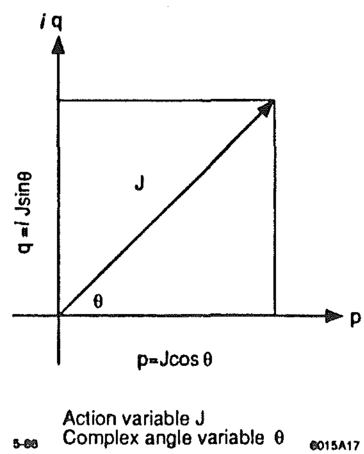

Fig. 17. Relation between p, q and angle-action variables.

Since the coordinate in a rotation periodicity is invariably an angle, such a cyclic Q always has a natural period of $2\pi(N)$. Accordingly, the length of the path in QP discrete 2–space evaluated from 0 to $2\pi(N)$ is just $2\pi(N)$ and QP becomes:

$$J = 2\pi(N) \times i \times p \ , \tag{78}$$

for all cyclic variables. Note that we evaluate π for cardinality N here. That is, we construct the combinatoric valuation of π on the global d–space of cardinality N, and not on the local d–subspace; then we map minimum Q to 0 and maximum Q to $2\pi(N)$. The value of Q measured as an angle is then discretized in increments of $2\pi/N$ from 0 to 2π by the mapping.

Given Eqs. (74) and (76), we may now express the commutation relation between J and Q:

$$[J, Q] = [2\pi(N)iP, Q] = \text{Constant} \ ,$$

or

$$[P, Q] = +i(\text{Constant})/2\pi(N) \ .$$

QED

From our earlier result, however, the general commutation to first order is just

$$[P, Q] = \frac{L}{\epsilon^2} \,. \tag{79}$$

If P and Q are linearly related, then the higher-ordered derivates vanish and Eq. (79) is exact. If we then take ϵ to be the minimum nonvanishing discrete value, with suitable reparameterization, we have

$$[P, Q] = L \,, \tag{80}$$

for the least increment in the complex angle variable.

5. A DISCRETE CONSTRUCTIVE MODELING METHODOLOGY

5.1 DEFINITIONS

Having developed the elements of a discrete, finite and computational formalism via the ordering operator calculus, we proceed to a mathematical foundation for a discrete and constructive modeling methodology. Such a methodology will allow us to use the ordering operator calculus to model various phenomena which do not have the intrinsic properties required by continuum mathematics.

We motivate the modeling methodology through a variation of a dictum issued by Bastin and Kilmister in 1973 [33] concerning the separability of syntax and semantics in a mathematical system, which we refer to here as the Separability Lemma.

> Separability Lemma:
> *A system has a mathematical structure (syntax) which can be expounded separately from the interpretation of it (semantics), provided that it is understood that the mathematics describes a process which can be represented as a computer program.*

Clearly, the ordering operator calculus meets the criteria of the Separability Lemma as demanded by Principles I–V. We are now ready to define a modeling methodology which consists of three broadly-defined structures: an epistemological framework, a representational framework and a procedural framework.

An **epistemological framework** or **E–frame** is a d–set of loosely-defined agreements made explicit by those engaging in the process of modeling (i.e., by injecting information into the model formulation).

1) *Agreement Upon Intent*

   The intent of the modeling effort must be agreed upon. The practice being modeled must be identified. It is also desirable to establish agreement regarding the conditions under which the effort will have been determined to fail, means of validation, the degree of accuracy required of the model (a stop rule) and rules for evaluation.

2) *Agreement On Observations*

   The ensemble of objects O, which constitutes the observations of and about the practice must be agreed upon.

3) *Agreement of Cooperative Communications*

   - commonly defined terms as fundamental

     Fundamental terms, as used in describing the practice, must be understood. They CANNOT be defined.

   - fundamental versus derived terms

     An operational distinction between fundamental and derived terms must be practiced.

   - agreement of pertinence

     Engaging in attempts to communicate about the practice being modeled must be founded on an agreement to assume and attempt pertinence.

4) *Agreement of Explicit Assumptions*

There must be an agreement to make assumptions explicit, rather than allowing them to be implicit.

5) *The Razor*

- agreement of minimal generality

  The "scope" of the modeling effort at any point in the evolution of the model should be constrained to manageable proportions.

- agreement of elegance

  The model should display a consistent and transparent structure, which minimizes the statement (size) of the model, while maximizing its explanatory (and in the event of a theory, its predictive) power.

- agreement of parsimony

  The model should contain as little as possible that is either (a) sufficient but not necessary, or (b) necessary but not sufficient in modeling the intended practice.

A **representational framework** or **R–frame** is an abstract formalism FS, consisting of a set of symbols F and a set of rules of manipulation I. It is an uninterpreted typography.

A **procedural framework** or **P–frame** is an algorithm which serves to establish rules of correspondence C between the observations O (as agreed upon in the E–frame) and the symbols of the R–frame F, and which then, through recursion, serves to modify the rules of correspondence and the E–frame and R–frame, until a sufficient level of agreement concerning accuracy is achieved or the model fails. Kuhn [34] would call such a failure a "crisis," which in the fullness of time will lead to a "paradigm shift."

Thus, we see a relationship between two d–sets being established (the O and F), with two d–sets of rules (I and C) for modification and/or information extraction.

We now cast this in terms of the ordering operator calculus and, specifically, of the finite differential geometry which we have constructed within it.

An **ob set** O is an ensemble of observations. The obs are differentiated (altered from a d–sort to a d–set) by one or more ordering operators, which serve to establish the lattice structure of the obs.

An **ob subset** is a d–set of obs, defined by at least one ordering operator. They may be multiordered and multiply-connected.

A **d–sort of formal symbols** F is an ensemble of labels which may be ordered (converted to a d–set) by a d–set of rules of manipulation I. The resulting d–set FS of formal symbols F with rules of manipulation I is called a **formalism** or **representational framework**, and may be either closed or open under the rules. Generally, this serves to form a abstract combinatorial system.

A **P–frame rule of correspondence** is a binary d–map between an element of F and an element of O. In practice the d–set of all rules of correspondence established up to some step in the modeling procedure are expressed as a *dictionary:* given an element of O one may look up a procedure for finding the corresponding element of F.

A **procedure** P is a bounded, recursive algorithm which (a) provides a recursive and exhaustive enumeration of the elements of O and the elements of F, such that there exists a smooth d–map between O and F in the sense given above, constructed from the d–set of rules of correspondence and which (b) provides a recursive reparameterization of the d–map, such that there exists a 1–1 d–map between a d–subset of O and a d–subset of F.

Ideally, the cardinality of these d–subsets increments with each recursion of the P–frame procedure, up to the cardinality of O itself.

5.2 OBSERVATION SPACE

We begin with a number of observations which may be clustered* (grouped into prearranged classes) into d–sets O_i. These observation d–sets are said to cover the observation d–space O in the sense that $\bigcup_i O_i = O$. Because our O must have boundaries—for any hypothetical O_u (O is a d–subset of O_u)—and is discrete, O is non-Hausdorf.

Clearly, for any finite O, there are a finite number of possible disjoint partitions of O; namely, $\sum_k k!/n!(k-n)!$, where n is the cardinality of O, and k ranges from 0 to n. However, the partitions need not be disjoint—we allow dependent observations, and any ob to be in more than one partition. Thus, the number of partitions may be as large as we wish, being determined by the bound we place on the combinatorics of repetitive sampling with replacement.

It is often convenient, in the absence of any constraints, to take a discrete version of R^N as the image space, thus allowing an analytic interpolation for functions defined on the space. We map each partition O_i of O to some subspace S_i of R^N by some d–map R_i. If each such subspace S_i of R^N is arbitrarily "labeled" with some formal symbol F_i, then the partitions O_i of O may be taken as "objects" in O and referred to by the F_i. The R_i then form rules of correspondence.

We define relationships between the O_i objects in terms of the coordinate transformations between the S_i.

Note that our definitions tell us immediately that there is no *a priori* parameterization on S which gives a preferred reference frame. In fact, there is no structure at all on S without a parameterization. There exists no metric, only local topology induced on O by F via R. The global topology is given by the cardinalities of O and F and by the partitioning of O, as well as by coordinate transformations between partitions induced by requiring that the formal rules of manipulation I_i map isomorphically to O, via the rules of correspondence R_i, giving connectivity to the topology. The image in O under R of I may leave invariant certain attributes of O, the study of which provide an understanding of the structure of the formal model of O.

5.3 THE MODELING METHODOLOGY ALGORITHM, MODELS AND THEORIES

We now give a specific P–frame algorithm, which meets the criteria established in the preceding section, and which establishes and guides the evolution of the model.

1. Choose the ob set O with n elements. This is a recursively enumerable d–set with cardinality n.
2. Partition the ob set O.
   a. Define the n obs (labels) by partitioning the d–set O into disjoint d–subsets.
   b. Choose a set of symbols O_i for these partitions, labeling them.
3. Select or develop an abstract formalism FS meeting the criteria of an R–frame.
4. Choose a set of rules of correspondence R between the symbols O_i of O and the formal symbols F.
5. Map to some space such as R^N. (We can always choose our discrete version of R^N locally for the d–map, although we must then define the obs on open d–sets.)
6. Determine relationship between obs vis-a-vis the formalism. In particular, determine the image of the I in O under R.
7. Establish a set of coordinate transformations and determine the induced structural invariances, in order to identify the interpreted global properties of the model.
8. We say that this procedure establishes a model, if the cardinality of the O_i is the same as that of F and if R is an isomorphism between O and FS. If the isomorphism fails, we call the result a **theory**, in that it has predictive power. In empirical practice, we will rarely obtain a formal model.

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* We will discuss methods of clustering compatible with the modeling methodology and the ordering operator calculus in a later paper. Note that if a distance function or a norm is definable on the O, the method of minimal distances may be used as a clustering algorithm for the partitioning. Methods based upon a general attribute distance function are closely connected to a general theory of computational measurement, in development.

9. If a model is not established because the isomorphism fails, then recursive application of the P–frame procedure is required to evolve the model. While no deterministic algorithm may be given which prescribes how the model should be altered, given a certain failure of the isomorphism, the P–frame procedure P allows one to develop heuristic knowledge about the modeling practice and how best to proceed in modifying the model. This heuristic knowledge may be made explicit within the E–frame from the outset and, indeed, becomes a part of the E–frame via P–frame recursion.†

Keep in mind that through P–frame recursion, one has many options: we may alter the partitions of O, the range of the maps R, the coordinate parameterizations on O, the d–maps R, the rules I, and so on. Each recursion of the procedure P modifies one and only one such aspect of the model; in so doing, the entire model must be reexamined for consistency and completeness of the representation, as each change alters the definition of one or more ordering operators. These modifications are necessarily inductive, and therefore have unpredictable consequences.

The revolutionary step is taken based on an inductive decision that a Kuhnian crisis [35] has developed. This is largely based upon *subjective* criteria concerning the viability of the model and, in some sense, an *intuitive* measure of the relative benefits of proceeding, starting over or opting for a radical revision. It is important to note that such criteria can be agreed upon as part of the E–frame; namely, agreeing in advance how much and what kind of deviation from the required isomorphism will be tolerated and how the validity of the modeling effort will be judged.

We halt the classical infinite regression of analysis of terms in modeling by recognizing the effect of the epistemological framework. We deny the validity and the value of any attempt to analyze "theory-laden" [36] language as used in the E–frame. Such an analysis lies outside the purported task of generating a specific model, and would require us to generate a model containing the specific model, as an instance. In particular, analysis of fundamental terms involves treating these terms as the ob set for a modeling effort. In keeping with the agreed upon intent of the modeling practice and our methodology, we can not engage in such analysis. The practice would necessarily involve nonconstructive methods: the analyst would have to work from the specific model by generalization, having failed to construct the general model first. The transition from the specific to the general is not only inductive in nature, but not recursively definable, and constitutes a revolutionary redefinition of the modeling effort as specified in the agreement of intent.

Note the implication here that it is possible to work from the general to the specific. It is possible to constructively "model the model" or even the modeling process. Indeed, part of the power of our modeling methodology lies in the constructive and recursive nature of the process.

In practice, we always bootstrap into the modeling process with a set of loose agreements and definitions (we don't really know what we are talking about), but the ordering operator calculus gives us a consistent mechanics of typography and the procedural framework gives us a recursive method of evolving toward an acceptable model and definitions. Once the process has begun, each pass through the P–frame may generate a modified, but nonetheless well-founded and well-defined, E–frame and R–frame. Constructively, we may keep records of our efforts and review these at will. On starting the effort, we have no record of earlier effort and no way of (re)constructing one; we may make no constructive claims regarding either the earlier effort or the results of that earlier effort. In some sense, we, thus, have a "fixed past and uncertain future," but with a fixed starting point.

5.4 Hierarchical Models

We will frequently have cause to deal with hierarchical structures. For this reason, we give a P–frame algorithm for constructing hierarchical models as a constructive definition.

1. Start with a model.
2. Specify a many-to-one d–map from the formalism F to ob labels O_i.
3. Redefine the partitioning via the process of refinement, mapping from the image in O to the representation d–set F with new mutually disjoint partitions, using inverse d–map of the R_i. This insures consistency for next step.

† We call the process of exercising P *recursion*, rather than *iteration*, because it operates on itself, as well as the model. In some sense, P, together with the modeling participant operating on the model, constitute a self-organising system.

4. Remap the formalism from new partitions induced in F under the inverse of the R_i to the image space O, using old mapping R.
5. Keep in mind the constraints of a many-to-one d-map. This d-map provides inclusion relations on the d-set F; thus partitions contain partitions or parts thereof, forming a lattice of partitions.

Theorem 54: For each model with multiple partitions mapped to a representational framework without disjoint refinement, there exists a hierarchical model with an equivalent local topology.

6. AN INTERPRETATION: LABORATORY PHYSICS

6.1 ESTABLISHING THE E-FRAME

We start on the route to physical interpretation by adopting the constructive modeling methodology developed in the previous chapter. We must, therefore, state explicitly the E-frame, the R- frame and the P-frame. Within the E-frame, we adopt as our agreed upon intent the modeling of the current practice of physics. We take as fundamental the commonly defined terms of laboratory physics, treating terms denoting nonobservables as derived or theoretical terms. Our understanding of the current practice of laboratory physics is guided by the "counter paradigm" [37].

Any elementary laboratory event, *under circumstances which it is the task of the experimental physicist to investigate, can lead to the firing of a counter.*

In this context, by "can lead to the firing of a counter," we implicitly allow for any measurement apparatus which involves discrete and finite measures, i.e., counting. Inasmuch as all laboratory measurements are normally viewed as bound by limitations of precision and resources—which bounds for us are evidence of the intrinsic finite and discrete character of the practice—few, if any, laboratory measurements are excluded by the counter paradigm; one must make the connection to counting explicit. We take laboratory events as a sufficient set of observations to be modeled, without requiring the standard theoretical interpretation. We take as understood that an experimental (laboratory) measurement may encompass many acts of observation and, thus, that our obs may be complex (e.g., multiply-connected). In other words, we are not committed to accept the how and why of the observations, only the observations themselves, operationally understood.* If the internal structure of an act of measurement is to be examined, then there must exist a finite procedure for carrying out the measurement (i.e., the measurement must be operational), so that the internal structure is transparent. Otherwise, we are required by Principle I to plead ignorance of the apparent internal structure.

We have now satisfied the requirements of establishing an E-frame, inasmuch as the requirements have to do with making explicit various aspects of the modeling effort. As to whether or not we are faithful to the other strictures of the E-frame, we shall leave it to the reader to decide, this being the very nature of consensual validation of the value of our effort.

6.2 ESTABLISHING THE R-FRAME

As our R-frame formalism, we adopt the ordering operator calculus. Inasmuch as quantum events, as understood within the current practice of physics, are unique, discrete, irreversible, nonlocal and yet indivisible, the principles upon which development of the ordering operator calculus was based make this an appropriate formalism.

6.3 ESTABLISHING THE P-FRAME

As our P-frame procedure, we select the algorithm given in the preceding chapter. We note in advance that some detailed aspects of the model are evolving. In particular, we are in the process of refining the specification of the d-space generator required by our formalism. This will have consequences regarding the detailed specification of any distance function on any attribute we identify. In addition, any global invariants are likely to be affected. Thus, the detailed identification of physically conserved quantities within the theory is tentative, though their existence is not.

* Note the distinction between E-terms and R-terms. Von Neumanns "observation" is, at best, only an R-term. Criticism of von Neumann's representation of quantum mechanics can start there, because his R-term is not necessarily consistent with Schrodinger continuity.

As noted in Section 5.1, the rules of correspondence may now be elucidated in the form of a dictionary. If we establish rules of correspondence between obs from the E–frame and symbols in the R–frame, any relationship between the symbols in the R–frame must reflect relationships within the context of the E–frame, whether known at this time or not. We, therefore, adopt rules of correspondence which are more useful than current practice in relating observations to the R–frame, and then see how the practice of *discrete physics* will differ from the current wisdom. In other words, we hope to see how the E–frame (and perhaps the R–frame) should be modified. Bridgman tried long ago to get rid of the representational framework by "operational" rules of procedure that reflected directly back into the E–frame. We expect that it would be conceded by most physicists that this heroic effort failed in its initial intent, and even Bridgman was led to modify it by including "mathematical operations" within the allowed procedures. One related effort was to reduce everything in physics to "pointer readings." Our methodology is even stricter in that sense, since we require every E–frame procedure and every R–frame construct to be reducible, at least in principle, to *counting* and finitely computable algorithms. We hope to have accounted for the philosophical and technical problems which led to the failure of Bridgman's operationalism.

Spatial Distance

For us, an attribute distance is the only thing in the R–frame that can correspond to a datum (E–frame) achieved by an experimental measurement within the practice of physics (E–frame). From the R–frame, however, we see that attribute distance has no computational meaning or significance outside the context of a particular reference frame, or without some ordering parameter (R–frame symbols). We do not make an absolute rule of correspondence between attribute distance and spatial distance; spatial distance will be a particular attribute distance. For us, however, any quantifiable experimental measurement must correspond to some attribute distance.

Cosmological and Proper Time

As noted above, we take the notion of sequence and counting in the laboratory as fundamental, so that the very character of observation in time (E–frame) is bound to the R–frame notions of counting, synchronization and both local and global ordering. We establish a rule of correspondence between laboratory proper time (E–frame) and the ordering parameter t_i (R–frame), associated with the generation of any particular reference frame F_i, via an ordering operator O_i. Similarly, we must establish a rule of correspondence between cosmological time and the global ordering T, associated with the generation of all reference frames within the model. That the global ordering may be specified in terms of the R–frame synchronization of attributes identically to the E–frame synchronization of events, establishes a requirement that events be specifiable as some particular kind of attribute. A significant portion of this section will be devoted to establishing the required nature of event attributes.

Three Dimensional Physical Space

As seen in Theorems 43 and 44, for any attribute space, no matter how simple or complex, there is some attribute which has the greatest number of attribute states of all the attributes which may be defined on the d–space. From Theorem 46, it is also clear that the corresponding attribute velocity for this attribute will be the infimum of the d–set of maximal attribute velocities. Finally, from Theorem 44 and by definition, this maximum attribute velocity will be the first bound encountered in any function involving more than one attribute. For these reasons, we identify this unique attribute velocity with the (E–frame) speed of light c, and the corresponding attribute states with the points or "4–positions" of physical space. Note that these points are events in the sense of the geometric view of general relativity.

As demonstrated in Theorem 13, for any attribute distance function, there are at most three independent runs of the ordering operator which generates these attribute states, if the global character of the d–space so generated is that it not have a preferred coordinate. Thus, the d–dimensionality of the attribute space is three, and we establish a rule of correspondence with the three–dimensionality of laboratory space.

The Global Structure of d–Space Generator

The next rule of correspondence must specify an ordering operator U, which generates the coordinate d-bases and a reference frame (R–frame) suitable for identification with the spatio-temporal reference frame (E–frame). This ordering operator U must provide the appropriate global invariances, if the identification is to be successful.

The relevant E–frame global invariances include the fundamental constants, the scale constants and the quantum numbers. For these invariances to be generated via a discrete algorithm suggests a hierarchical structure with a stop rule. For further justification of these requirements, see Bastin, 1966 [38] and Bastin, 1956 [39]. We may interpret the generators of each level of such a hierarchy to be coupled ordering operators; then the coupling scale may be calculated by definition, together with probabilities of coupling between the levels, which must be the coupling constants of laboratory physics.

We allow multiple, independent, but synchronized, runs of the U in order to generate a discrete space, without a preferred axis, and preserving translational invariance (i.e., having a homogenous distance function). By Theorem 13, the dimensionality of this d–space will then be three; that is, we need only three independent runs of U or any other generator of the d–space, as additional runs will not produce additional global structure. The unobservable universal (cosmological) and locally consequential (proper) time will then* be given by the universal ordering parameter associated with U.

As noted previously, an ordering operator U may be understood as generating bit strings, instead of labels which we take as abstract representations of physical attribute states. Here, we invoke the principles requiring that any specified attributes of a finite and discrete ensemble can be mapped onto an ordered sequence of 1's and 0's, by asking whether they are present or absent in a reference ensemble. Such an ordered sequence is called a **bit string**, and may combine with other sequences of the same bit length by an operation such as XOR ("exclusive or"), symmetric difference, addition (mod 2), $+_2$ .... When Noyes treats the symbols "0," "1" as bits and/or as integers, the more general *discrimination* operation "$\oplus$" defined by

$$S^a \oplus S^b \equiv (ab)_n \equiv \left[\ldots,\ \left(b_i^a - b_i^b\right)^2,\ldots\right]_n$$

$$= (\ldots,\ b_i^a +_2 b_i^b,\ \ldots)_n;\ b_i^\ell \in 0,1;\ i \in 1,2,\ldots,n;\ \ell \in a,b,\ldots,$$

is used. Note that discrimination meets succinctly the requirements for combining serializable ordering operators, if the bit strings are linearly independent; i.e., there is no information loss regarding the distance function on discrimination, if the resultant bit string is given a dual Hamming measure—one counts the 0's instead of the 1's—and discrimination is then a length preserving operation. For us, this is a required property of U.

U is further required, by the definition of ordering operator, to consist of an incompletely specified (though, in principle, specifiable) part, and a completely specified part. The incompletely specified part must not have an effect on the global structure, nor on the combinatoric complexity of its generation. As long as the structure generated, and the order in which it is generated, is compatible with the knowable Universe (E–frame), the unspecified part can be any algorithm, whatsoever.

Non–Local, Discrete Events

What is now required is an R–frame definition of event. For us, this definition must follow the geometrodynamic point of view, in that the existence of an event depends upon an operation defined on strings (or similar representation of attribute states) and a distance function defined on these strings, which satisfies the so-called "triangle inequality." That is, the distance function must be a norm (see Section 2.4). Note that the definition of a norm requires a minimum of three independent strings. We establish a rule of correspondence which identifies the satisfaction of these conditions on the points of our physical space representation with the unique, nonlocal, yet indivisible and irreversible, events of quantum mechanics, since they meet the minimal conditions for nonlocalized operations on localized d–points which have a norm.

* This result was anticipated conceptually by E. W. Bastin [40].

Defining $k_i^x(n) = \Sigma_{i=1}^n b_i^x$, $x \in a,b,c$, with n being the number of generations of the ordering operator, from the definition of a norm (Section 2.4), it is easy to see, for any three strings $(a)_n$ $(b)_n$ $(c)_n$ which satisfy the constraint $(abc)_n = (0)_n$ where $b_i^0 = 0; \in 1, \ldots, n$, that $|k^a - k^b| \leq k^c \leq k^a + k^b$ (cyclic on a,b,c) for any event. Thus, k, the number of "1" s in a string, can serve as a discrete distance function; in fact, this is just the Hamming distance. Note that the our definition of events necessarily will make them nonlocal. That is, a minimum of three independent and distance function ordered bit strings is required, although some attribute distance exists between them.

In order to locate the required reference frame "origin" (which in the R–frame corresponds to a reference ensemble) of our metric symmetrically in the finite and discrete interval allowed, we define an attribute distance $q_a \equiv f(k,n,\lambda_a)$—a linear function of k,n,λ_a, where λ_a has the dimensions of attribute distance and is identified via a rule of correspondence with a physical length. At each generation of the ordering operator, q_a changes by $\pm\lambda_a$, which we associate via a rule of correspondence with the minimum attribute distance increment of the R–frame, with the sign + or − being determined by whether a "1" or a "0" is concatenated with the extant string; i.e., whether the distance is increasing or decreasing, with respect to the reference ensemble. Note that if perfect synchronization is possible, λ_a is just $1/n$. This factor serves to normalize the distance on the [-1, 1] interval.

If we define the local event time (proper time) as a linear function of the ordering parameter $t = n\Delta t$, we see that we can define a velocity $v_a \equiv f(k,n,\lambda_a)V_x = \beta_a V_x$ where $V_x = \lambda_a/\Delta t$ is a *maximal velocity* of magnitude identified with the speed of light c, achieved when all the steps have the same sign (i.e., are in the same direction) and $f(k,n,\lambda_a)$ is a linear transformation of the Hamming distance k. We also have an *event horizon* that grows with the number of steps the generating operator has taken.

Lorentz Invariance

It is clear that q satisfies the definition of an attribute distance and satisfies β, as required in Theorems 23 and 41. We formally establish a rule of correspondence between that β and the usual β of special relativity. The specific dependence of λ on the generation of attribute states in the sequence given by the ordering operator is unknown, and, for our purposes, not required, as long as sufficient variety is produced. From Chapter 3 (and independent of the particular generator of the d–space), we have immediately a 3+1† discrete—and locally flat—space with distance function, which is invariant with respect to the coordinate transformations of Theorems 23 and 41 and with the previously stated rule of correspondence that the maximal attribute velocity for this "position" attribute corresponds to the velocity of light c; i.e., to the minimum of the maximal attribute velocities. We now identify the coordinate transformations of Theorems 23 and 41, *when applied to the position attribute*, as the Lorentz Transformations.

That the definition of velocity is indeed a first derivate of the position q is obvious. If q is linear in t, then we have $(q/n) \times (\lambda/\Delta t)$, where $\lambda/\Delta t$ is just the "slope." If q is not linear in t, then λ is a function of t, so that we obtain $(q/n) \times (\Delta\lambda(t)/\Delta t)$, which (evaluated at some q and t) give the "instantaneous" velocities. Furthermore, not only these velocities, but any attribute velocities, thus satisfy Theorem 35, which is now identified as the relativistic composition law for velocities.

Persistence Effects and de Broglie Wavelengths

By evolution of a system, we mean that some attribute states are invariant under some transformations on the system, and nothing more. When such attribute states are jointly identified and are invariant together, we say that they constitute an "object" which persists or is stable. We now note that if we consider a system that evolves with constant velocity—i.e., by a linear d–map, $\beta_0 \equiv f(k_0, n_0, \lambda_0)$—strings which grow subject to this constraint—i.e., $n = n_T n_0$, $k = n_T k_0$, $1 \leq n_T \leq n/n_0$—will have a periodicity $T \equiv n_T \Delta t = n_T \lambda / V_x$, specifying the events in which this condition can be met. Hence, in more complicated situations, where there can be more than one "path" connecting strings with the same velocity to a single event, this event can occur only when the paths differ by an integral number

† Our use of 3+1, here, is meant only to emphasise the evolution of the ordering operator which locally distinguishes the ordering parameter, and not to deny the validity of the 4–space geometric view which is globally valid after the generation has taken place.

of attribute distance increments. We, therefore, establish a rule of correspondence between λ and the "de Broglie wavelengths." Thus, our construction already contains the seeds of "interference" and an explanation of the "double slit experiment."

The Relativistic Doppler Shift

From Theorem 33, and independent of the particular d–space generator, we obtain the relativistic doppler shift, as required from the laboratory evidence.

Supraluminal Correlations

Because the derivations in the development of the ordering operator calculus do not depend upon any particular interpretation, particularly those which could be read as referring to "physical distance," it is clear that the principles and axioms suffice to imply relativistic and quantum effects which could be identified with physical characteristics other than distance.

On the face of it, this is a surprising conclusion. However, for us, it demands that we treat the universe as a multiply-connected attribute space. If it is not the case that nonspatial attribute distances behave as does the spatial attribute distance, then either conventional or discrete theory must supply some reason for this difference. To our knowledge, making such a distinction has yet to be motivated in current analyses. Clearly, not all the attributes which may be generated in a discrete space will satisfy the precise definition given for q. Therefore, regardless of the generator of the d–space, we must conclude that the d–space is multiply-connected, with the consequences derived in Theorems 43–46. We show in this section that the theory encourages us to accept as "obviously possible" the disturbing facts demonstrated by the laboratory experiments of Clauser, Frye, Aspect and others [41]. Indeed, the theory predicts that such results could be obtained for quantum attributes other than spin and polarization. These results are predicted in the following way.

Theorem 43 describes the essential character of Aspects EPR experiments, where E is electromagnetic and P is polarization, S represents the source, L the left detector and R the right detector systems. The time-of-flight experiment does not alter the model, since this only serves to verify the "instantaneous" character of the anticorrelations. The results of such experiments are readily understood in this context.

Note that supraluminal communication is not allowed, since the connection between E and P is not 1:1 and is, in fact, locally "random." Furthermore, the theory is not a hidden variable theory, nor is it a nonlocal theory in the usual sense in which these are understood. We do not provide hidden variable extensions to quantum mechanics or to special relativity in order to understand the correlations: we provide a theory which reduces to quantum mechanics or special relativity under certain restricted interpretations (e.g., the existence of the continuum). We do not postulate an absolute nonlocal quantum multiple-connectedness, as is implied, for example, by Bohm's implicate order. Neither is the multiple-connectedness like that proposed by the branching universe of Wheeler and DeWitt. Rather, we postulate a topology which admits multiple, usually independent, distance functions and metrics.

For Aspect's experiments in particular, the global relation between polarization angle and electromagnetic propagation must be identified as some cosine-squared function. This function must be independent of the electromagnetic attribute distance identified as q, but dependent upon the polarization attribute distance—i.e., the difference between the polarization angles—by hypothesis. Since the least increment for polarization angle is defined by the event horizon N (i.e., from a computation of $\pi(N)$ via the method given in Chapter 1), we may expect that the number of spatial attribute states is approximately the square of the number of polarization attribute states. This suggests that the correlations seen by Aspect will fall off as the time for propagation of changes in the optical switches approaches the square root of the propagation delay for light.

We are led by the formalism to predict that there is a correlated rate of change of the optical switch, which destroys the correlation between the arms; namely, $V(P)$. That is, when the time T between switching in one arm versus switching in the other arm is short compared to $d(P : LR)/V(P)$, the correlation should be destroyed by our analysis. An examination of the correlation with T would show stronger correlation as T approaches $d(E : LR)/V(E)$ from below. One might reasonably expect the distribution to be exponential. Unfortunately, T is likely to be extremely short for any practical distance $d(E : LR)$.

The global topology of the discrete finite attribute space is multiply-connected. There is a unique attribute which serves to define a global metric; in our case, conventional 3–space as provided by the electromagnetic attribute. Globally, our d–space is necessarily limited to 3–space. However, locally a nonisotropic n–space may be defined. That is, if we no longer require translational invariance, there is no preferential coordinate, or if synchronization is not required locally in transforming between reference frames, one may define more than three independent, short runs of the parameterized bases which will behave (locally) as coordinates.

This topology, together with the fact that events as defined have intrinsic quantum interference properties, leads one to suspect that superluminal correlations should display quantum interference; namely, the "measurement" in the right and left detectors constitute events in both E and P attribute space. Suppose that the events are arranged in such a way that they are separated in E–space, but not in P–space. Furthermore, suppose that in P–space the events have wavelengths such that interference can occur. This interference should then modulate the correlation in E–space. Such a "correlation interference pattern" would be striking evidence of the proposed topology, since this cannot occur in distant (in E) events in the conventional theories.

Computer Models

We may model our system with the required topology on a computer* In particular, the violation of Bell's Inequalities and related effects may be demonstrated in the computer model, since our formalism is strictly computable. Care must be taken in establishing the functional connection between E and P in the computer model, however. The connection must be sufficiently complex computationally to lead to the appearance of local (i.e., restricted memory) "random" behavior. This is just the problem of precision in computer modeling, used in reverse to establish certain statistical properties of the model. Indeed, it would appear that the model may be set up to demonstrate physical supraluminal correlations between physically separated computer systems in a distributed processing, shared memory environment.†

Mass and the Law of Relativistic Mass Change

We can associate a parameter m with the total size S [Eq. (44)] of the ensemble, and establish a rule of correspondence which identifies m (R–frame) with *mass equivalent* or *energy* (E–frame). Note

* As has been partially done for a particular 3–space generator [42].

† Related Work: The relationship between this model and cryptographic techniques is interesting as well. A recent paper by Goldreich [43] considers a constructive approach to random bit strings based on computational complexity which is similar, though more specific and restrictive than that introduced in the present paper. In particular, the authors introduce programs that run in polynomial time and which lead to identical results when fed with either a set S of strings or elements randomly selected from the set of all strings.

Such poly-random collections can be shown to enable many parties to share efficiently a random function f in a distributed environment, by which we mean that if f is evaluated at different times by different parties on the same argument x, the same value $f(x)$ will be obtained. Such sharing can be achieved by selecting k-bits to specify a function in a poly-random collection. These k-bits are then communicated to and stored by each party. No further messages need be exchanged between parties to share f. It is a trivial matter to make the sharing either correlated or anticorrelated if $f(x)$ is two-valued. The physical communication of the k-bits may be dispensed with in a multiply-connected attribute space, as the k-bits may be "local" through some particular attribute. Thus, the k-bits are always available in "local" shared memory.

These results are, of course, familiar in terms of so-called public key encryption systems. Here, a public key is distributed for encryption of messages to the key distributor. Although the encryption key is public, the cryptographic function does not allow decryption without access to the private key. And the number of possible private keys is too large to be determined by trial and error.

Actually, the entire scheme of shared random number generators has been put into effect. One can purchase a plastic card which contains a microprocessor. This processor produces an apparently random sequence of bit strings. When interrogated by a system which shares the random function, a match is produced and thus the card serves as a "key." Each card contains a k-bit code for the particular function and this serves to identify the particular user. Clearly, two cards with the same k-bit code would be perfectly correlated regardless of separation and yet would produce apparently random output.

that we differentiate between the mass and the energy. For a bit string in an evolving system to have an invariant mass at constant attribute velocity, the mass may be defined as the energy divided by some normalization factor, which depends on the cardinality of the attribute states which might be generated, and on the cardinality of the attribute Universe (R–frame). In this way, adding a distinguishable state (a '1') to the Universe and to the bit string do not alter the "mass" parameter in a measurable way, and results in a statistically invariant mass. For consistency with our finite principle, we must require $0 < k < 1$; thus, no massive event can lie on the event horizon.* Independent of the particular generator of the d–space, Theorem 34 is interpreted as showing that the definition of this parameter follows the law of relativistic mass change.

Momentum Conserving Events

We require the existence of a norm for an attribute which can be identified with momentum, and in this way obtain momentum conservation. Once we have shown that the attributes of position and momentum can be identified (or equivalently, position, velocity and an invariant mass), and a norm in each of these spaces defined for a configuration which we identify as a quantum mechanical event, the generator of the d–space can be any algorithm whatsoever.

Defining $p_a \equiv m_a v_a = m_a \beta_a V_z = \beta_a m_a \lambda_a / \Delta t$ and establishing the rule of correspondence which identifies this as momentum, we see that $|p_a - p_b| \leq p_c \leq p_a + p_b$, provided only (as is required for consistency) $m_a \lambda_a / \Delta t$ is any finite constant independent of a. Thus, there is a norm in momentum attribute space. As Noyes would put it, the "triangle" thus closes in "momentum space," as well as "configuration space." Our d-events can now be interpreted as 3–momentum conserving, 3-particle scattering events in the zero momentum frame, with the "center-of-mass" of laboratory physics at rest.

Zitterbewegung

We have already seen that any system with "constant velocity" (i.e., at those generations of the ordering operator when events can occur) evolves by discrete increments $\pm\lambda$ in q *between* d–events. These steps occur in the *void* where space and time are undefined. Since $\lambda/\Delta t = V_z$, each step occurs forward or backward with the limiting velocity. Thus, we deduce a discrete *Zitterbewegung* from our theory. If we think of this as a "trajectory" in the traditional pq phase space, each time step induces a step $\pm\lambda$ in q correlated with a step $\pm m V_z$ in p. Even in the case of a particle "at rest," this must be followed by two steps of the opposite sign to return the system to "rest;" see Figure 18.

Thus, there is, minimally, a four-fold symmetry to the "trajectory" in phase space, corresponding to the generation periodicity we discovered above.

Commutation Relations, Uncertainty, Planck's Constant

From the E–frame definitions of the obs corresponding to p and q, and consistent with the present example, we see that p and q are not independent. It follows from Theorem 50 that p and q do not commute, and from Theorem 42 that there is an uncertainty associated with the product of the variances in p and q. We establish a rule of correspondence between the constant in Eq. (76) and Plancks constant. By definition, the least step in p is just mc, since this step occurs at the maximal attribute velocity. Once again, these results are independent of the particular d–space generator chosen.

Since the least change in the product of the variances is h by the rule of correspondence, it follows that the least step in q is appropriately identified as just $L = h/mc$. To go on to the commutation relations, we take the usual step in the geometrical description of periodic functions, of taking the $q_J J$ plane to be the complex plane $(q, 2\pi i p)$; then the steps around the cycle in the order $qpqp$ are proportional to $\pm 2\pi(1, i, -1, -i)$, where $\pm$ depends on whether the first step is in the positive or negative direction or, equivalently, whether the circulation is counterclockwise or clockwise.

We have now shown that $qp - pq = \pm i\hbar$ for free particles; this result holds for any theory which uses a discrete free particle basis.

* From the definition of maximal attribute velocity, we should be led to the mass conversion law.

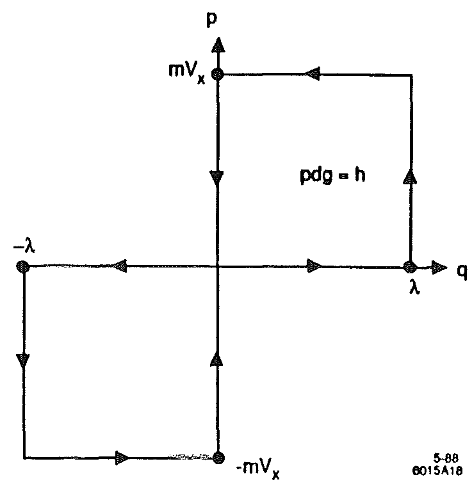

Fig. 18. *Zitterbewegung* in phase space for a particle "at rest."

The Angular Momentum Commutation Relations

Going to three dimensions, the commutation relations for angular momentum (as usually defined) follow immediately. Following T. F. Jordan [44], we may now derive the angular momentum commutation relations. Suppose we have P and Q in a discrete 3–space (i, j, k), related by a basis vector L, which we will call the angular momentum:

$$L = Q \times P \ ,$$

which is shorthand for three equations

$$L^i = Q^j P^k - P^j Q^k \ ,$$

with i, j, k taking all values from 1 to 3, and not equal to each other.

From the previous derivation of the P, Q commutation relations, we have

$$Q^j P^j - P^j Q^j = \frac{ih}{2\pi(N)} \ ,$$

$$L^i \times L^j = \frac{ihL^k}{2\pi(N)} \ .$$

For example,

$$L^1L^2 - L^2L^1 = (Q^2P^3 - Q^3P^2)(Q^3P^1 - P^1Q^3) - (Q^3P^1 - Q^1P^2)(Q^2P^3 - Q^3P^2)\ ,$$

$$= Q^2P^3Q^3P^1 + Q^3P^2Q^1P^3 - Q^3P^1Q^2P^3 - Q^1P^3Q^3P^2\ ,$$

$$= Q^1P^2(Q^3P^3 - P^3Q^3) + Q^2P^1(P^3Q^3 - P^3Q^3)\ ,$$

$$= (Q^1P^2 - Q^2P^1)\frac{ih}{2\pi(N)} = \frac{ihL^3}{2\pi(N)}\ .$$

Similar results follow for each of the relationships involving other coordinates (Q^1, Q^2, Q^3, P^1, P^2, P^3, L^1, L^2, L^3).

We have now shown that

$$L^i \times L^j = \frac{ihL^k}{2\pi(N)}\ ,$$

for free particles; this results holds for any theory which uses a discrete free particle basis.

Complete Identification of Laboratory Units

Now that we have shown, once given a specific generator of the 3+1–space, how to compute two ($\hbar$ and c) of the three dimensional constants needed to connect a fundamental theory to experiment in the 3–space in which physics operates, and which we have proved must be the asymptotic space of our theory, all that remains is to determine a unit of mass. Theorem 34 allows us to specify that the mass of an object is just the size S, although it does not tell us what object determines the fundamental unit. This can only be done once a specific generator of the d–space has been selected.

Scattering Range Computation

Once a unit of mass has been identified, we can show how to compute the classico-quantum scattering range from attribute distance. Note that for $h/(2m_pc)$, from the existing rules of correspondence for c, m_p and h, one obtains the following. Define an attribute such that the minimum attribute distance increment is I, with the following definitions holding: $h = I^2$ (minimum possible "area" in "phase space," $m_p = S = I + D$, and $c = v_{max} = I - D/I + D$. Thus, $h/(2m_pc) = I^2/2(I - D)$ when $v = c = v_max$; i.e. when $D = 0$. Therefore, we have $I^2/2(I) = I/2$, where I is just the minimum attribute distance increment for the attribute corresponding to the ensemble A with invariant size (mass) $I + D$. Clearly, since the 3–space is homogenous, we may interpret I as a diameter. Suppose a second ensemble B "approaches" with the first. Take two cases for the (generalized) attribute distance between them; $r > I$ and $r \leq I$. If $r > I$, then ensemble A may "travel" a distance I without any states in the generalized attribute distance being shared with the attribute states of B. If, however, $r \leq I$, then there exists the possibility of shared generalized attribute states between A and B, and thus nonindependence, exactly as described in the explanation of commutivity.

Note further, that if ensembles A and B do not have the same size and the same attribute distance definition for velocity computation, then the minimum interaction distance is not just the minimum of the "minimum attribute distance increments" for A and B, as compared via the generalized attribute distance. This is because the operation of addition is no longer well-defined: ensembles A and B are no longer independent, and this alters the generalized attribute distance definition.

Wheeler–Feynman and Massless Particles

Along other lines, we also have indications of how to compute transition probabilities from the ratios of the number of ensembles in given states, as determined by the combinatorial hierarchy. Let there be two attributes, such that the enumeration of states generated by the corresponding ordering operators are just the inverse enumeration of each other; that is, the last state generated by one is indistinguishable from the first state generated by the other, the next to the last state indistinguishable from the second state, etc. Further, let the representation of the states be duals ('0' in one represents the same thing as '1' in the other, and vice versa). From our rules of correspondence, these then correspond to particle and antiparticle.

This geometry suggests that zero mass particles are anomalous: no photon can be observed without both emission and absorption, and the path length in the photon frame is zero.* In the rest frame of the photon, any point on the photon trajectory can be treated as an electron/positron pair without violating relativity or the conservation laws. It would appear that photon emission/absorption is then modeled in our formalism as an electron emitted by the "emitter" and a positron (i.e., electron traveling backward in time) emitted by the "absorber," so that the photon can be treated as a virtual particle. From the reference frame of the photon, this exchange, and the evolution of the corresponding state vector, takes place atemporally. It is outside of time, happening everywhere along the photon path "at once." There is a difference in the energy of the two ends of the trajectory which is given by the torsion of the space—this being related to the constant identified above as Planck's constant, and to the minimal attribute distance increment exactly as in the (five-dimensional) Kaluza-Klein model. Thus, there is an apparent "transfer of energy" in the electron/positron pair exchange. This structure can *not* be detected locally. A similar argument holds for massless particles, in general.

6.4 Related Results: The Combinatorial Hierarchy and Program Universe

Bastin, Kilmister, Amson, Noyes and Parker-Rhodes have shown that there exists a unique finite hierarchy, combinatorially generated, which constructs at least some of the properties we require. This structure is referred to in the literature as the **combinatorial hierarchy**. Without developing the details here, we point out the essential features which make this structure interesting. First, the cardinalities of the primary objects (discriminately closed subsets) at each level of the structure are identifiable with the number of (E-frame) objects which may participate in the fundamental forces: to first order (which in our terms assumes first degree coupling only), they are the scale constants of laboratory physics (which we would identify computationally with the coupling scale of the relevant ordering operators). Second, Parker-Rhodes has shown that the construction leads to an amazingly accurate *computation* of the ratio of the mass of the proton to the mass of the electron (consistent with the present work). Third, Noyes et al. have developed a particular algorithm, known as **Program Universe**, for generating the combinatorial hierarchy, and have shown that the quantum numbers may be specified in such a manner as to make appropriate identification with the first generation of leptons and quarks. We refer to this algorithm as *PU*.

This last algorithm is of particular interest for our purposes, since it has all the characteristics of an ordering operator, including the fact that it is too complex to be deterministicly knowable from partial generation. The algorithm has two degrees of freedom, that is, two points at which an appeal to an arbitrariness generator is necessary. These two steps in the algorithm do not affect the global structure of the combinatorial hierarchy thus ultimately produced. Rather, the specifics of these steps will determine the dynamic evolution of the structure and the statistics during this evolution. Once the structure has been completely generated, the statistics are no longer affected.

For these reasons, we point out that PU [46] is, as an algorithmic definition, exemplary of the type of ordering operator which will generate the three-dimensional d-space, as described in Theorem 13. We caution the reader, however, to keep in mind that PU, the specific distance functions which are defined on it and the related derivations *are simply an example* of how we may proceed in detail. We identify U with PU, subject to falsification and subsequent modification. We are not dependent upon these details for the results presented here, which deal primarily with a physical interpretation of the ordering operator calculus. Nonetheless, we believe that either the details are valid, or that these aspects of the model can evolve smoothly (via the P-frame) to become valid.

* This is just the Wheeler-Feynman rule. Indeed, the work of Cramer's transactional interpretation is in full agreement, and is an extension of the Wheeler-Feynman interpretation. That such an interpretation results in a time-symmetric, self-renormalising QED with no singularities or second-quantisation problem is indeed encouraging [45].

For example, PU generates a universe of such strings which grows, sequentially, in either number (SU) or length (NU). The main program starts with PICK, an arbitrariness generator that picks two arbitrary strings from memory and discriminates them. This is one of the degrees of freedom mentioned. If this produces a novel string, an operation called ADJOIN results, which adjoins the string to the universe (SU:=SU+1). If the string produced by PICK is already in the universe, an arbitrariness generator called TICK is triggered which increases each string independently, by concatenating it with one arbitrary bit (NU:=NU+1). After either ADJOIN or TICK, the algorithm then recourses to PICK. The arbitrariness which occurs in selected strings from memory (in PICK) or in selecting bits to concatenate (in TICK), serves to guarantee that the algorithm represented by Program Universe is incompletely specified (though in principle specifiable) and, hence, we may treat the output as a Bernoulli trial (as required by Theorem 13), and PU as an ordering operator. If these are fully specified in an algorithmic sense, PU becomes deterministic, and the full evolution of the cosmology becomes known. However, much of the phenomena of laboratory physics arises specifically because we do not have the information. Indeed, we claim that the finite system represented by laboratory physics lacks the space complexity required to fully represent such an algorithm. Thus, some free parameters in the algorithm may not be determined from the recorded output of PU to date. At best then, PU represents a class of algorithms, each of which is sufficient, but not necessary, to account for the phenomena of laboratory physics. We propose that the class encompasses the necessary conditions.

That PU meets the conditions outlined in previous chapters for an ordering operator which is a metric generator, is easy to see. When the operation TICK of PU occurs, there will be three strings connected with the generation process which satisfy the conditions

$$S^a \oplus S^b \oplus S^c = (0,0,...,0)_{N_U} .$$

When NU is large, these conditions will be satisfied by many combinations. We can now identify the free function $f(k,n,\lambda)$, presented in the discussion entitled Nonlocal, Discrete Events in defining the attribute distance q_a, subject to a possible scaling factor. For PU, $f(k,n,\lambda) \equiv [2k^a(n) - n]\lambda_a$, and the conditions required in the preceding paragraphs are satisfied automatically. Therefore, PU is consistent with—and can legitimately appeal to—the results presented in this paper, without further derivation.

In an earlier work by Noyes et al. [47], a propagator for relativistic quantum scattering theory was derived. Now that we have shown how to explicitly construct the commutation relations, the interpretation or use of complex notation and how to construct the exponentiation operator, we claim that this work is well founded in all its detail.

Noyes has subsequently shown how to provide the interaction terms of the theory, by identifying our 3–momentum conserving events as "Yukawa vertices." Additionally, a tentative identification has been given of the first three levels of the hierarchy with (1) chiral electron-type neutrinos, (2) electrons, positrons and photons and (3) up and down quarks in a color octet, and with level four to provide weak-electromagnetic unification, with weak coupling to the first three levels.

That the overall mass scheme should come out right, is clearly suggested by the success of the Parker–Rhodes calculation: [48] $m_p/m_e = 137\pi/[(3/14)[1+2/7+(2/7)^2](4/5)] = 1836.151497\ldots$, which was later reformulated by Noyes to be consistent with the present theory. As Noyes has pointed out [49], the cosmology of Program Universe appears to have a charged lepton and a baryon number consistent with current observation, and, hence, with a locally flat space. These results can be understood as following immediately from establishing rules of correspondence between laboratory practice in high energy physics and performing the appropriate computations. Indeed, this author believes that there are few degrees of freedom available in establishing that interpretation, and perhaps none whatsoever. For example, if PU is selected, we must compute the largest to the smallest mass ratio; but this has already been done for us by the combinatorial hierarchy result $2^{127} + 136 \simeq 1.7 \times 10^{38} \simeq \hbar c/Gm_p^2 = (M_{Planck}/m_p)^2$, which tells us that we can either identify the unit of mass in the theory as the proton mass—in which case we can calculate, to about 1% in this first approximation, Newton's gravitational constant—or, if we take the Planck mass as fundamental, calculate the proton mass.

CONCLUSIONS

The ordering operator calculus has provided a formalism compatible with, and having explanatory and predictive power regarding, the current practice of physics. Indeed, a discrete and unified model of quantum mechanics and special relativity has been made possible.

Much work remains to be done. Not only is considerable effort required in establishing and validating the rules of correspondence, but extensions of the ordering operator calculus to other domains of mathematical investigation are desirable—we have mentioned some of these efforts along the way—and, of course, we would clearly like to incorporate a discrete version of general relativity in our theory. We have laid the foundation for doing so with the definitions of manifolds, neighborhoods, one-forms and other relevant mathematical objects. The reader should note that ours is always a "locally Lorentz invariant" theory and that local frames are, by construction, "inertial," meaning that the geometry is locally flat and exhibits no accelerations. Indeed, accelerations can only arise between the kinds of events we have constructed nonlocally, via the global topology (the connection), even though any dynamics are completely determined from the local geometry. Also in keeping with the geometric picture, our coordinate space has been constructed (from the beginning) from attribute "events," which locate an event by "what happens there," the ordering operator calculus being context sensitive. We already have some indication that "local (gravitational) distortion" of our distance function by a mass can be shown, and work we have recently encountered in the domain of cellular automata is relevant to, our corresponding notion of a field.

A number of experimental predictions have been made. According to P. Suppes [50], there are many generalized inequalities concerning joint probabilities, among which Bell's Inequalities are but a specific example. We have suggested a means of using these inequalities to test whether the nonlocality which violation of the inequalities demonstrates is absolute (along the lines of Bohm's Implicate Order), or, in fact, due to a multiply-connected topology.

We also suggest several other tests of the topology. Our theory predicts that the correlation in Aspect's time-of-flight experiments must be sensitive to the time between changes in the randomly shifted Brewster mirrors, and that the correlation will disappear for data taken arbitrarily close in time to one or the other shift. We should also be able to calculate the shape of an expected distribution curve for the fall-off in correlation, and might be able measure the slope experimentally. These experiments will be quite difficult because of the accuracy in measurement required.

Finally, we have suggested that this phenomena is NOT necessarily microscopic, or limited to spin and polarization quantum variables. The theory is sufficiently general that macroscopic violations of Bell's Inequalities should be constructable. Certainly, the effect can be modeled on computers and, indeed, is used today in publicly key encrypted security (access) cards.

As pointed out in the introduction, the ordering operator calculus is intended as a formalism for modeling diverse phenomena, and not just physical phenomena. Work along these lines is proceeding, and as yet unpublished applications to computational linguistics and computer science have been quite successful.

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DISCRETE PHYSICS:*
Practice, Representation and Rules of Correspondence

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1. INTRODUCTION

The practice of physics cannot get off the ground without essential agreement among the practitioners as to what they are about, how to go about it, and what constitutes progress in their common effort.

We adopt David McGoveran's modeling methodology [1]. This has three critical elements:

(1) an epistemological framework ("E–frame"), which is a set of loosely defined agreements made explicit by those injecting information into the model formulation—Gefwert [2] would call this a *practical understanding* of physics;

(2) a representational framework ("R–frame"), which is an abstract formalism consisting of a set of symbols and a set of rules for manipulation—to formulate such a frame is, for Gefwert, to practice *syntax*;

(3) a procedural framework ("P–frame"), which is an algorithm that serves to establish *rules of correspondence* between the observations agreed on in the E–frame and the symbols of the R–frame. Gefwert would describe this activity as the practice of *semantics*. Through recursion, the P–frame serves to modify the rules of correspondence, the E–frame and the R–frame, until a sufficient level of agreement concerning accuracy is achieved—or the model fails. Kuhn [3] would call such a failure a "crisis," which in the fullness of time could lead to a "paradigm shift."

Note that we halt the infinite regress of the analysis of terminology in constructive modeling by recognizing the epistemology. We deny the validity and the value of any attempt to analyze "theory-laden" language. Such an analysis lies outside our task when we engage in generating a specific model. Attempting to make such an analysis would require us to generate a model which would contain the specific model as an instance. We *cannot* do so within our methodology. Analysis of that sort would involve nonconstructive methods: the analyst *must* work from a specific model by generalization—having failed to construct the general model first.

The methodology implies iteration in the EPR or ERP sequence, or any interleaving of such sequences. Comparison with our diagram showing how the *participator* engages in a research program in physics [4] is given in Fig. 1. The comparison with McGoveran's modeling methodology is supposed to bring out the fact that the possible legal walks of the diagram are the same, but that the research program is contained *within* the methodology and that the methodology contains routes (arrows)that are *outside* the program. Thus the entry of the participator from a direction outside the box, and of the empirical confrontation (represented by Posiden's pitchfork Ψ) from a different direction, remain the same; so does the fact that corroboration leaves the participator inside, while falsification takes him outside, in yet another direction. The practitioner (and hopefully the reader of this volume) should therefore ask how far we have gone toward meeting his problems with contemporary physics. We assume that we agree on the following criteria:

1. agreement of cooperative communications
   * commonly defined terms as fundamental
   * fundamental vs. derived terms
   * agreement of pertinence
2. agreement of intent
3. agreement on observations
4. agreement of explicit assumptions
5. The Razor
   * agreement of minimal generality
   * agreement of elegance
   * agreement of parsimony

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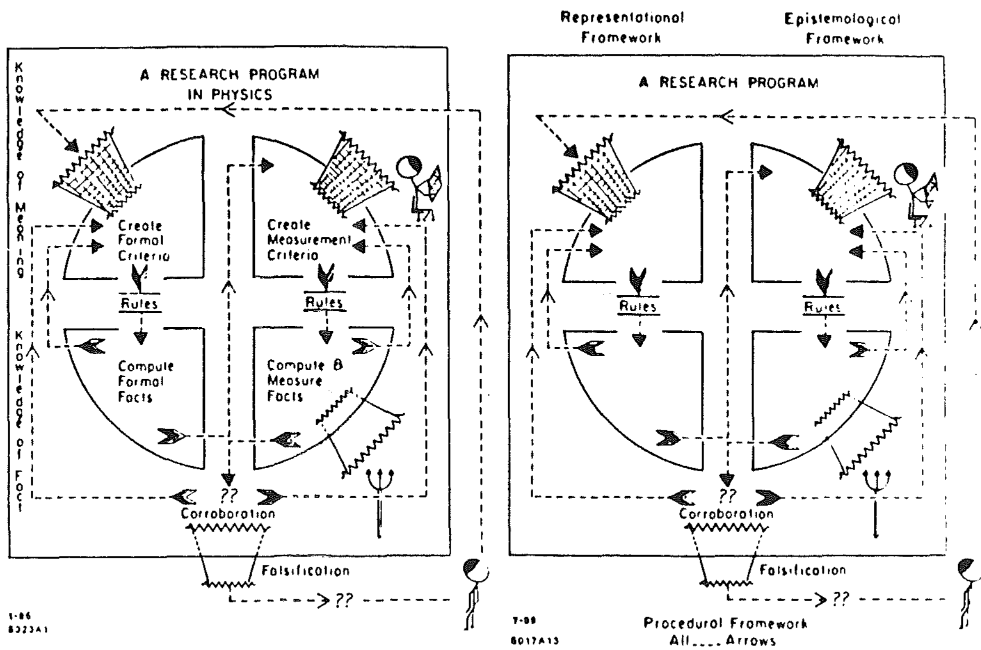

Fig. 1. Comparison between McGoveran's modeling methodology and Gefwert's participator model.

Our agreed upon intent is to model the practice of physics. We take as fundamental the commonly defined terms of laboratory physics, treating terms denoting nonobservables as derived or theoretical terms. We take laboratory events as a sufficient set of observations to be modeled, without requiring the standard theoretical interpretation. We take as understood that an experimental (laboratory) measurement may encompass many acts of observation. In other words, we are not committed to accept the how and why of the observations, only the observations themselves, operationally understood.*

In the next chapter we make a brief historical review of some aspects of modern physics which we find most significant in our own endeavor. In Chapter 3 we discuss the "Yukawa Vertices" of elementary particle theory as used in laboratory practice, second quantized field theory, analytic S–Matrix theory and in our own approach. In Chapter 4 we review the conserved quantum numbers in the Standard Model of quarks and leptons. This concludes our presentation of the "E–frame."

In Chapters 5–8 we try to develop a self-consistent representation of our theory. We have already claimed that this approach provides a discrete reconciliation between the formal (representational) aspects of quantum mechanics and relativity [5].

Chapters 9–13 provide rules of correspondence connecting the formalism to the practice of physics by using the *counter paradigm* and event-based coordinates to construct relativistic quantum mechanics

* Note the distinction between E–terms and R–terms. Von Neumann's "observation" is, at best, only an R–term. One line of criticism of von Neumann starts there, because his R–term is not necessarily consistent with Schrödinger continuity.

in a new way. The process comes to a temporary halt with a sequence of questions which could be answered in this framework.

2. THE HISTORICAL PRACTICE OF PHYSICS

Physics was a minor branch of philosophy until the seventeenth century. Galileo started "physics" in the contemporary sense. He emphasized both mathematical deduction and precise experiments. Some later commentators have criticized his *a priori* approach to physics without appreciating his superb grasp of the experimental method which he created—including reports of his experiments that still allow replication of his accuracy using his methods. He firmly based physics on the *measurement* of *length* and *time*, and established the uniform acceleration of bodies falling freely near the surface of the earth.

A century later, Newton entitled what became the paradigm for "classical" physics, "*The Mathematical Principles of Natural Philosophy,*" recognizing the roots that physics has in both disciplines. He also was a superb experimentalist. To a greater extent than Galileo, Newton had to create "new mathematics" in order to express his insight into the peculiar connection between experience, formalism and methodology that still remains the core of physics. To length and time, he added the concept of *mass* in both its inertial and its gravitational aspect, and tied physics firmly to astronomy through universal gravitation. For philosophical reasons, he introduced the concepts of absolute space and time, and thought of actual measurements as some practical approximation to these concepts.

It is often thought that Einstein's special relativity rejects the concept of absolute space-time, until it is smuggled back in through the need for boundary conditions in setting up a general relativistic cosmology. The concept of the homogeneity and isotropy of space, used by Einstein to analyse the meaning of distant simultaneity in the presence of a limiting signal velocity, in fact is very close to Newton's absolute space and time. What Einstein shows is rather that it is possible to use local, consequential time to *replace* this concept. This was pointed out to me by David McGoveran in the context of our fully finite and discrete approach to the foundations of physics, and our derivation of the Lorentz transformations without using the concept of continuity (cf., Ref. [1]). This same analysis shows that in a discrete physics, the universe has to be multiply connected. The space-like separated "supraluminal" correlations predicted by quantum mechanics—and recently demonstrated experimentally to the satisfaction of many physicists—can be anticipated for spin and for *any* countable degrees of freedom.

Nineteenth century physics saw the triumph of the electromagnetic field theory. That "classical" physics was still firmly based on historical units of mass, length and time; it provided no way to question *scale invariance*. Quantum theory and relativity were born at the beginning of this century. Quantum mechanics did not take on its current form until nearly three decades of work had passed. Although one route to quantum mechanics (that followed by deBroglie and Schrödinger) started from the continuum relativistic wave theory, the currently accepted form breaks the continuity by an interpretive postulate due to von Neumann sometimes called "the collapse of the wave function."

Criticism of this postulate as conceptually inconsistent with the time reversal invariant continuum dynamics of wave mechanics has continued ever since. This criticism was somewhat muted for a while by the near consensus of physicists that Bohr had "won" the Einstein–Bohr debate and the continuing dramatic technical successes of the theory. Scale invariance is gone because of the quantized units of mass, action and electric charge. These specify in absolute (i.e., countable) terms what is meant by "small." Explicitly $r_{Bohr} = \hbar^2/m_e e^2$ (with m_e the electron mass) specifies the atomic scale, $\lambda_{Compton} = (e^2/\hbar c) r_{Bohr} = \hbar/m_e c$ specifies the quantum electrodynamic scale and the "classical electron radius " $e^2/m_e c^2 = (e^2/\hbar c)\lambda_{Compton} \simeq 2\hbar/m_\pi c \simeq 14\hbar/m_p c$ specifies the nuclear scale; here m_p is the proton mass, and $m_\pi \simeq 2 \times 137 m_e$ is the neutral pion mass. The elementary particle scale $\hbar/m_p c$ is related to the gravitational scale by $\lambda_G = (G\hbar^3/c)^{\frac{1}{2}} = \hbar/M_{Planck}c = (Gm_p^2/\hbar c)^{\frac{1}{2}}(\hbar/m_p c)$

The expanding universe and event horizon specify what is meant by "large." Here the critical numbers any *fundamental* theory must explain are: "Age" of the universe as about 15 billion (15×10^9) years; "Mass" of the universe as about $3 \times 10^{76} m_p$—or at least ten times that number if one includes current estimates for "dark matter" ; "Size" of the universe or *event horizon*—naively the maximum radius which any signal can attain (or arrive from) transmitted at the limiting signal velocity c during the Age of the universe. Backward extrapolation using contemporary "laws of physics" to the energy and matter density when the radiation breaks away from the matter (size of the "fireball") is consistent with the observed 2.7°K cosmic background radiation. The cosmological parameters are numerically related to the elementary particle scale by the fact that the visible mass in the currently observable universe is approximately given by $M_{vis.U} \simeq (\hbar c/Gm_p^2)^2 m_p$, and that linearly extrapolating backward from the fireball to the "start of the big bang" gives a time $T_{fireball} \simeq (\hbar c/Gm_p^2)(\hbar/m_p c^2)$ =3.5 million years. It is clear that any theory which can calculate all these numbers has a claim to being a fundamental theory.

For a while it appeared that reconciliation between quantum mechanics and special relativity would resist solution, since the uncertainty principle and second quantization of classical fields gave an infinite energy to each point in space-time! During World War II, Tomonoga, and afterwards Schwinger and Feynman, developed formal methods to manipulate away these infinities and obtain finite predictions in fantastically precise agreement with experiment. Recently the non-Abelian gauge theories have made everything calculated in the "standard model" finite. Weinberg recently asserted at the Schrödinger Centennial in London that there is a practical consensus—but no proof—that second quantized field theory is the *only* way to reconcile quantum mechanics with special relativity. However, he also pointed out that the finite energy due to vacuum fluctuations is then 10^{120} too large compared to the cosmological requirements; the universe should rap itself up and shut itself off almost as soon as it starts expanding [6]. Even if one is willing to swallow this camel, there is no clear way to include strong gravitational fields in the theory. So continued attention to foundations seems fully justified.

The concept on which most of elementary particle physics rests has moved a long way from the mass points of post-Newtonian dynamics. For us, a paraphrase of the concept used by Eddington [7] is more useful: **A PARTICLE** is "A conceptual carrier of conserved 3-momentum and quantum numbers between events." This definition applies in the practice of elementary particle physics, both (1) in the high energy particle physics laboratory and in the theoretical formulations of either (2) second quantized field theory or (3) analytic S-matrix theory. In (1), the experimental application, "events" refer to the detection of any number of incoming and outgoing "particles" localized in macroscopic space-time volumes called "counters," or some conceptual equivalent. In (2), "events" start out as loci in the classical Minkowski 4-space continuum at which the "interaction Lagrangian" acting on a state vector creates and destroys particle states in Foch space. Since this prescription, naively interpreted, assigns an infinite energy and momentum to each space-time point, considerable formal manipulation and reinterpretation is needed before these "events" can be connected to laboratory practice. In (3), "events" refer to momentum-energy space "vertices" which conserve 4-momentum in the "Feynman diagrams" originally introduced in context (2) as an aid to the systematic calculation of renormalized perturbation theory. S-matrix theory makes a strong case for viewing continuous "space-time" as a mathematical artifact produced by Fourier transformation. Like any scattering theory, or any application of second quantized field theory to discrete and finite particle scattering experiments, S-matrix theory includes rules for connecting amplitudes calculated from these diagrams directly to laboratory practice (1).

For "events" generated by *Program Universe* [8] connecting bit strings (see Chapter 6), the "carrier" connects shorter to longer strings, or for strings of the same length connects two "3-events" to form a "4-event." We prove below that in this context the conservation of 3-momentum and quantum numbers consistent with laboratory practice (1) (thanks to the "counter paradigm," Chapter 9) can be derived within our construction of discrete physics, and serves the same purposes as the theoretical constructs in second quantized relativistic field theory (2) or analytic S-Matrix theory (3).

3. YUKAWA VERTICES

With the exception of *gluons*, the standard model of quarks and leptons starts from conventional interaction Lagrangians of the form $g\bar{\psi}\psi\phi$, into which various finite spin, isospin, ... operators may be inserted. Here g is the "coupling constant" which measures the strength of the interaction relative to the mass terms in the "free particle" part of the Langrangian, ψ ($\bar{\psi}$) is a fermion (antifermion) second quantized field and ϕ a boson or "quantum" field. All three fields can be expanded in terms of creation and destruction operators in "particle" or "Foch" space states, which in the momentum space representation contain separate 4-momentum vector variables for each fermion, antifermion or quantum.

Fortunately for us, in one of the first successful efforts to tame the infinities in this theory, Feynman introduced a diagrammatic representation for the terms generated by such interaction Lagrangians in a perturbation theory (powers of g) expansion of the terms which need to be calculated and summed in order to obtain a finite approximation for the predictions of the theory. These "Feynman Diagrams" have taken on a life of their own; they bring out the symmetries and conservation laws of the theory in a graphic way. This can be a trap, particularly if they are reified as representing actual happenings in space time; but if used with care, they can short circuit a lot of tedious calculation (or suggest viable additional approximations) and provide a powerful aid to the imagination.

In the usual theory, Minkowski continuum space-time is assumed and any interaction Lagrangian is constructed to be a Lorentz scalar. Consequently, the quantum theory conserves 4-momentum at each 3-vertex. Here one must use care because of the uncertainty principle. If 4-momentum is precisely specified, the uncertainty principle prevents any specification of position, and the vertex can be anywhere in space-time. This is the most obvious way in which the extreme nonlocality of quantum mechanics shows up in quantum field theory. However, if we use a momentum space basis, we can still have precise conservation at the vertices. In practical application of the theory, of course, momentum cannot be precisely known; quasi-localization is allowed as long as the restrictions imposed by the uncertainty principle are respected. In a thorough treatment, this is called "constructing the wave packet"; this requires some care, as can be seen, for instance, by consulting Goldberger and Watson's *Collision Theory*. In practice, one usually works entirely in momentum space, knowing that the orthogonality and completeness of the basis states will allow the construction of appropriate wave packets in any currently encountered experimental situation. We have made a start on the corresponding construction in our theory [4].

Although 4-momentum conservation is now insured in the conventional treatment, this is not the end of the problem. All this insures is that for a particle state with energy ϵ and 3-momentum $\vec{p}$, that $\epsilon^2 - \vec{p}\cdot\vec{p} = M^2$; here M is any invariant with the dimensions of mass and need not correspond to the rest mass of the particle m. In the usual perturbation theory this is simply accepted. The dynamical calculations are made "off mass shell," and the specialization to physical values appropriate to the actual laboratory situations envisaged is reserved to the end of the calculation. S-Matrix theory sticks closer to experiment, in that all amplitudes refer to physical (realizable) processes with all particles "on mass shell." The dynamics is then supposed to be supplied by imposing the requirement of flux conservation ("unitarity")—a nonlinear constraint—and relating particle and antiparticle processes by "crossing." The analytic continuation of the amplitudes for distinct physical processes which gives dynamical content to the theory then makes the problem a self-consistent or "bootstrap" formalism. There is no known way to guarantee a solution of this bootstrap problem, short of including an infinite number of degrees of freedom—if then; of course, it is also well-known that there is no known way to prove that quantum field theory possesses any rigorous solutions of physical interest. Consequently, one again has recourse to finite approximations which may or may not prove adequate to particular situations.

The finite particle number scattering theory [9–12] keeps all particles on mass shell and, hence, has 3-momentum conservation at 3-vertices. This theory then insures unitarity for finite particle number systems by the form of the integral equations; these also provide the dynamics. The uncertainty principle is respected because of the "off-energy-shell" propagator, as it is in nonrelativistic scattering theory; the approximation is the truncation in the number of particulate degrees of freedom.

If we put the "Feynman Diagrams" of the second quantized perturbation theory on mass shell, we can talk about 3-vertices and 4-events using a common language for all three theories. The rules are easy to state, particularly if we do so in the "zero momentum frame." We are justified in using this frame in the mathematical models* because we have restricted ourselves to free-particle, mass-shell kinematics. We can use a corresponding statement in the laboratory because this frame is empirically specified as the frame at rest with respect to the 2.7°K background radiation.† Then the Poincaré invariance of the theories allows us to go from this description to any other convenient Galilean frame.

* This a "Representational framework" statement in McGoveran's terminology.

† That is, again in the language of McGoveran's modeling methodology, we have a rule of correspondence ("Procedural framework" statement) connecting this zero momentum frame to laboratory practice ("Epistemological framework"), including the way calculations are performed in setting up and interpreting experiments.

As we show in Chapter 7, the 3–momenta at a 3–vertex add to zero. Diagrammatically we have three "vectors" which are "incoming" or "outgoing." By putting one of each together we obtain the generic 4–event, as indicated in Fig. 2. Clearly, for 4–events the total momentum of the two outgoing lines has to equal the total momentum of the two incoming lines, but the plane of the outgoing 3–event can be any plane obtained by rotating the outgoing vectors in the planar figure about the axis defined by the single line connecting them. By associating quantum numbers with each line, we can extend this description of 3–momentum conservation in Yukawa vertices and the 4–events constructed from them to the conservation of quantum numbers which "flow" along the lines. The idea of associating physical particles with the lines as carriers of both momentum and quantum numbers which comes from this pictorial representation is almost irresistible. The reader is warned once again to resist this temptation. The diagram is in 3+1 momentum-energy space and *not* in space time. In fact, if we insist on interpreting it as a space-time diagram representing the motion of particles, the quantum theory will blow up! It will force us to assign an infinite energy and momentum to each point of that space time, and simplicity of interpretation becomes elusive.

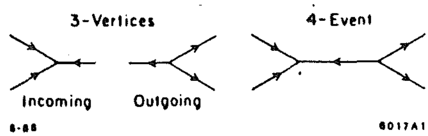

Fig. 2. The connection between 3–vertices and 4–events.

Once we have this picture in hand, "crossing" is easy to define. Since reversing a line and at the same time changing all quantum numbers to their negatives does not alter the conservation laws, the new diagram also represents a possible physical process. The "particle" whose quantum numbers are the negative of another is called its "antiparticle." So "crossing" can also be stated as the requirement that the reversal of a vector and the simultaneous change from particle to antiparticle represents another possible physical process. The manner in which a single diagram in which momenta and quantum numbers add to zero at a general 3–vertex generates emission, absorption, annihilation and decay vertices by this rule is illustrated in Fig. 3. The manner in which a single diagram, in which momenta and quantum numbers add to zero in a general 4–event, generates six physically observable processes by this rule is illustrated in Fig. 4.

Since one of the quantum numbers ("spin") is a pseudovector, "time reversal," which changes the sign of velocity and, hence, the direction, is not the same as the "parity" operation which changes all coordinates to their negatives. In quantum electrodynamics or QED, the theory in which the diagrams originated, the quantum number which distinguishes particle from antiparticle is electric charge; these rules are a consequence of the "CPT invariance" of the theory. They generalize to other types of "charge"; e.g., "color charge" in quantum chromodynamics (QCD). Spin is of great interest since it has a "space-time" significance as well as sharing the discrete, quantized character of other quantum numbers.

Before going on to the other quantum numbers, we note that the form of the Yukawa vertex couples the particle and antiparticle field in such a way that in the "time ordered" interpretation of the diagrams the number of fermions minus the number of antifermions is conserved; this is called the conservation of fermion number. Clearly, the diagrams respect this conservation law; so far as we know, f–number conservation is followed in nature.

4. THE STANDARD MODEL

The fermions encountered in nature fall into two classes: leptons and baryons. So far as we know to date, lepton number and baryon number are separately conserved. The lifetime for the decay of the proton into leptons and other particles has been shown to be greater than 10^{35} years; the experimental upper limit for the value depends on which decay mode was searched for. This fact has already ruled out many proposed schemes for "grand unification." The existence of the enormous underground detectors

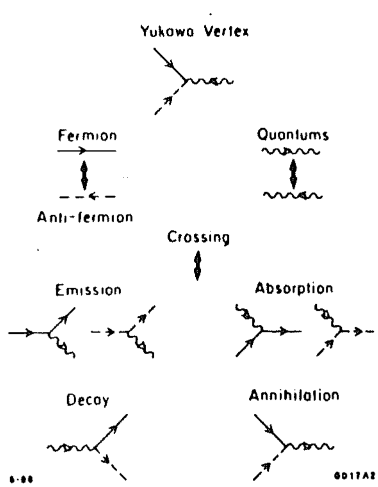

Fig. 3. The generic Yukawa vertex and crossing.

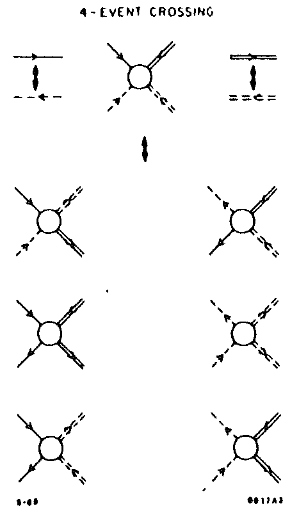

Fig. 4. Four-leg crossing.

constructed to test the hypothesis of proton decay had an unexpected payoff when two of them detected, "simultaneously," neutrino bursts from a supernova explosion 50,000 parsecs (1 parsec = 3.3 light-years) away. Individual neutrinos within the burst were cleanly resolved, but the time spread of the burst itself was so short that no information about the mass of the neutrinos was obtained. Although the time for the actual production of the neutrinos is supposed to be very short, the spread induced by the subsequent diffusion of the neutrinos out through the bulk of the star makes the calculation sensitive to the model used for calculating the explosion. It appears unlikely that limits on how much the neutrino mass might depart from zero better than those already established by terrestrial methods will be forthcoming from the analysis of this exciting event. Empirically, we can take electron-type neutrinos to be massless.

The quanta which couple via elementary Yukawa vertices in the standard model all have spin–1. The earliest coupling explored in quantum field theory was the electromagnetic coupling between electrons (e^-), positrons (e^+) and the massless electromagnetic quanta; the theory, which can be extended to other charged fermions, is called quantum electrodynamics (QED). The masslessness of the electromagnetic quanta is imposed within the second quantized relativistic field theory by requiring the theory to be "gauge invariant." A lower limit to the mass of either fermions or quanta with specified quantum numbers defines a well-understood experimental problem; if all such lower limits had to be finite, this would kill "gauge invariance." The requirement of gauge invariance is not compelling for me prior to some rough consensus as to what additional, independent tests (at an accuracy specified in advance) are relevant. I know of no proposed experimental program that could test gauge invariance within realistic error bounds. However, the upper limits on the mass of electromagnetic quanta are very good; empirically, we can assume photons to be massless.

The skepticism just implied makes my explanatory problem difficult. The current fashion in high energy elementary particle physics starts from "non–Abelian" gauge theories. Their broken "symmetries" generate "mass" from a "spontaneous breakdown of the vacuum." With care, this mechanism is claimed to be a guaranteed way to remove the infinities from a tightly constrained version of second quantized field theory. Without those constraints, which start from the necessity to get rid of the "classical" infinity of the e^2/r potential (infrared divergence) and the "second quantized" infinity of energy–momentum at each space-time point forced on us by the uncertainty principle (ultraviolet divergence), these theories are *prima facie* nonsensical. Self-consistency *within* the mathematical theory (R–frame) is contested by some who take the "rigour" of continuum mathematics seriously.

Following a conventional route in a 4-dimensional formalism one runs into trouble because a massless photon with momentum has only two chiral states (γ_{LL} and γ_{RR}), while the formalism requires four components for a 4-vector. For a massive spin-1 "particle" (i.e., something that can "carry" 3-momentum between two events in any coordinate system, and whose mass defines a rest system) there is no problem. The three states which quantum mechanics requires for spin 1 can be resolved along, against or perpendicular to the direction of motion, while the fourth component of the 4-vector is related to these three components "on-shell" by the invariant mass. When the invariant mass is zero, we are left with only two chiral 3-momentum carrying states. For fermions this is no problem, once parity conservation is abandoned. But for spin-1 massless bosons, the "third" and "fourth" component of the "4-vector" have to combine to yield an undirected $1/r$ "coulomb potential" in a gauge invariant and manifestly covariant manner. In a classical theory with extended sources this was no problem because the transformation between the 4-vector notation and the "coulomb gauge" was always well-defined, although coordinate system dependent; but in second quantized field theory, consistency between the classical substrate and the Feynman rules requires all kinds of technical artifices (indefinite metrics and the like). In a finite particle number theory, one can avoid some of these technical difficulties by always using transverse photons and the coulomb interaction in a well-defined coordinate system, provided the (no longer manifest) "covariance" can be maintained. Of course, this removes some of the (we believe superficial) formal simplicity of the "manifestly covariant" 4-vector formalism. Since the theory we have developed commits us to 3-momentum conservation as fundamental, this is a natural route for us to take.

Once this is understood, the $e_L^-(Q = -e, s_h\hbar = -\frac{1}{2}\hbar)$, $e_R^-(Q = -e, s_h\hbar = +\frac{1}{2}\hbar)$ crossing symmetric Yukawa vertices specifying massive leptonic QED for a single flavor (in this case e) coupled to $\gamma_{LL}, \gamma_{RR}, \gamma_c$ are given in Fig. 5. We note that for electromagnetic coupling, charge and lepton number go together, so the conservation law for one implies the conservation law for the other. We represent the combined conservation laws of $2s_h \in 0, \pm 1, \pm 2$ and $\ell = -Q/e \in 0, \pm 1$, by the vector states in a plane by Fig. 6. A Yukawa (QED) vertex requires three quantum number "vectors" consisting of a fermion, an antifermion and a quantum which add to zero, plus the temporally ordered processes derived from the fundamental diagram by crossing. The field theory notation for this QED coupling is [13] $-i\bar{e}\gamma_\lambda e A_\lambda$, with $Q^2/\hbar c = e^2/\hbar c \simeq 1/137$.

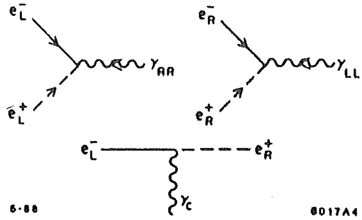

Fig. 5. Quantum electrodynamics.

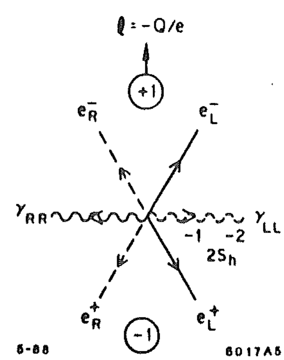

Fig. 6. Quantum electrodynamic conservation laws as planar vectors.

In contrast to the parity conserving electromagnetic vertices, the "weak" interactions violate parity conservation maximally. The easiest way to represent this is to use a massless neutrino (ν_L), conventionally called "left-handed." Consider an arrow in front of you with the head on the right. If you slip your right hand under the arrow to pick it up, your thumb will point in the same direction as the head; if you pick it up by slipping your left hand under the arrow, your thumb will point in the opposite direction to the head. The latter case is called "left-handed." By the Feynman rule, the antineutrino $\bar{\nu}_L$ is then right-handed. The charged quantum which couples to the electron and neutrino is called W (the weak vector boson) and is also chiral, since in the zero momentum frame $e_L^- + \bar{\nu}_L \to W_{LL}^-$; in field theory notation the coupling is

$$-i(G_F M_W^2/\sqrt{2})^{\frac{1}{2}}\ \bar{\nu}\gamma_\lambda(1-\gamma_5)\ eW_\lambda\ .$$

The Weinberg–Salam–Glashow "weak-electromagnetic unification" requires, in addition to this electrically charged weak boson, which was a convenient way to parameterize the parity-nonconserving theory of β–decay, the neutral weak boson Z_0 responsible for "neutral weak currents." The reasons had to do initially with the removal of infinities from the theory, and go through a complicated sequence of arguments that predict, in addition, one or more scalar "Higgs bosons," for which there is at present no laboratory evidence. Since our theory is born finite and cannot produce the infinities of second quantized field theory, we have no need for these hypothetical particles in the first place. If they should be discovered (thanks to current efforts at many laboratories which are now consuming a large fraction of their experimental and computational resources), we will be faced with some difficult conceptual problems in our discrete theory. Fortunately, for the moment, we can ignore them, which makes our presentation of the conservation laws in the leptonic sector considerably simpler.

The coupling of the Z^0 to neutrinos is chiral and is given by

$$(-i/\sqrt{2})(G_F M_Z^2/\sqrt{2})^{\frac{1}{2}}\ \bar{\nu}\gamma_\lambda(1-\gamma_5)\ eZ_\lambda\ .$$

The coupling to electrons is more complicated because it brings in the "weak angle" θ_W that distinguishes the coupling to left- and right-handed electrons in the following way:

$$(-i/\sqrt{2})(G_F M_Z^2/\sqrt{2})^{\frac{1}{2}}\ \bar{e}\gamma_\lambda[R_e(1+\gamma_5)+L_e(1-\gamma_5)]\ eZ_\lambda\ .$$

Here $R_e = 2sin^2\theta_W$, $L_e = 2sin^2\theta_W - 1$. If $sin^2\theta_W = 1/4$, which is not too bad an approximation to the experimental value, Z couples to electrons like a heavy gamma ray, except that it is a pseudovector rather than a vector. The mixing angle is not independent of the masses of the weak bosons, because

$$M_W \sin\theta_W = [\pi e^2/\hbar c G_F\sqrt{2}]^{\frac{1}{2}} = 37.3\ \mathrm{Gev}/c^2 = M_Z \sin\theta_W \cos\theta_W\ .$$

Since there were estimates of the weak mixing angle available before the discovery of the weak bosons, their masses could be estimated to be around 84 and 94 Gev/c^2 respectively, which aided greatly in their experimental isolation. Since the W's are charged, they couple to photons and also directly to the Z. These couplings are given in Ref. [13], p. 116. Eventually the more complicated 4–vertices given in the same reference should provide a critical test of the standard model, and conceivably might also distinguish between our theory and the standard model, even in the absence of experimental evidence for the Higgses. We ignore this complexity in what follows.

The conservation law situation is now considerably more complicated than it was for electromagnetic quanta. Charge, lepton number and helicity are still conserved, but the pattern is not easy to follow if written in those terms. Following a strategy that was first introduced into nuclear physics to describe the approximate symmetry between neutron and proton as an "isospin doublet," we form a "weak isospin doublet" from the left-handed electron ($i_z = -\frac{1}{2}$) and left-handed neutrino ($i_z = +\frac{1}{2}$) and, assuming lepton number conservation, can talk about either charge conservation or "z–component of isospin conservation," by introducing an appropriate version of the Gell–Mann–Nishijima formula, namely $Q = \ell/2 + i_z$, for the left-handed doublet. To include the right-handed electron, which does not couple to neutrinos, we make it an isospin singlet. To couple it to γ–rays, we assign it a "weak hypercharge" $Y = -2$ and modify Gell Mann–Nishijima formula to read $Q = Y/2 + i_z$. Our conservation laws are now conveniently described in the 3–space picture given in Fig. 7. The numerical specifications are given in Table 1.

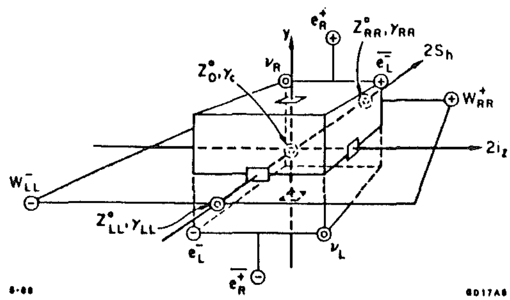

Fig. 7. Weak-electromagnetic unification in terms of weak hypercharge, weak isospin and helicity.

Table 1. Quantum numbers for weak-electromagnetic unification.

Particle		Q	Y	$2i_z$	ℓ	$2h$	m in Gev/c^2
fermion	ν_L	0	−1	+1	−1	−1	0
	$\bar{\nu}_L$	0	+1	−1	+1	+1	0
	e_L^-	−1	−1	−1	−1	−1	$.511 \times 10^{-3}$
	$\bar{e}_L^-$	+1	+1	+1	+1	+1	"
	e_R^-	−1	−2	0	−1	−1	"
	$\bar{e}_R^-$	+1	+2	0	+1	+1	"
quantum	W_{LL}^-	−1	0	−2	0	−2	$37.3/\sin\theta_W$
	$\bar{W}_{LL}^-$	+1	0	+2	0	+2	"
	Z^0_{LL}, γ_{LL}	0	0	0	0	−2	$37.3/\sin\theta_W \cos\theta_W, 0$
	$\bar{Z}^0_{LL}, \bar{\gamma}_{LL}$	0	0	0	0	+2	"
	Z^0_0, γ_c	0	0	0	0	0	"

Although the type of spatial representation of the quantum numbers presented in Fig. 7 suggests that there might be rotational invariance in this space, actually only the values on the axes have precise meaning in terms of conservation laws. Total isospin is only approximately conserved; it is a "broken symmetry." Perhaps this should not be a surprise in a relativistic theory; if we take as the four independent generators of the Poincaré group mass, parallel and perpendicular components of 3-momentum and helicity (or the component of angular momentum along the parallel direction), the total angular momentum cannot be simultaneously diagonalized. People often forget that "total spin" is not a well-defined concept in a relativistic theory.

Now that we have explored in detail the weak-electromagnetic unification of electrons, whose mass is 0.511 Mev/c^2, and their associated massless neutrinos, the full weak-electromagnetic unification scheme is easy to state. In addition to the electrons, we have two systems of leptons with much larger masses, the muon with mass 105.66 Mev/c^2 and the tau lepton with mass 1784 Mev/c^2. Associated with each are left-handed $(\nu_\mu)_L$ and $(\nu_\tau)_L$ neutrinos whose interactions can be experimentally distinguished from those of the electron neutrinos $(\nu_e)_L$ and from each other. They may well be massless, but the upper limits on their masses are much higher than for the electron type neutrinos. The coupling scheme is the same as that we have already discussed above within each "generation" $(e, \mu, \tau = 1^{st}, 2^{nd}, 3^{rd})$ and the coupling between generations, specified by the Kobiyashi–Maskawa mixing angles, is weak.

To complete the scheme for the weak interactions we must bring in the quarks. There are two "flavors" (up and down) for the first (electron) generation, and two (charmed and strange) for the second (muon) generation; there are supposed to be two more in the third (tau) generation to complete the picture. The existence of the beautiful (or bottom) quark is well-established, but searches for the true (or top) quark are still under way. It is the only particle missing from the scheme, other than the Higgses, if you stick to three generations. The quarks are fermions and have electric charge $Q_{u,c,t} = \pm\frac{2}{3}$, $Q_{d,s,b} = \mp\frac{1}{3}$ and baryon number $\frac{1}{3}$. Each forms a weak isodoublet and an isosinglet in the now familiar pattern. This completes the weak-interaction picture at the level we will discuss it here.

The quarks differ markedly from the leptons in several respects. To begin with, they carry a conserved "color charge" with three colors, three anticolors and an eightfold symmetry we will describe in more detail in Chapter 8. They couple strongly at low energy to eight spin–1 colored "gluons." Color conservation is given a vector representation in Fig. 8.

Remarkably, both quarks and gluons are "confined": they show up like internal particulate degrees of freedom in high energy experiments (parton model), but never have been liberated to be studied as free particles. Hence, the definition of their masses is indirect; recent calculations would seem to indicate that the "mass" of an up or down quark is about one-third the mass of a proton at low energy, but falls off like $1/p^2$ as the momentum with which they interact increases [14]. One up quark combined with an up-down pair in a spin-singlet state to forsm an overall color singlet state form a proton with charge 1, while a down quark combined with the pair in the same way forms a neutron with charge 0. Consequently, the β–decay properties of the neutron can be related to the weak isodoublet description given above.

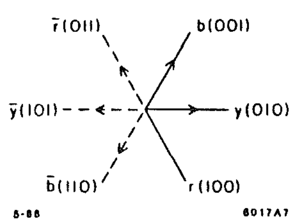

Fig. 8. Colors and anticolors as discrete vectors.

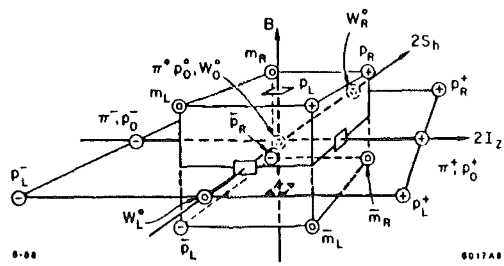

Fig. 9. Spin, isospin and baryon number conservation for color singlet neutrons and protons $p = u(ud), n = (d(ud)$.

So far as quantum number conservation goes, we can talk about baryon number (B) spin and (strong) isospin with charge conservation given by $Q = B/2 + I_z$ in the same way we talked about weak hypercharge and weak isospin conservation above. Quark-antiquark pairs describe the mesons (pions, etc.,) which older theories used to explain nuclear forces, but the details of how the quark-nuclear physics interface actually works quantitatively is a very controversial field of research. The easiest way to picture all this is to write the "color" vertices separately as vectors in a plane and assume that they add to form a color singlet (which can be a neutral colored or anticolored triplet, or any one of the color-anticolor pairs). Then we can return to the familiar picture of neutron, proton, and their antiparticles and associated mesons in the (s_h, I_z, B) space pictured in Fig. 9. Note the symmetry of the diagram for these parity-conserving strong interactions, in contrast to the asymmetric diagram which pictures the parity nonconserving weak-electromagnetic unification.

We will show in Chapter 8 how this whole picture can be reproduced at this level by our discrete physics construction. To get the quantitative details right is obviously a major research program, comparable (until we can find short cuts) to the hard work that is engaging many particle physicists every day in many laboratories. A useful reference that gives some idea of the magnitude of the task is the Proceedings of the 1986 SLAC Summer Institute [15]. Clearly, we must stop at some point short of that effort in this volume; we choose to do so when we have reached the same degree of desciption explained in this chapter.

5. THE COMBINATORIAL HIERARCHY AND THE LABEL–CONTENT SCHEMA

The overall status of the research [16–17], here aimed at providing a common explanatory theory for both quantum mechanics and relativity in a discrete and finite framework, has been provided a historical context in Chapter 2. The early thinking in this program did not approach the problem with such an explicit objective. Bastin realized that when we go to the very large (distant galaxies, early times...) or the very small (quantum events, elementary particles...) the information available to us becomes extremely impoverished compared to the phenomena modeled by classical physics. He concluded that this fact should be reflected in the theory in such a way that this restriction is respected.

The route into the theory initially followed by Bastin and Kilmister concentrated on the problem of modeling discrete events [18]. Ordered strings of zeros and ones gave a powerful starting point for analyzing this problem. Attention eventually centered on the question of whether bit strings were the same or different. Define a bit string by

$$(a)_n \equiv (\ldots,\, b_i^a, \ldots)_n;\ b \in 0,1;\ i \in 1,2,\ldots,n\,.$$

An economical way to compare an ordered sequence of two distinct symbols with other sequences of the same bit length is to use the operator XOR ("exclusive or," symmetric difference, addition (mod 2) $= +_2, \ldots$). Since we sum (or count) the one's in the string to specify a measure, we must treat the symbols "0," "1" as integers, and only in some contexts can we think of them as bits; hence, our "bit strings" are more complicated conceptually than those encountered in standard computer practice. We therefore use the more general *discrimination* operation "$\oplus$," and a short hand notation for it. Define the symbol $(ab)_n$ and the discrimination operation $\oplus$ by

$$(ab)_n \;\equiv\; S^a \;\oplus\; S^b \;\equiv\; [\ldots,\, (b_i^a - b_i^b)^2, \ldots]_n = [\ldots,\, b_i^a +_2 b_i^b, \ldots]_n\,.$$

The name comes from the fact that the same strings combined by discrimination yield the null string, but when they differ and $n \geq 2$ they yield a third distinct string which differs from either; thus the operation discriminates between two strings in the sense that it tells us whether they are the same or different.

We define the *null string* $(0)_n$ by $b_i^0 = 0, i \in 1,2,\ldots,n$ and the *antinull string* $(1)_n$ by $b_i^1 = 1, i \in 1,2,\ldots,n$. Since the operation $\oplus$ is only defined for strings of the same length, we can usually omit the subscript n without ambiguity. The definition of discrimination implies that

$$(aa) \;=\; (0);\quad (ab) \;=\; (ba);\quad [(ab)c] \;=\; [a(bc)] \;\equiv\; (abc)\,,$$

and so on.

The importance of closure under this operation was recognized by John Amson. It rests on the obvious fact that $[a(ab)] = (b)$, and so on. We say that any finite and denumerable collection of strings, where all strings in the collection have a distinct tag $i, j, k \ldots$, are *linearly independent* iff

$$(i) \neq (0):\ (ij) \;\neq\; (0),\ (ijk) \;\neq\; (0), \ldots\ (ijk\ldots) \;\neq\; (0)\,.$$

We define a *discriminately closed subset* of nonnull strings $\{(a),(b),\ldots\}$ as the set with a single string as member or by the requirement that any two different strings in the subset give another member of the subset on discrimination. Then two linearly independent strings generate three discriminately closed subsets, namely

$$\{(a)\},\ \{(b)\},\ \{(a),(b),(ab)\}\ .$$

Three linearly independent strings give seven discriminately closed subsets, namely

$$\{(a)\},\ \{(b)\},\ \{(c)\}\ ,$$

$$\{(a),(b),(ab)\},\ \{(b),(c),(bc)\},\ \{(c),(a),(ca)\}\ ,$$

$$\{(a),(b),(c),(ab),(bc),(ca),(abc)\}\ .$$

In fact, x linearly independent strings generate $2^x - 1$ discriminately closed subsets because this is simply the number of ways one can take x distinct things one, two, three,..., x at a time. This is critical to the construction of the combinatorial hierarchy, as we now discuss.

The discovery of the combinatorial hierarchy [19] was made by Parker–Rhodes in 1961. The story as I recall hearing it a decade after the facts, which Bastin now informs me is somewhat misleading,* was that the challenge posed to Frederick was how to generate a sequence with one or two small numbers, something of the order of a hundred, some very large number and *stop.*† Frederick (P–R) did indeed generate the sequence $3, 10, 137, 2^{127} + 136 \simeq 1.7 \times 10^{38}$ in suspiciously accurate agreement with the "scale constants" of physics. This was a genuine discovery; the termination is at least as significant!• The sequence is simply $(2 \Rightarrow 2^2 - 1 = 3), (3 \Rightarrow 2^3 - 1 = 7)$ $[3 + 7 = 10], (7 \Rightarrow 2^7 - 1 = 127)$ $[10 + 127 = 137], (127 \Rightarrow 2^{127} - 1 \simeq 1.7 \times 10^{38})$. The real problem is to find some "stop rule" that terminates the construction.

The original stop rule was due to Parker–Rhodes. He saw that if the discriminately closed subsets at one level, treated as sets of vectors, could be mapped by nonsingular (so as not to map onto zero) square matrices having uniquely those vectors as eigenvectors, and if these mapping matrices were themselves linearly independent, they could be rearranged as vectors and used as a basis for the next level. In this way the first sequence is mapped by the second sequence $(2 \Rightarrow 2^2 = 4)$, $(4 \Rightarrow 4^2 = 16)$, $(16 \Rightarrow 16^2 = 256)$, $(256 \Rightarrow 256^2)$. The process terminates because there are only $256^2 = 65{,}536 = 6.5536 \times 10^4$ linearly independent matrices available to map the fourth level, which are many too few to map the $2^{127} - 1 = 1.7016 \ldots \times 10^{38}$ DCsS's of that level. The (unique) combinatorial hierarchy is exhibited in Table 2.

Although this argument proves the *necessity* of the termination (which is no mystery in the sense that an exponential sequence must cross a power sequence at some finite term), it did not establish the existence of the hierarchy. This was first done by me by creating explicit constructions of the

* Quoting a recent letter by Bastin to HPN, 3 March 1988, "Frederick had come very recently into the discussions about hierarchies and level relationships [among Amson, Bastin, Kilmister, Pask], and couldn't come on a second trip with me to the analog computer at Brussels [where they were being explored experimentally?] because of 'flu. When he had his mapping relation giving an 'information preserving' (as we should then have said) relation between levels; and the numbers. He and the rest of us knew that the numbers had to be the primary step to physics, but we planned to avoid any attempt at deduction of them thinking it probably impossible. It was a morning or two later that Frederick arrived very crestfallen because he had found the breakdown of the algorithm." [The "breakdown" referred to is the termination of the sequence at the fourth level, which turns out to be a critical success of the basic theory when we come to explaining gravitation.]

† Continuing the quote from Bastin, "I never proposed that sort of challenge to Frederick, though I can see you may have wanted a quick way to be fair to everyone and hit on that."

• According to Parker–Rhodes, in "Agnosia," *Proc. ANPA 7*, p. 74: "Somewhere around 1962 I hit upon a series of numbers of which Ted Bastin noticed that the last two (the generating procedure could not produce more than four) were close to two well-known physical constants, the reciprocals of the fine-structure constant and the gravitational coupling constant." The somewhat different history given in the "Preface and Acknowledgements" to Parker–Rhodes' *The Theory of Indistinguishables* does not give this credit to Bastin. I know that this preface was an afterthought, and that Frederick did not prepare it with care.

Table 2. The combinatorial hierarchy.

Hierarchy Level ℓ	$B(\ell+1)=H(\ell)$	$H(\ell)=2^{B(\ell)}-1$	$M(\ell+1)=[M(\ell)]^2$	$C(\ell)=\Sigma_{j=1}^{\ell}H(j)$
(0)	-	2	(2)	-
1	2	3	4	3
2	3	7	16	10
3	7	127	256	137
4	127	$2^{127}-1$	$(256)^2$	$2^{127}-1+137$

Level 5 cannot be constructed because $M(4) < H(4)$.

mapping matrices [20] and later more elegantly by Kilmister [21]. That the termination, and indeed the combinatorial hierarchy itself, is much more than the apparently *ad hoc* mapping procedure which first led to it might suggest, can be seen either by Kilmister's latest derivation [22] or by the very different way Parker–Rhodes now gets it out of his *Theory of Indistinguishables* [23]; a useful discussion of that theory entitled "Agnosia" is given in Ref. [17].

For some time, the only operation used in the theory was discrimination. Kilmister eventually realized that one should also think about where the strings came from in the first place. He met this problem by introducing a second operation which he called "generation." As he and I realized, this operation eventually generates a universe which goes beyond the bounds of the combinatorial hierarchy. Once this happens, we can separate the strings into some finite initial segment that represents an element of the hierarchy, which we call the *label*, and the portion of the string beyond the label which we now‡ call the *content*. It is clear that from then on the content ensemble for each label grows in both number and length as the generation operation continues. Since it takes $2+3+7+127=139$ linearly independent basis strings to construct the four levels of the combinatorial hierarchy, the labels will be of at least this length; if we use the mapping matrix construction, they will be of length 256. Call this *fixed* length L, the length of any content string n, and the total length at any TICK (see next section) in the evolution of the universe $N_U = L+n$. Then the strings will have the structure $S^a = (L_a)_L \| (A_x^a)_n$ where a designates some string of the $2^{127}+136$, which provide a representation of the hierarchy, and x designates one of the 2^n possible strings of length n; the symbol "$\|$" denotes string concatenation.

6. PROGRAM UNIVERSE

In order to generate a universe of strings which grows, sequentially, in either number (SU) or length (N_U) Mike Manthey and I created *program universe*. Recently Manthey realized that the criterion used to increase the string length (TICK) was unjustifiably selective. The previously published version of the program [8], called *program universe 1*, is compared with Manthey's new proposal in Fig. 10. The most significant effect of the change, other than simplification (using "The Razor" in McGoveran's terminology), is to allow the bit string universe to contain, ephemerally in many cases, distinct strings which are indistinguishable under discrimination. This will not affect anything in this paper, but might eventually provide alternative cosmological models that make observationally different predictions.

‡ The term Kilmister and I first used was "address" rather than "content." This has turned out to be unfortunate from the point of standard computer science usage. It has been proposed that "address" be replaced by "content," and I adopt that new usage in what follows. Kilmister and I used "address" because we envisaged (as has now happened) the use of this portion of the string to construct our discrete version of "space-time;" thus the address is like that on an envelope, with the label being the name. This has the advantage that neither is meaningful without the other. On the other hand the "contents" of a label describe the relevant states which are occupied at a given TICK of PROGRAM UNIVERSE, and their order of production—if known or knowable—would serve to enumerate them. We should try to stabilise the terminology at ANPA 10.

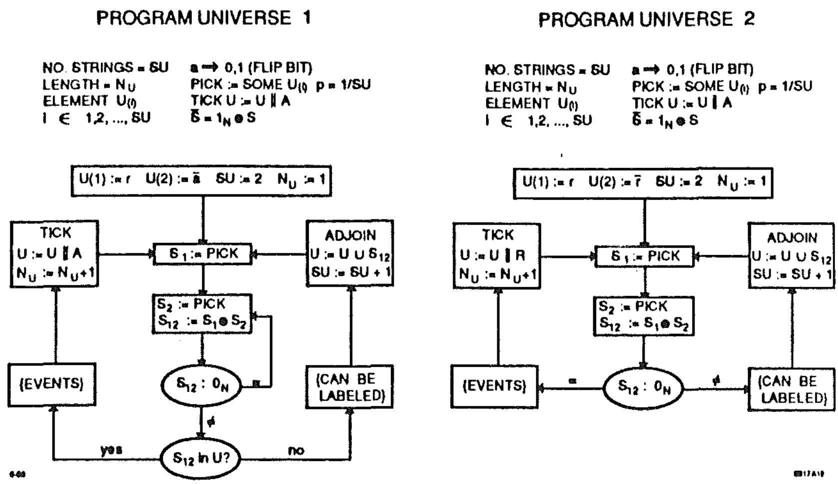

Fig. 10. Program Universe 1 and 2 compared.

The program is initiated by the arbitrary choice of two distinct bits, which become the first two strings in the universe. Whether insisting that one be "0" and the other "1," as in done in the flow chart, rather than allowing both to be arbitrary, will eventually produce a significantly different cosmology (or choice among cosmologies) at our epoch is an open question. Entering the main routine at $PICK$, we choose two strings (i) and (j) and discriminate them: $(ij) \equiv (i) \oplus (j)$. Whenever the two strings picked are identical, $(ij) = (0)_{N_U}$ and we go to TICK. TICK concatenates a single bit, arbitrarily chosen for each string, to the growing end, notes the increase in string length and the program returns to $PICK$. The alternative route, which occurs when discrimination generates a nonnull string, simply ADJOINs the newly created string to the universe and the program returns to $PICK$.

In the older version, we proved that TICK had to be "caused" (in the computer simulation) either by the occurrence of the "3–event" configuration $S^a \oplus S^b \oplus S^c = 0_{N_U}$ or by the configuration $S^a \oplus S^b \oplus S^c \oplus S^d = 0_{N_U}$, which we called a "4–event." But this implied a uniqueness which has no known demonstrable counterpart in nature, as modeled by contemporary physics; there can be many "simultaneous" events. At ANPA 9, I extended the definition of "event" to include all cases in which, at a given string length (or TICK), three or four strings combine under discrimination to produce the null string. This definition is retained here, but in Program Universe 2 is no longer the "cause" of TICK. Instead, we TICK whenever two strings "interact" without producing any novelty. This is as close as we need to get to defining what would be called a "point" in a continuum theory. We will see in Chapter 10 that this construction of a "point" is consistent with our development of Einstein synchronization and, hence, to the extent possible in our discrete theory, consistent with the conventional use of the term "event" in relativity theory.

The constraints $(abc)_{N_U} = (0)_{N_U} = (abcd)_{N_U}$ at each TICK are our model for the unique, nonlocal, yet indivisible and *irreversible* events of quantum mechanics. We have a lot more work to do before we can show that they have the requisite properties. In particular, we have to demonstrate that they can act like the 3–vertices and 4–vertices of the Feynman Diagrams discussed in Chapter 3. When N_U is large, these constraints will be satisfied by many combinations and—because of McGoveran's Principle IV[♮] —*all* must be viewed as "simultaneous" events.

♮ "The theory possesses the property of absolute nonuniqueness," cf., Ref. [1].

The method Manthey and I use to "construct" the hierarchy is much simpler than the original matrix construction given by Parker–Rhodes; in fact, some might call it "simple-minded." We claim that all we have to do is to demonstrate explicitly (i.e., by providing the coding) that any run of PROGRAM UNIVERSE contains (if we enter the program at appropriate points during the sequence) all we need to extract some representation of the hierarchy and the label content scheme from the computer memory *without* affecting the running of the program. The obvious intervention point exists where a new string is generated, i.e., at ADJOIN. The subtlety here is that if we assign the tag i to the string $U[i]$ as a *pointer* to the spot in memory where that string is stored, this pointer can be left unaltered from then on. It is, of course, simply the integer value of $SU + 1$ at the "time" in the simulation [sequential step in the execution of that run of the program] when that memory slot was first needed. Of course, we must take care in setting up the memory that *all* memory slots are of length $N_{max} > N_U$; i.e., can accommodate the longest string we can encounter during the (necessarily finite) time our budget will allow us to run the program. Then, each time the program TICKs, the bits which were present at that point in the sequential execution of the program when the slot $[i]$ was first assigned will remain unaltered; only the growing head of the string will change. Thus, if the strings $i, j, k \ldots$ tagged by these slots are linearly independent at the time when the latest one is assigned, they will remain linearly independent from then on.

Once this is understood, the coding Manthey and I gave for our labeling routine should be easy to follow. We take the first two linearly independent strings and call these the basis vectors for *level 1*. The next vector which is linearly independent of these two starts the basis array for *level 2*, which closes when we have three basis vectors linearly independent of each other and of the basis for *level 1*, and so on until we have found exactly $2 + 3 + 7 + 127$ linearly independent strings. The string length when this happens is then the *label length* L; it remains fixed from then on. During this part of the construction, we may have encountered strings which were *not* linearly independent of the others, which up to now we could safely ignore. Now we make one *mammoth* search through the memory and assign each of these strings to one of the four levels of the hierarchy; it is easy to see that this assignment (if made sequentially passing through *level 1* to *level 4*) has to be unique.

From now on, when the program generates a new string, we look at the first L bits and see if they correspond to any label already in memory. If so, we assign the content string to the *content ensemble* carrying that label. If the new string also has a new label, we simply find (by upward sequential search as before) what level of the hierarchy it belongs to and start a new labeled content ensemble. Because of discriminate closure, the program must eventually generate $2^{127} + 136$ distinct labels, which can be organized by us into the four levels of the hierarchy. Once this happens, the label set cannot change and the parameters i for these labels will retain an *invariant* significance, no matter how long the program continues to TICK. It is this invariance which will later provide us with the formal justification for assigning an invariant mass parameter to each string. We emphasize once more that *what* specific representation of the hierarchy we generate in this way is irrelevant; any "run" of PROGRAM UNIVERSE will be good enough for us.

It should be noted that in a strict sense this way of arriving at the hierarchy is not "constructive." What we do is to go through a procedure which allows us to recognize that the program has generated some bit string representation of the hierarchy. This recognition program is internal to a part of the computer memory, and is not used explicitly in the way we go on to set up rules of correspondence and physical interpretation; it in no way affects the running of the basic program and was coded only in order to show that we could do it. The new *Universe Program* being written by McGoveran will, instead, be strictly constructive and will generate its own stop rule for the label-content separation, rather than putting it in from the outside. This has no immediate consequences other than satisfying the rule of parsimony, but will tie down our cosmology more firmly than the current *Program Universe* does. The *event* definition which we have explained above $[(abc) = (0); (abcd) = (0)]$ will continue to be rigorously applicable.

Each event occurs in a TICK, which increases the complexity of the universe in an irreversible way. Our theory has an ordering parameter (N_U) which is conceptually closer to the "time" in general relativistic cosmologies than to the "reversible" time of special relativity. The arbitrary elements in the algorithm that generates events preclude *unique* "retrodiction," while the finite complexity parameters (SU, N_U) prevent a combinatorial explosion in *statistical* retrodiction. In this sense, we have a *fixed*—though only partially retrodictable—*past* and a necessarily *unknown future* of finite, but arbitrarily increasing, complexity. Only structural characteristics of the system, rather than the bit strings used in computer simulations of pieces of our theory, are available for epistemological correlations with experience.

What was *not* realized when this program was created was that this simple algorithm provides us with the minimal elements needed to construct a finite particle number scattering theory. The increase in the number of strings in the universe by the creation of novel strings from discrimination is our replacement for the "particle creation" of quantum field theory. It is not the same, because it is both finite and irreversible; it also changes the "state space." Note that the string length N_U is simply the number of TICKs that have occurred since the start up of the universe; this order parameter is irreversible and monotonically increasing, like the cosmological "time" of conventional theories. Our events are unique, indivisible and global, in the computer sense; consequently, events cannot be localized and will be "supraluminally" correlated.

7. "VECTOR" CONSERVATION LAWS

So far we have a gross structure based on bit strings, and two operations which generate them via a specific program: (1) ADJOIN, which adjoins a nonnull string produced by discrimination to the extant bit string universe, and (2) TICK which increases the string length by concatenating a single bit, arbitrarily chosen for each string, at the growing end of each string. We have two kinds of connectivity which result from this construction. One is the label-content schema. Once the label basis has closed under discrimination to form 2+3+7+127 linearly independent strings, program universe will necessarily generate some representation of the combinatorial hierarchy at that label length; this will close with $3 + 7 + 127 + 2^{127} - 1$ labels of that length. Once the label basis (and label string length) is fixed, program universe assigns each novel content string to a specific label when it is created by discrimination, and augments each content string by an arbitrary bit at each TICK. The second is the connectivity between strings of the same length (i.e., "between ticks") which we have characterized as 3–vertices $(abc)_{L+n} = (0)_{L+n}$ and 4–events $(abcd)_{L+n} = (0)_{L+n}$.

To come closer to what we need for physics in the sense of relating the (R–frame) model to measurement ("counting") in the laboratory, we need to introduce a quantitative measure and a norm for such measures. Once we have done this, we can introduce a third operation connecting bit strings ("inner product") that supports *relative* conservation laws. Define a measure $\|x\|$ on (x) by

$$\|x\| \equiv \Sigma_{i=1}^{n} b_i^x, \qquad x \in a, b, c \ldots .$$

This is the usual Hamming measure. $\|x\|/n$ is McGoveran's normalized attribute distance relative to the reference string (0) ($b_i^0 = 0$ for all i; $\|0\| = 0$), and $(n - \|x\|)/n$ is the distance relative to the antinull string (1) ($b_i^1 = 1$ for all i; $\|1\| = n$).

Consider a *3-vertex* defined by $(abc) = (0)$ or, equivalently, by $\|abc\| = 0$.

Theorem 1: The measure $\|x\|$ is a norm, i.e.,

$$(abc) = (0) \Rightarrow | \; \|a\| - \|b\| \; | \leq \|c\| \leq | \; \|a\| + \|b\| \; |, \quad \text{cylic on } a, b, c \; .$$

Argument:

From the definition of discrimination, if we consider the three bits at any ordered position i in the three strings of a 3–vertex, we can only have either one zero and two ones's in the three strings, or three zeros. If the single zero is $b_i^a = 0$, call the number of times this occurs n_{bc} (cyclic on a, b, c) and the number of times we have three zero's n_0. Clearly, $n_{bc} + n_{ca} + n_{ab} + n_0 = n$ and $\|a\| = n_{bc} + n_{ca}$, cyclic on a, b, c, from which the desired inequalities follow.

Note that this theorem depends on a computer memory. It is *static* in that it depends only on a particular type of configuration that is "wired in" by the program. It is *dynamic*, in the sense that the three strings are brought together as a consequence of past sequences that are *arbitrary* from the point of view of the local vertex. It is *global* in that any single 3–vertex (or 4–event) *could* lead to a TICK which affects the whole bit string universe.

If we now define the inner product $\langle (x)\cdot(y)\rangle$ between two strings $(a),(b)$ connected by a 3–vertex $(abc)=(0)$ with the equality

$$2\langle (a)\cdot(b)\rangle \equiv \|a\|^2+\|b\|^2-\|c\|^2 \;;$$

it follows immediately that

Corollary 1.1:

$$\|ab\|^2 = \langle (a)\cdot(ab)\rangle + \langle (b)\cdot(ab)\rangle = \langle (ab)\cdot(ab)\rangle \;,$$

$$\|a\|^2 = \langle (ab)\cdot(a)\rangle + \langle (b)\cdot(a)\rangle = \langle (a)\cdot(a)\rangle \;,$$

$$\|b\|^2 = \langle (ab)\cdot(b)\rangle + \langle (a)\cdot(b)\rangle = \langle (b)\cdot(b)\rangle \;.$$

If we define a *4-vertex* by $(abcd)=(0)$, or equivalently by $\|abcd\|=0$, with an obvious extension of the notation, it also follows that

Theorem 2:

$$(abcd)=(0) \Rightarrow \|a\| = \|bcd\|, \quad \textit{cyclic on abcd} \;.$$

$$\|ab\| = \|cd\|;\; \|ac\| = \|db\|;\; \|ad\| = \|bc\|$$

Argument:

$(abcd)=(0) \Rightarrow (abc)=(d)$, etc., and $\Rightarrow (ab)=(cd)$, etc., from which the result follows.

Corollary 2.1: For any pair taken from the ensemble $abcd$, the appropriate version of Corollary 1.1 follows.

Corollary 2.2:

$$\langle (a)\cdot(cd)\rangle + \langle (b)\cdot(cd)\rangle \;=\; \|ab\|^2 \;=\; \|cd\|^2 \;=\; \langle (c)\;\cdot\;(ab)\rangle + \langle (d)\cdot(ab)\rangle \;,$$

and so on, for any of the three pairs.

Theorem 3:

$$\|abcd\| = 0 \Rightarrow \|a\|^2 \;=\; \langle (b)\;\cdot\;(a)\rangle + \langle (c)\cdot(a)\rangle + \langle (d)\cdot(a)\rangle \;, \quad \textit{cyclic on abcd} \;.$$

Argument:

This follows by standard (finite!) algebra.

It is tempting to go from these results for the inner product to the conclusion that a 4–vertex defines the vector conservation law

$$\vec{a}+\vec{b}+\vec{c}+\vec{d} \;=\; 0 \;,$$

and that with $\vec{d}=0$, the same is true at a 3–vertex. This, however, depends on a convention. If some vectors are "incoming" and some are "outgoing," the same algebraic relations can be interpreted as also supporting the three interpretations

$$\vec{a}+\vec{b} \;=\; \vec{c}+\vec{d};\; \vec{a}+\vec{c} \;=\; \vec{b}+\vec{d};\; \vec{a}+\vec{d} \;=\; \vec{b}+\vec{c} \,,$$

and the four interpretations

$$\vec{a}+\vec{b}+\vec{c} \;=\; \vec{d}\;; \quad \text{cyclic on } a,b,c,d \;.$$

We can base our version of "crossing invariance on these eight interpretations.

To go from what we have proved above to the usual definition of directions and angles in a "vector space" would require us to derive (among other things) rational fractions for the sines of angles whose cosines are also given by rational fractions. As Pythagorous is often credited with discovering, this problem cannot always be solved in the space of rational fractions (though, of course, he didn't put it that way). We have conservation laws at vertices; they are not *vector* conservation laws in the continuous, directional sense.

We can use "directions" to model experimental (laboratory) facts with reasonable precision. This amounts to a *rule of correspondence*: the *counter paradigm*, Chapter 9. Here we assume that at some string length N, we have either a 3–vertex or a 4–vertex involving a labeled string $(a)_N$ and that we have a second vertex involving the same label and a string $(a)_{N+n}$. From now on, we consider the latest portion of the string of length n and interpret it as a "random walk" in which a "1" represents a step in the + direction and a "0" a step in the − direction. Then the "distance" between the two vertices can be defined as $2\|a\| - n$. The direction is established macroscopically by thinking of the vertices as sequential "counter firings" involving the same "particle" separated by distance L and a (positive) time interval T. Since, empirically, such events always define a velocity $V = L/T$ less than or not measurably different from the limiting velocity c, we relate this type of laboratory fact to our bit string model by taking $V = \beta c$ with $\beta_a = 2\|a\|/n - 1$, and the positive direction along this line defined by the positive sign for β.

Since a 4–vertex $(abcd) = (0)$ can be decomposed in seven different ways, namely

$$(ab) = (cd)\ ; \quad (ac) = (bd)\ ; \quad (ad) = bc)\ ;$$

$$(a) = (bcd)\ ; \quad (b) = (cda)\ ; \quad (c) = (dab)\ ; \quad (d) = (abc\ ;)$$

we can—by appropriate identification of the directions with sequential counter firings in the laboratory—make seven different temporally ordered interpretations of the single 4–vertex given above: three (2,2) channels, four (3,1) channels and the unobservable (4,0) channel. Note that all eight relationships are generated by one 4–vertex.

Our next step is to recall that we can always separate a string into two strings $(a)_{L+n} = (L_a)_L\|(A_a)_n$ where "$\|$" denotes string concatenation. We call the first piece the *label* and the second the *content*. There is a simple correlation between the two pieces. If we take some content string A_a with velocity $\beta_a = 2\|A_a\|/n - 1$, the string $(a1)$ has the opposite velocity. Further, if we use the string (a) as the reference string for a conservation law defined by the inner product relations given above, the reversal of the velocity achieved by discrimination with the antinull string also reverses all the label conservation laws. However any system of bit strings has a *dual* system with all zeros and ones interchanged, but precisely the same algebraic structure. Thus, the theory is invariant under the *arbitrary* choice of reference direction and the *arbitrary* choices of the dichotomous reference symbols in the label, provided they all reverse on this same interchange.

8. THE STANDARD MODEL FOR QUARKS AND LEPTONS USING COMBINATORIAL HIERARCHY LABELS

Physical interpretation of the labels naturally starts with the simplest structures, which are the weak and electromagnetic interactions. We can get quite a long way just by looking at the leading terms in a perturbation theory in powers of $e^2/\hbar c \simeq 1/137$ for quantum electrodynamics and of $G_F \simeq 10^{-5}/m_p^2$ for the low energy weak interactions, such as beta decay. As Lee and Yang saw, if the neutrino is massless and chiral, the Fermi β–decay theory will violate parity conservation maximally; this is still the simplest accurate description of low-energy, weak interactions.

Since *level 1* has only two basic entities, we identify these with the neutrino ν and the antineutrino $\bar{\nu}$. Their closure is the zero helicity component of the spin–1 neutral weak boson Z^0, defining the 3–vertex $(\nu\bar{\nu}Z^0)$. If we follow the usual convention of defining the chirality of the neutrino as "left-handed," once we have added content strings and defined directions, we still need a convention as to whether the label is to be concatenated with the string $(1)_n$ with velocity $+c$ or the string $(0)_n$ with velocity $-c$. We can take the bit string state $(\nu_L)_{L+n} = (\nu_\lambda)_L\|(1)_n$ and the right-handed (i.e., anti) neutrino $(\nu_R)_{L+n} = (\bar{\nu}_\rho)_L\|(0)_n$. Then, if we use a representation in which $(\nu_\rho)_L = (1\nu_\lambda)_L$, the Feynman rules will be obeyed. The vertex can be interpreted as representing the physical processes $\nu_L + \nu_R \leftrightarrow Z_0^0$, $\nu_L \leftrightarrow \nu_L + Z_0^0$ or $\nu_R \leftrightarrow \nu_R + Z_0^0$, depending on context. Taking all three particles as incoming (or outgoing), the quantum numbers add to zero—as they should—while, if we reverse the direction of either neutrino to make it outgoing, it becomes the same as the incoming neutrino. Note that for massless particles ($\beta = \pm c$), we cannot specify a direction until we connect them to slower particles whose directions can be assigned. Thus we are forced to adopt a Wheeler–Feynman type of theory, in which all massless "radiation" emitted by charged particles must be absorbed; we will see later that charged particles must be massive and, hence, must have $|\beta| < c$.

Interpretation of *level 2* as modeling the vertices of quantum electrodynamics for electrons, positrons and photons is almost as easy. We take as the linearly independent basis strings $(e_\lambda^+), (e_\lambda^-), (\Gamma_{\lambda\lambda})$ and define the nonnull string which guarantees their independence as $(\Gamma_c) = (e_\lambda^+ e_\lambda^- \Gamma_{\lambda\lambda})$. The remaining three label strings which close essi level 2 are then defined by

$$(e_\rho^+) = (\Gamma_c e_\lambda^-)\ ; \qquad (e_\rho^-) = (\Gamma_c e_\lambda^+)\ ; \qquad (\Gamma_{\rho\rho}) = (\Gamma_c \Gamma_{\lambda\lambda})\ .$$

We take the same convention for positive direction and chirality as we did for *level 1*, using the negative, left-handed electron as our reference string and the velocity $\beta_{e_L^-} = 2k_{e_L^-}/n - 1$ as positive when this number is positive. The physical states, where we omit the subscripts on β, are then given by

$$(\gamma_c)_{L+n} = (\Gamma_c)_L \| (1)_n\ ; \quad (e_L^-) = (e_\lambda^-)\|(-\beta)_n\ ; \quad (e_L^+) = (e_\lambda^+)\|(-\beta)_n\ ,$$

$$(e_R^+) = (e_\rho^+)\|(\beta)_n = (\gamma_c e_L^-)\ ; \quad (e_R^-) = (e_\rho^-)\|(\beta)_n = (\gamma_c e_L^+)\ ,$$

$$(\gamma_{RR}) = (\Gamma_{\rho\rho})\|(1)_n\ ; \quad (\gamma_{LL}) = (\Gamma_{\lambda\lambda})\|(0)_n = (\gamma_c \gamma_{RR})\ ,$$

and the Feynman rules are obeyed for all 3–vertices.

The 4–vertex $(e\bar{e}\gamma\gamma_c) = (0)$ cannot be readily discussed until we have the configuration space theory nailed down. It is related to our finite treatment of Bremstrahlung in a "coulomb field." The vertex $(\gamma_{LL}\gamma_{RR}\gamma_c) = (0)$ would seem to imply an interaction between photons and the "coulomb field,"—a vertex that vanishes in the conventional theory because of the masslessness of the photon and gauge invariance.

A related problem arises with the vertices implied by our connection between particles and antiparticles, namely

$$(\nu\bar{\nu}\,1) = (0)\ ; \quad (e\bar{e}\,1) = (0)\ ; \quad (\gamma\bar{\gamma}\,1) = 0\ .$$

A little thought shows that such vertices will occur for *any* particle-antiparticle pair. Hence, the antinull label string "interacts" with everything and must be assigned to *level 4*. This unique label string, which occurs with probability $1/(2^{127} + 136)$, is identified with Newtonian gravitation. It leads to the bending of light in a "gravitational field," as we will show at a later stage in the development of the theory. Of course, to get the experimentally observed result, we will have to identify the "spin–2" gravitons as well, and show that they double this deflection.

These problems will have to be deferred until we have articulated the theory further. We conclude this article by identifying the *level 3* structure with the quarks and gluons of quantum chromodynamics. This discussion follows along the lines already laid down in discussing the first two levels. We take as our basis label strings a quark part $(u^+), (u^-), (d^+)$ or (d^-) concatenated with a color part $(r), (y), (b)$, which gives us the seven independent strings needed to form *level 3*. The color strings are linearly independent, so we can define (analogous to what we did at *level 2*)

$$(ryb) = (w) ; \quad (\bar{r}) = (rw) ; \quad (\bar{y}) = (yw) ; \quad (\bar{b}) = (bw) ,$$

from which it follows that

$$(ry\bar{b}) = (0) ; \quad (r\bar{y}b) = (0) ; \quad (\bar{r}yb) = (0) ; \quad (\bar{r}\bar{y}\bar{b}) = (0) .$$

Similarly, the linear independence of the quark parts allows us to define

$$(u^+u^-d^+d^-) = (Q) ; \quad (\bar{q}) = (qQ) , \qquad q \in u^+, u^-, d^+, d^- .$$

Then a colored quark label $(q_c^{\pm}) = (q^{\pm})\|(c)$ and a colored gluon label $(g_c) = (Q)\|(c)$, $c \in r, y, b$ allow us to recognize the label part of the Yukawa vertex for QCD as $(q_{c_1}\bar{q}_{c_2}g_{c_3}) = (0)$. The essential point here is that, as proved above, $(c_1\bar{c}_2c_3) = (0)$ for any three distinct colors. We can then attach content labels and helicity in the same way we did in QED, and once again the Feynman rules apply. Any one familiar with lowest order QCD can now immediately derive from our formalism the "valence quark" structure of the proton and neutron in terms of three quarks, and the structure of the π, ρ and ω in terms of quark-antiquark pairs. In contrast to the *level 2* situation, the 3–gluon vertex does not vanish and implies a 4–gluon vertex, so we find that we have constructed *all* the lowest order vertices of QCD with the correct conservation laws.

The problem of "color confinement" is solved, in principle, by *McGoveran's Theorem* [24,25]; i.e., the conclusion that in any finite and discrete theory there can be no more than three "homogeneous and isotropic dimensions" that remain indistinguishable as the (finite and discrete) cardinals and ordinals keep on increasing. (We discuss this theorem with more care in Chapter 9.) Because our labels are tied to contents and, hence, via the counter paradigm to macroscopic directions, we can only have three quantum number "dimensions" asymptotically. These are saturated by the three absolutely (so far as we know currently) conserved quantum numbers: lepton number, baryon number and charge (or "z–component" of isospin), leaving no room for free quarks or gluons conserving asymptotic "color charge." They can occur at short distance as degrees of freedom in the scattering theory—as we showed above—but eventually they have to "compactify" and become distinguishable from free particle quantum numbers. We can conclude this immediately without any detailed dynamical argument.

9. THE COUNTER PARADIGM

Bastin has insisted for decades that the primal contact between a (computable) formalism and the empirical "world" can only be made once. This was a basic reason why he and Kilmister [18,19] fastened on steps of a scattering process as a likely point at which to investigate the connection between finite mathematics and physical theory. I started thinking of the elementary scattering process as fundamental, thanks to my early involvement in Chew's S–Matrix theory; for me this gave specific content to Bridgman's operationalism and Heisenberg's very early ideas. At ANPA 2 and 3 some of us saw that Stein's "random walk" derivation of the Lorentz transformation and the Uncertainty Principle [26] must somehow connect to scattering processes; others recognized the seminal nature of his work because of his ontological viewpoint.

The specific genesis of the "counter paradigm" occurred after my presentation [27] at the conference honoring deBroglie's 90^{th} birthday. Fortunately, I had an opportunity to start working on the final version of that paper [28] in consultation with Ted Bastin before it was published. I realized that if I thought of Stein's "random walk" as a model for two sequential events in two spatially separated laboratory counters with the discrete step length being the deBroglie relativistic phase wavelength, that by representing Stein's random walks as bit strings with the bit 1 taken as a step toward the final counter and the bit zero a step away from it, I had the right point of contact between the bit strings used in the *combinatorial hierarchy* and the start of a scattering theory.

So far we have only discussed 3– and 4–vertices for a fixed value of n, but each time program universe TICKs, each content string in each labeled ensemble acquires an arbitrary bit at the growing end. In the absence of further information, each content string therefore represents a sequence of Bernoulli trials, with 0 and 1 representing the two possibilities. This has an extremely important consequence, which we call *McGoveran's Theorem* [24,25]. As has been noted by Feller [29], if we have D *independent* sequences of Bernoulli trials, the probability that after n trials we will have accumulated the same number (k) of one's is $p_D(n) = (\frac{1}{2^{nD}})\Sigma_{k=0}^{n}\binom{n}{k}^D$. He then shows that the probability that this situation will repeat N times is strictly bounded by

$$P_D(N) \;=\; \Sigma_{n=1}^{N}\, p_D(n) < \left[\frac{2}{\pi D}\right]^{-\frac{1}{2}}\; \Sigma_{n=1}^{N}\, n^{\frac{1}{2}(D-1)}\ .$$

Consequently, for $D = 2,3$ where $p_D(n) < n^{-\frac{1}{2}}, n^{-1}$, such repetitions can keep on occurring with finite probability; but for four or more independent sequences, this probability is strictly bounded by zero in the sense of the law of large numbers.

McGoveran uses finite attributes, which can always be mapped onto ordered strings of zeros and ones, as the starting point for his ordering operator calculus. As is discussed in more detail in Ref. [1], these can be used to construct a finite and discrete metric space. In order to introduce the concept of *dimensionality* into this space, he notes that we need some metric criterion that does not in any way distinguish one dimension from another. (In a continuum theory, we would call this the property of "homogeneity and isotropy"; we need it in our theory for the same reason Einstein does in his development of special relativity.) McGoveran discovered that by interpreting the coincidences $n = 1, 2, \ldots, N$ in Feller's construction as "metric marks," the metric space so constructed has precisely the discrete property corresponding to "homogeneity and isotropy" as just defined. Consequently, Feller's result shows that in *any* finite and discrete theory, the number of independent "homogeneous and isotropic" dimensions is bounded by three! If we start from a larger number of independent dimensions using *any* discrete and finite generating process for the attribute ensembles, we find that the metric will, for large numbers, continue to apply to only three of them, and that what may have looked like another dimension is not; the probability of generating the next "metric" mark in any of the others (let alone all of them) is strictly bounded by $1/N_{MAX}$!

Of course, the argument depends on the theory containing a *universal ordering operator* which is isomorphic to the ordinal integers. Further—since we know empirically that "elementary particles" are *chiral*—we will need three, rather than two "spatial" dimensions. Thus *any* discrete and finite theory such as ours, when applied to physics, must be globally described by three dimensions and a monotonically increasing order parameter. Consequently, we are justified in constructing a "rule of correspondence" for our theory which connects the large number properties of our R–frame to *laboratory* (E–frame) 3 + 1 space-time. Earlier treatments of the "counter paradigm" simply took this possibility for granted. McGoveran's Theorem fills this serious logical gap.

We begin with the paradigmatic case of a single particle entering a space-time volume (detector) $\Delta V \Delta T$, causing a count and a time T, later entering a second detector with similar resolution a macroscopic distance L from the first and causing a second count. We then say that the (average) velocity of the particle between the two detectors is $V = L/T$; empirically, this number is always less than or indistinguishable from the limiting velocity c.

This language is well-understood by the particle physics experimentalist, but raises a number of problems for others. To begin with he uses "cause" in a philosophically vague but methodologically precise sense, which includes a host of practical experience about "background," "spurious counts," "real counts," "goofs," "GOK's" (i.e., "God only knows"), ...

The actual practice of experimental particle physics implies the concept of *indistinguishability* in a critical way; the experimentalist uses, often without conscious analysis, finite collections whose cardinal number may exceed their ordinal number; this fact is diagnostic for *sorts* that are not reducible to *sets* [24]. To put it more formally in terms of "background" and "counts," in the absence of a constructive definition of the two subsets—which is often unavailable in practice, and in our theory we would claim can be unavailable in principle—the two collections are *sorts* rather than *sets*.

The rule of correspondence in the counter paradigm case (two sequential counts spatially separated) applies to a labeled string with label L_a, which at the TICK with the content string length n_0 was part of a 3- or 4-vertex and, again, part of a vertex at content string length n_0+n_a, AND WHICH IS APPROPRIATELY ASSIGNED TO THEORETICALLY RELEVANT DATA RATHER THAN TO BACKGROUND. We ask how many one's were added to the content string; we call these k_a. We identify the (average) laboratory velocity of the particle ($V = L/T$) with the R-frame quantity by the equation $V = [(2k_a/n_a) - 1]\,c$. The sign of this velocity defines the positive or negative sense of the direction between the counters in the laboratory (or visa versa: a choice must be made *once*). Since the evolution of the bit string universe will provide many candidates for the strings which meet these criteria within the time and space resolution of the counters, we will have to provide more and more precise definitions of these criteria as the analysis develops.

10. EVENT-BASED COORDINATES AND THE LORENTZ TRANSFORMATIONS

As is discussed with much more care in Ref. [1], any theory satisfying our principles can be mapped onto ensembles of bit strings simply because, with respect to *any* attribute, we can say whether a collection has that attribute or does not. To introduce a metric, we need a distance function *relative* to some reference ensemble. Because of our finite and discrete principles, any allowed program can only take a finite number of steps to bring any ensemble into local isomorphism with the reference ensemble *in respect to that attribute.* Note that there can be many attributes, many distance functions, and that the space can be multiply connected. Note also that this definition also provides a (dichotomous, e.g., $\pm$) *sense* to the computation steps: they must increase the attribute distance or decrease it. Calling the number of increments I and the number of decrements D, using a well-defined computational procedure, the attribute distance is, clearly, $D_A = I - D$, and the total number of steps $N = I + D$. Then we can also define the *attribute velocity* with which the two ensembles are "separating" or "coming together," $V_A = (I - D)/(I + D)$. Thus, there always is a "limiting velocity" for each attribute, which is attained when all steps are taken in the same direction.

If we wish to model the events of which contemporary physics takes cognizance, we know that all physical attributes are directly or indirectly coupled to electromagnetism. Therefore, the limiting velocity of physics, c, will be the *smallest* of these limiting attribute velocities, simply because it refers to the attribute with the maximum cardinality. Any ensemble of attributes specified by a more limited description involves a "supraluminal" velocity, without allowing supraluminal communication of information. Hence, we can expect to find correlation between and synchronization of events in space-like separated regions; from our discrete point of view, the existence of the effects demonstrated in Aspect's and other EPR-Bohm experiments is anticipated and in no way paradoxical. We guarantee Einstein locality for *causal* events; that is, for those initiated by the transfer of *physical* information [30].

In order to go from this general proof of the limiting velocity to the laboratory practice of relativistic particle quantum mechanics, we need a more specific formalism than the general derivation given in Ref. [1]. We start from the 3- and 4-vertices already mentioned and consider how they can be used to model the "laboratory" situation given in Fig. 11. The initial 4-vertex $(abcd)_{L+n_0} = 0$ is followed sequentially by five vertices involving "soft" photons. In the laboratory neither vertices, nor elementary events, nor soft photons can be observed; limiting cases in which the disturbance caused by the firing of counters connected with these five events is negligibly small are easy to envisage. We use a specific

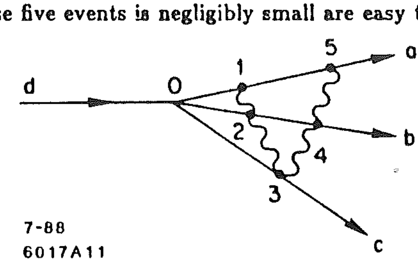

Fig. 11. A 4-event followed by five events involving limiting velocity signals which can be used to establish the Lorentz transformations for Event 3.

example of labels that can, if we wish, be given a specific interpretation in which particles a, b, c have spin-1/2 and the photons have left or right spin-1 helicity.

We assume that it takes n_i TICKs of program universe beyond $L+n_0$ to generate the strings involved in the i^{th} event. Since all strings will have the portion through content string length n_0 unaltered, we need use only these *relative* values: $n_i = N_U(i) - L - n_0$ and the corresponding terminal pieces of the strings for our contents. For Event 1, we take the three strings to be

$$(a) = (1000)\|(A_1^a)_{n_1} \; ; \quad (a') = (0100)\|(A_1^a)_{n_1} \; ; \quad (\bar{\gamma}) = (1100)\|(0)_{n_1} \; .$$

Hence, $(aa'\bar{\gamma}) = (0)$ defines a 3-vertex in which the velocity of a does not change; we could call it a "soft photon" vertex. By crossing (cf., Chapters 3 and 7 above), this also can be interpreted as a vertex in which a flips its spin and emits a photon with the appropriate helicity; i.e., $(\gamma) = (0011)\|(1)_{n_1}$. The laboratory direction between Events 1 and 2 then defines the reference direction for all subsequent discussion. The remaining vertices can be consistently represented by using

$$(b) = (1000)\|(A_2^b)_{n_2} \; ; \quad (\gamma) = (0011)\|(1)_{n_2} \; ; \quad (b') = (0111)\|(A_2^b)_{n_2} \; ;$$

$$(\gamma') = (1100)\|(1)_{n_2} \; ;$$

$$(c) = (1000)\|(A_3^c)_{n_3} \; ; \quad (\gamma') = (1100)\|(1)_{n_3} \; ; \quad (c') = (0111)\|(A_3^c 1)_{n_3} \; ;$$

$$(\bar{\gamma}') = (0011)\|(0)_{n_3} \; ,$$

$$(b') = (0111)\|(A_4^{b'})_{n_4} \; ; \quad (\bar{\gamma}') = (0011)\|(0)_{n_4} \; ; \quad (b'') = (1000)\|(A_4^{b'})_{n_4} \; ;$$

$$(\bar{\gamma}) = (1100)\|(0)_{n_4} \; ;$$

$$(a') = (0100)\|(A_5^{a'})_{n_5} \; ; \quad (a'') = (1000)\|(A_5^{a'})_{n_5} \; ; \quad (\bar{\gamma}) = (1100)\|(0)_{n_5} \; .$$

We now trust that our rule of correspondence between 3- and 4-vertices and a standard "laboratory" situation used in the derivation of the Lorentz transformations is clear.

For simplicity, we consider here that particle a is, *on the average*, "at rest" between Events 0, 1 and between Events 1, 5:

$$k_0^a = \frac{n_0}{2} \; ; \qquad k_1^a = \frac{n_1}{2} \; ; \qquad k_5^a = \frac{n_5}{2} \; .$$

We also assume, again *on the average*, that b and c have constant velocity over the appropriate intervals:

$$\beta_b = 2\frac{k_0^b}{n_0} - 1 = 2\frac{k_2^b}{n_2} - 1 = 2\frac{k_4^b}{n_4} - 1 \; ,$$

$$\beta_c = \beta = 2\frac{k_0^c}{n_0} - 1 = 2\frac{k_3^c}{n_3} - 1 \; .$$

Our next simplification is to assume that all the events lie on a single "line," reducing this to a 1+1 dimensional problem. None of these simplifications are needed, as can be seen from the general discussion in Ref. [1].

In conventional terms, we are asking the question of how the coordinates of an event at $x = \beta ct$ in one coordinate system (the one in which particle a is at rest) transform to the coordinate system in which particle b is at rest. We assume, as in conventional treatments, that the velocity of light is the same in all coordinate systems and that the time at which Event 3 occurs is the average between when the light signal that defines Event 3 was emitted by a and returns to it. Introducing a parameter with the dimensions of length, whose value we will discuss later, these statements follow immediately from the definitions of attribute distance and velocity, since

$$\frac{x}{\lambda} = 2k - n \; ; \qquad \frac{ct}{\lambda} = n \; ; \qquad \beta = \frac{2k}{n} - 1 \, ,$$

for any particle, and $k = 0$ or n specifies a connection with the limiting velocity for any set of strings. This is even clearer when we introduce "light cone" coordinates:

$$d_+ = n + (2k - n) = 2k \; ; \qquad d_- = n - (2k - n) = 2(n - k) \; .$$

The relationship between the two descriptions is illustrated in Fig. 12.

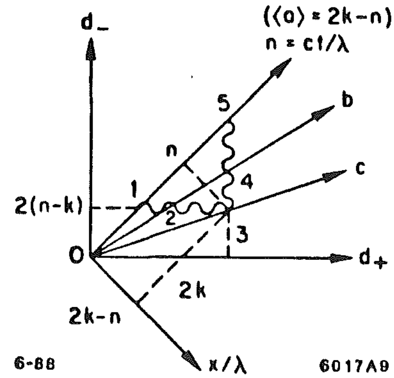

Fig. 12. The connection between space-time and light-cone coordinates in terms of bit string distances and velocities for the physical situation envisaged in Fig. 1.

One way to derive the Lorentz transformations is to require that the interval s between Events 0 and 3 be invariant, where

$$\frac{s^2}{\lambda^2} = \frac{c^2t^2 - x^2}{\lambda^2} = n^2 - (2k - n)^2 = 4k(n - k) \; .$$

In light cone coordinates this relationship becomes

$$d_+ d_- = 4k(n - k) = \frac{s^2}{\lambda^2} \, ,$$

which makes one way of insuring the invariance requirement particularly simple, namely

$$k' = \rho k \, , \quad n' - k' = \rho^{-1}(n - k) \Rightarrow 4k'(n' - k') = 4k(n - k) \; .$$

Note that if we are to compare the integer *bit string* coordinates, this restricts k' to be a rational multiple of k. One of the great successes of our theory is precisely this restriction that keeps events an integral number of deBroglie wavelengths apart. A fundamental explanation of why our theory can contain "interference" phenomena starts here.

If we now note that

$$d_{\pm} = (1 \pm \beta)\, n \,,$$

the invariance requirement gives us

$$\frac{k'}{k}\,\frac{n-k}{n'-k'} = \rho^2 = \frac{1+\beta'}{1+\beta}\,\frac{1-\beta}{1+\beta'} \,.$$

Hence,

$$\beta_\rho = \frac{\beta'-\beta}{1-\beta\beta'} \iff \rho^2 = \frac{1+\beta_\rho}{1-\beta_\rho} \,.$$

From the fact that when transforming from a system at rest $d_+/d_- = 1$, we see that the relative velocity between the two systems is simply β_ρ; we have derived the velocity composition law for rational fraction velocities in any system. Tom Etter arrived at this composition law for attribute velocities on general grounds, as is discussed in Ref. [1]. With

$$\gamma = \frac{1}{2}\left[\rho + \rho^{-1}\right] \,,$$

we have that

$$x' = \gamma(x + \beta_\rho ct) \,; \qquad t' = \gamma(ct + \beta_\rho x) \,.$$

QED

11. QUANTUM MECHANICS

Program universe provides an *invariant* significance for the label strings, once they close (in some length, with at least 139 bits) to form some basis for some realization of the combinatorial hierarchy. For each of the $2^{127} + 136$ labels L_ℓ, we can assign a dimensional parameter λ_0^ℓ, which is the step length when the particle is "at rest"; i.e., when on the average $2k_\ell = n_\ell$. Since program universe increases the string length one arbitrary bit at a time, this requirement can at best be satisfied only at every other step. We have seen that when all steps are in the same direction (i.e., when the content string is either the null string or the antinull string), this corresponds to a "light signal." In any string evolution, all steps are executed at the limiting velocity c—a finite and discrete "zitterbewegung." The invariance of λ_0^ℓ allows us to associate with each label an invariant parameter with the dimensions of mass m_0^ℓ, and relate the two by $\lambda_0^\ell = h/m_0^\ell c$, where h is a universal constant with the dimensions of action. We will now show that h can indeed be identified with Planck's constant.

The extension of our Lorentz transformations to momentum space is now immediate. We simply define $E = \gamma m_0 c^2$, $p = \gamma\beta m_0 c$. For $p_\pm = E/c \pm p$, we have $p_+p_- = m_0^2c^2$, $p_+/p_- = k/(n-k)$ and $(p_+x_- + p_-x_+)/2 = Et - px$. The justification of calling this "momentum" is more than definitional; we showed above that 3- and 4-vertices support "vector" conservation laws and "crossing symmetry." We have 3-momentum conservation in any allowed event-based reference frame. Clearly, $m_0 c\lambda_0 = h = E\lambda/c$ in any allowed coordinate system, and we have recovered the initial identification of the step length in the "random walk" as $\lambda = hc/E$, the deBroglie phase wavelength with which our initial statement of the "counter paradigm" began. We can now *derive* the quantum mechanical commutation relations from our model.

We note that if we consider a system that evolves with constant velocity $\beta_0 \equiv 2k_0/n_0 - 1$, strings which grow subject to this constraint, i.e., $n = n_T n_0, k = n_T k_0, 1 \le n_T \le n/n_0$, will have a periodicity $T \equiv n_T \Delta t = n_T\lambda/c$ specifying the events in which this condition can be met. Hence, in more complicated situations where there can be more than one "path" connecting strings with the same velocity to a single event, this event can occur only when the paths differ by an integral number of "d-wavelengths" λ. Thus, our construction already contains the seeds of "interference" and a conceptual explanation of the "double slit experiment."

We have already seen that any system with "constant velocity"—at those "ticks" when events can occur—evolves by discrete steps $\pm\lambda_a$ in $x = q_a$ *between* ticks. McGoveran's ordering operator calculus [1] which specifies the connectivity between events allows these discrete happenings to occur in a *void* where space and time are meaningless. Since $\lambda/\Delta t = c$, each step occurs forward or backward with the limiting velocity; thus, we deduce a discrete *zitterbewegung* from our theory. If we think of this as a "trajectory" in the pq phase space, each time-step induces a step $\pm\lambda$ in q correlated with a step $\pm mc$ in p. Even in the case of a particle "at rest," this must be followed by two steps of the opposite sign to return the system to "rest." Thus there is, minimally, a 4-fold symmetry to the "trajectory" in phase space corresponding to the generation periodicity we discovered above.

If we now recall from classical mechanics [31] that for any momentum which is a constant of the motion, we can transform to angle and action variables, with $\oint p_J dq_J = J$ where J has the dimensions of action, $p_J = J/2\pi$ and q_J is cyclic, we have an immediate interpretation. In the classical case, the "period" goes to infinity for a free particle; for us, we have already seen that we have a *finite* period $T = \lambda/c$. Therefore, we can immediately identify $m_a c\lambda_a = J = n_T h$; we have constructed Bohr–Sommerfeld quantization within our theory.

To go on to the commutation relations, we can replace the geometrical description of periodic trajectories in phase space by using complex coordinates $z = (q, ip)$ [or by $(q_J, in_T h/2\pi)$, where q_J is restricted to $2n+1$ values with $-n_T \leq n \leq +n_T$]. Then the steps around the cycle in the order $qpqp$ are proportional to $\pm 2\pi\,(1, i, -1, -i)$, where $\pm$ depends on whether the first step is in the positive or negative direction or, equivalently, whether the circulation is counterclockwise or clockwise. We have now shown why $qp - pq = \pm i\hbar$ for free particles in our theory; this result holds for any theory satisfying our principles which uses a discrete free particle basis.

In order to go to a detailed 3–dimensional description, we must supply three linearly independent reference strings, define inner products with respect to them (cf., Chapter 7) and go to a "coordinate" description. There will then be three independent periodicities (velocities and momenta) which will commute with each other but not with their conjugate position variable. The commutation relations for angular momentum follow immediately. Since this has already been shown in quite general terms in Ref. [1], we will leave the details to future publications. An alternative is to develop the "radial coordinate" (n, l, m) description using "bound states" as the basis.

Now that we have two ($\hbar$ and c) of the 3–dimensional constants needed to connect a fundamental theory to experiment in the 3–space in which physics operates, and which we have proved must be the asymptotic space of our theory, all that remains is to determine the unit of mass. This has already been done for us by the combinatorial hierarchy result $2^{127} + 136 \simeq 1.7 \times 10^{38} \simeq \hbar c/Gm_p^2 = (M_{Planck}/m_p)^2$, which tells us that we can either identify the unit of mass in the theory as the proton mass, in which case we can calculate (to about 1% in this first approximation) Newton's gravitational constant or—if we take the Planck mass as fundamental—calculate the proton mass. From now on, we have to compute everything else. If we fail to agree with experiment to the appropriate accuracy (one of the rules of correspondence!), we must either revise or abandon the theory.

12. A DISCRETE MODEL FOR THE BOHR ATOM

We have seen that any bit string has the deBroglie periodicity h/mc^2 for each digital "time step" $\Delta n = 1$ and that, when it evolves with "constant velocity," also has the longer digital period n_0 connected to the velocity by $\beta = 2k_0/n_0 - 1$ at each finite "position" $N_{ph} n_0 \beta = N_{ph}(2k_0 - n_0)$, where an event can (but need not) occur after the initial vertex at $N_{ph} = 0$. Note that we are not interested in particles "at rest." We define $\Delta k_0 = k_0 - n_0/2$ and, hence, $\beta = \Delta k_0/n_0$. Only one integer can be added to the string at each step. This must happen Δk_0 times before the periodic pattern can be completed. Therefore, the number of step lengths in the periodic pattern—the *coherence length*—is $n_0 = 1/\beta$. Since, as we saw above, the step length is $\lambda = hc/E$, we find that the coherence length required for periodic phenomena at constant velocity is $\lambda_g = hc/\beta E = h/p$.

By adding a constraint representing a second periodicity, we can now model the periodicity representing a "closed orbit around some fixed center." Clearly, this periodicity must use the coherence length derived above if we are to have a stable, repeating pattern that starts from some "origin" and closes after N_B coherence lengths. This model, which only describes the average "motion," will persist from the time when we start the model off to the time when some vertex—for example the absorption of a "hard" photon—ends the finite sequence of periods. Of course, this can only occur at one of the positions allowed for events. In the average sense, we can image this "trajectory" as a regular polygon with N_B sides of length λ_g. With the usual "geometrical" image in mind, we call the distance traversed in this period "$2\pi R$"$= N_B\lambda_g$ and, hence, $mvR = N_B\hbar$. Afficionados of the early history of quantum mechanics will recognize that we have constructed a digital version of deBroglie's analysis of the geometry of the Bohr atom and produced a reason for angular momentum quantization. For the meaning of "π" in a discrete and finite theory, refer to the discussion in Ref. [1].

Although this part of the derivation of the Bohr atom should be reasonably familiar, our introduction of the "electromagnetic interaction" will be radically different from the conventional approach. We have seen above that the coulomb interaction is represented by only one out of 137 labels in the combinatorial hierarchy construction, and that strings evolve by the arbitrary selection of strings from memory to calculate the vertices (thanks to the counter paradigm, these vertices have now become "events"). In the case at hand, 136 of these choices can only provide a "background" which will cause fluctuations of the position of our particle; on the average these must cancel out. Only once in 137 times will the step correspond to the vertex that serves to keep the particle in its orbit. We can think of this as happening at the vertices of the polygon; i.e., N_B times in one full period. So, compared to the basic evolution time, we find that for this electromagnetic orbit, $\beta = 1/137N_B$. Making the hierarchy identification $137 = \hbar c/e^2$, our quantization condition derived above then gives us the standard result $R = N_B^2\hbar^2/me^2$ and an explanation of the old puzzle of why the Bohr radius is 137 times the Compton wavelength!

To calculate the binding energy, consider the energy change between this average motion and the particle at rest caused, for example, by the emission or absorption of a photon. We must use the average velocity because, in the absence of other information, we cannot know "where" in the orbit the interaction occurs. Our theory can readily accommodate emission and absorption of photons—conserving both momentum and energy—as we have seen in our derivation of the Lorentz transformations, and can include the usual recoil correction, if we so desire. Thus, we argue that the binding energy ϵ_{N_B} is related to the velocity $\beta_{N_B} = 1/137N_B$ by $(\epsilon_{N_B} + m_0c^2)^2 = m_0^2c^4/(1-\beta_{N_B}^2)$, from which all the usual results for the Bohr atom follow to order β^2.

13. SCATTERING THEORY

To construct a scattering theory, we need to provide the connectivity between events. To obtain a statistical connection between events, we start from our counter paradigm and note that, because of the macroscopic size of laboratory counters, there will always be some uncertainty $\Delta\beta$ in measured velocities, reflected in our integers k_a by $\Delta k = \frac{1}{2}N\Delta\beta > 0$. A measurement which gives a value of β outside this interval will have to be interpreted as a result of some scattering that occurred among the TICK's that separate the event (firing of the exit counter in the counter telescope that measures the initial value of $\beta = \beta_0$ to accuracy $\Delta\beta$), which defines the problem and the event which terminates the "free particle propagation"; we must exclude such *observable* scatterings from consideration.

What we are interested in is the probability distribution of finding two values k, k', within this allowed interval, and how this correlated probability changes as we TICK away. If $k = k'$, it is clear that when we start, both lie in the interval of integral length $2\Delta k$ about the central value $k_0 = \frac{N}{2}(1+\beta_0)$. When $k \neq k'$, the interval in which both can lie will be smaller and will be given by

$$[(k+\Delta k) - (k' - \Delta k)] = 2\Delta k - (k' - k) ,$$

when $k' > k$, or by $2\Delta k + (k' - k)$ in the other case. Consequently, the correlated probability of encountering both k and k' in the "window" defined by the velocity resolution, normalized to unity when they are the same, is $f(k,k') = [2\Delta k \mp (k' - k)]/[2\Delta k \pm (k' - k)]$, where the positive sign corresponds to $k' > k$. The correlated probability of finding two values k_T, k_T' after T TICKs in an event with the same labels and same normalization is $[f(k_T, k'_T)]/[f(k,k')]$. This is one if $k' = k$ *and* $k'_T = k_T$. However, when $k' \neq k$, a little algebra allows us to write this ratio as

$$\frac{1 \pm \frac{2(\Delta k - \Delta k_T)}{(k'-k)} + \frac{4\Delta k \Delta k_T}{(k'-k)^2}}{1 \mp \frac{2(\Delta k - \Delta k_T)}{(k'-k)} + \frac{4\Delta k \Delta k_T}{(k'-k)^2}} \ .$$

If the second measurement has the same velocity resolution $\Delta\beta$ as the first, since $T > 0$, we have that $\Delta k_T < \Delta k$. Thus, if we start with some specified spread of events corresponding to laboratory boundary conditions and tick away, the fraction of connected events we need consider diminishes. If we now ask for the correlated probability of finding the value β', starting from the value β for the sharp resolution approximation (i.e., ignoring terms smaller than $1/T$ or proportional to $1/T$ and smaller), this is one if $\beta = \beta'$, and bounded by $\pm 1/T$ otherwise. That is, we have shown that in our theory a free particle propagates with constant velocity with overwhelming probability—our version of Newton's first law, and Descartes' principle of inertia.

Were it not for the $\pm$, the propagator in a continuum theory would simply be a δ–function. In our theory, we have already established relativistic "point particle" scattering kinematics for discrete and finite vertices connecting finite strings. We also showed that the order in which we specify position and velocity introduces a sign that depends on which velocity is greater, which in turn depends on the choice of positive direction in our laboratory coordinate system and, hence, in terms of the general description on whether the state is incoming or outgoing. In order to preserve this critical distinction in our propagator and keep away from the undefined (and undefinable for us) expression *const.*/0, we write the propagator as

$$P(\beta,\beta') = \left[\frac{-i\eta\lambda}{\beta' - \frac{\beta \mp i\eta}{T}}\right] ,$$

where η is a positive constant less than T. The normalization of the propagator depends on the normalization of states, and is best explored in a more technical context, such as the relativistic Faddeev equations for a finite particle number scattering theory in the momentum space continuum approximation, being developed elsewhere [9–12].

14. A TEMPORARY HALT

Each paper I have written for *ANPA Proceedings* has had to stop at an unsatisfactory point for me, and I fear for any reader who has persisted to the end. In the past, I have clobbered together a synopsis—fortunately, often prophetic—of where we might be headed. This time I wish to put the burden on the reader. I ask some questions which I believe might be answered by pursuing the lines already laid down. I am working on all of them, and would appreciate some company!

Queries

We take $\hbar, c$ and G as measured by current *scale invariant* techniques, and define our dimensional units of mass [M], length [L] and time [T] by

$$[M] \equiv \left(\frac{\hbar c}{G}\right)^{1/2} ; \qquad [L] \equiv \frac{\hbar}{[M]c} ; \qquad [T] \equiv \frac{[L]}{c} .$$

It is taken as understood in our work that a fundamental theory such as ours must compute everything else as pure numbers in terms of ratios to these units, and provide rules of correspondence consistent with the current practice of physics that will enable us to say how successful we have been in making such calculations.

Query 1:

To what extent do you agree or disagree with this statement? What arguments would you advance in support of it? What experimental or logical evidence would convince you that this is a bad starting point for a fundamental theory?

It is often thought by people who have followed the ANPA program that we have, by now, predicted up to a factor of $[1 \pm O(1/137)]$, the following physical consequences, where the symbols have their usual significance:

$$[M] = (2^{127} + 136)\ m_p\ ; \qquad \frac{\hbar c}{e^2} = 137 = 2^2 - 1 + 2^3 - 1 + 2^7 - 1\ .$$

Query 2:

What arguments would you advance to support this conclusion? What experimental or logical evidence would convince you that these results are wrong or misleading?

Query 3:

Can you explain why you believe in, or do not believe in, the Parker–Rhodes formula for the proton electron mass ratio

$$\frac{m_p}{m_e} = \frac{137\pi}{\frac{3}{14}\left(1 + \frac{2}{7} + \frac{4}{49}\right)\frac{4}{5}}\ .$$

Query 4:

Using the recent results establishing momentum conservation, can you

(*a*) calculate the "center-of-mass" correction to the Bohr formula $[m_e \to m_e/(1 + m_e/m_p)]$, and

(*b*) see if a consistent discrete calculation provides a new route to the Parker–Rhodes formula?

Query 5:

Can you, by using the relativistic discrete theory including angular momentum and "elliptical orbits," obtain the Bohr–Sommerfeld fine structure splitting for the hydrogen spectrum and, by using—instead—the spin degree of freedom, show that this is consistent with the Dirac calculation of the same quantity?

Query 6:

By treating the $(1)_{N_L}$ label (i.e., the unique label in the full, 4–level $2^{127}+136$ bit string representation of the hierarchy which interacts with everything) as the Newtonian "quantum" in the same way that the coulomb "quantum" is treated in the previous exercises, can you solve the Kepler problem?

Query 7:

Can you show that our theory predicts the gravitational red shift for light emitted from any massive object?

Query 8:

Can you show that Newtonian gravitation in our theory predicts only half the observed deflection of apparent stellar positions by the sun? Can you extend the gravitational theory to provide spin–2 gravitons, in addition to the Newtonian term, and show that one can then get the experimental result?

Query 9:

By using spin–2 gravitons in the Kepler problem (see Query 6)—in analogy to the Dirac version of the Bohr–Sommerfeld problem (see Query 5)—can you calculate the precession of the perihelion of Mercury?

Query 10:

Can you show that the mass of the neutral pion is approximately 274 times the electron mass (137 electron-positron pairs), and calculate the binding energy?

Query 11:

Is the identification of $(2^{127}+136)^2$ as an estimate of the baryon number (and charged lepton number) of the universe, which seems natural in the context of *program universe*, a necessary consequence of theories of the type we are constructing?

Query 12:

Is the fact that particles currently known can only be identified with reasonable assurance at *level 3*, that all such particles are "visible" (interact electromagnetically, either directly or indirectly) and that, from the statistical point of view, labels that close on the first two levels will be 127/10 times more prevalent, an indication that there should be roughly ten times as much "dark" as "visible" matter in the universe? Realize that although these labels are not identified, they, like *any* label in the scheme, must interact gravitationally.

Query 13:

Does the success of the Noyes–Dyson argument for the mass of the neutral pion (see Query 10) take us far enough to calculate the 2–gamma decay lifetime of this particle (0.87×10^{-16} seconds)?

Query 14:

How do we calculate the mass of the W and the Z_0? If we can do this, the $\pi^{\pm}-\pi^0$ and neutron-proton mass splittings should follow.

Query 15:

Can we calculate some approximation to the "gluon condensate" which allows Namyslowski to get "running masses" for quarks and gluons? If so, most of strong interact physics should follow, in due course.

Query 16:

Are there quantum geons?

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COMBINATORIAL PHYSICS

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A group of physicists, mathematicians, logicians and computer scientists[1], has developed a form of quantum physics based on sequences of quantum processes without the assumption that there is automatically a spatial or temporal manifold underlying it. This combinatoric approach embodies the fewest assumptions possible in a physical theory. In it, many of the results of conventional theory have already been produced, and some very fundamental calculations have been made which are beyond the reach of current methods. Conceptual difficulties which beset the current theory and which stem from the intuitive spatio-temporal continuum assumptions accordingly lose their paradoxical character and can be looked at afresh. Everything to do with measurement naturally falls into this category: measurement, or observation, is universalized and de-personalized, so that though a conscious observer may participate in the process, it happens anyway, and not only when there is a need for a *deus ex machina* in the shape of an observer to step in and collapse a wave function. Other profound problems look quite different too; in particular those where the extremes of extrapolation are involved. The theory is constructive in the way a body of sequences generates itself, and there is an analogy between this generation and the big bang. However at the beginning of the process spatial relationships do not exist, and clarity on this point about the real operational situation in the circumstances of the big bang is overdue.

Early inspiration for the theory in the fifties concerned extrapolation and what we should now call the event-horizon. Eddington had thought that there must be reasons outside existing theory why the coupling constants, which imply connection between the very small and the very large, must have the values they do, and that these values ought to be seen as determining the relative values of e, h, c, m_p and R rather that the other way around. His attempt to deduce these values was wong, but the logic of his motivation is difficult to fault. An argument which seemed to be related, though one could not them see how, concerned the event horizon and its complement at the quantum end which measurement forced us to accept. How could one discover limits to the measurement process by applying that process? (The measurement process being understood on the classical model of rods and clocks.) The logical puzzle could be resolved—as it seemed—only if one were to adopt a thoroughgoing finitism in which measurement itself were defined in terms of the finite process, and were to be inseparable from that process.

The only way forward seems to be to postulate a combinatorial hierarchical structure in physics in which the famous dimensionless numbers (the coupling constants) could be basic characteristics of the successive levels, and in which the measurement processes provided the generation of the hierarchy. In this way the limits could comprehensibly be the results of measurements. The relation between the levels was at first imagined to be that the elements at one level were the set of operations on the elements of the adjacent simpler level taken pairwise. Analogue and digital models of the two simplest levels of the hierarchy were constructed, with the rest represented by a random input. It was hoped that if constraints on the model corresponding to the coupling constants were imposed artificially then other experimentally known quantities would emerge. There were many perplexities about this model. Did the construction represent a universe, or our knowledge of a universe—of an investigation into a universe?

Insight into these perplexities gradually came after the major single advance of the work, which was to represent elements by binary strings (using symbols '0' and '1' as in computer machine language). Elements at a new level were constructed from the operators (conveniently matrices) which preserved the closure of all sets of elements at the first level under the generation operation. From the combinatorial hierarchy construction in its abstract form as just described we get the level progression: $2^2 - 1 = 3 \rightarrow 2^3 - 1 + 3 = 10 \rightarrow 2^7 - 1 + 10 = 137 \rightarrow 2^{127} - 1 + 137$, which then terminated, and which we interpret (to order 1/137) as $\hbar c/e^2 \simeq 137$, $\hbar c/Gm_p^2 \simeq 2^{127} + 136 \simeq 1.7 \times 10^{38}$. It then appeared that the maximum size of each of the levels so defined agreed well with the known values of the dimensionless ratios.[2] (The theoretical quantities will henceforth be referred to as 'scale constants' to emphasize their true role in theory and to make it clear that a further step is required to relate them to the less general concept of a coupling constant and its experimental determination.) What is more, there were the right number of levels of construction, for the algorithm cannot be continued beyond the fourth level, in accordance with the fact that the gravitational is the weakest field. It seemed that we had stumbled on the reason for either the strange distribution of fundamental magnitudes or scales on which physical phenomena depend, and our failure to understand which has been described as a fatal shortcoming of current physics by several influential writers.

Nineteenth century physics saw the triumph of the electromagnetic field theory, but was still firmly based on arbitrary units of mass, length and time; it provided no way to question *scale invariance*. Although one route to quantum mechanics (that followed by de Broglie and Schroedinger) started from the continuum relativistic wave theory, the currently accepted form breaks the continuity by an interpretative postulate due to von Neumann sometimes called 'collapse of the wave-function'. Criticism of this postulate has continued ever since, although muted for a while by the near consensus of physicists that Bohr had 'won' the Einstein-Bohr debate — a consensus which may have been in deference to the continuing successes of the theory. But scale-invariance is gone, because of the quantized units of mass, action and electric charge. These specify what is meant by "small", while the expanding universe and event horizon specify what is meant by "large".

For a while it appeared that reconciliation between quantum mechanics and special relativity would resist solution, because the uncertainty principle and second quantization of classical fields gave an infinite energy to each point in space-time. During World War II, Tomonaga, and afterwards Schwinger and Feynman, created formal methods to manipulate away these infinities and to obtain finite answers in fantastic agreement with experiment. Recently the non-Abelian gauge theories have made everything calculated in the "standard model" finite. Weinberg recently asserted at the Schroedinger Centennial in London that there is a practical consensus—but no proof—that second quantized field theory is the *only* way to reconcile quantum mechanics with special relativity. However he also pointed out that the finite energy due to vacuum fluctuations is then 10^{120} times too large for the cosmological requirements; the universe would wrap itself up and shut itself down almost as soon as it had started expanding[3]. Even if one were willing to swallow this camel, there would still be the gnats of strong gravitational fields in the theory to strain at. So continued attention to the foundational problem of the origin of discreteness and of the fundament constants embodying it, seems justified.

To get to the point where we could apply the combinatorial model could be applied to these perplexities has taken us a long road. It is now clear that the combinatorial hierarchy structure arises in a universe of bit-strings in which construction of new strings is given free rein subject to finding by random search what strings exist already; that one constructs the all-important 'observation' of the quantum theory by giving special form to the search into the unknown background; and that the entities in terms of which we express the most fundamental observations are the constructs we provide for quantum numbers and particles and the relation between them. Discrete conservation laws provide invariant significance through the statistical selection of strings for the constructs which give rise to 'space', 'time' and 'particle'. The calculations of scale constants with consequences for the elementary particles is consistent with this order of development.

Recent work on the mathematics of the generation process has explained a lot about the dynamics which must apply as soon as multiple processes are considered. In particular, the asymptotic three-dimensionality of any discrete manifold capable of containing the processes, the Lorentz form for any representation of velocity, and the multiconnectedness of the topology has been deduced. The resemblance of the combinatorial hierarchy construction to a computer program suggested that we import the almost axiomatic distinction between label-and address-parts of strings. Accepting this a short cut, the number of bits in a string can be used to define the magnitude of momentum, and from this the treatment of the conservation laws already mentioned. It is then possible to deduce the uncertainty relation, and to speak of individual, yet non-local, quantum processes.

Without going outside this framework it has proved possible to identify the 137 elements that occur at the third level of the construction with the first generation of the standard model for quarks and leptons with their associated quanta and to schematize this description so as to reproduce their behavior by conserved quantum numbers. We provide a model for big-bang cosmology. We also claim[4] the result[5] $m_p/m_e = 1836.151497\cdots$, although the extreme accuracy in interpretation of this calculation is still something of a puzzle.

We are now working out the details of the coupling scheme which our model forces on quarks and leptons. We are sure that the precise way weak/electromagnetic unification is achieved will differ in detail from the way the standard model does it since *we* have no renormalization problem and hence no need for the Higgs mechanism. This work is bound to produce observable consequences, and hopefully, a decisive test.

The current era in high energy particle physics as well as in superconductivity and other cooperative phenomena where macroscopic jumps might appear, have set the stage for a renewed interest in foundational studies. Unfortunately the extant traces of the very early development of the big bang seem to be the only place to test modern theories of grand unification. Much more experimental scope would be allowed if the notions of process and general systems, with their techniques for introducing logical pluralism of interactions or observations (replacing the intuitive pluralism of placing them at separate spatial points) were used as the basis for a new synthesis. At present in standard practice we have only as it were the rump of this pluralism appearing negatively as the uncertainty principle.

We are proposing to subsume the very material stuff of the universe under information (something never attempted in the information- theoretic approaches — e.g. Brillouin — to quantum foundations). Such a 'new synthesis' would be at the expense of even greater conceptual change than the Newtonian synthesis of dynamics and astronomy achieved three centuries ago. We hope that it will appear from this report that dispassionate appraisal of our synthesis is warranted by the successes already achieved.

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ON DISCRIMINATION

The first A.Frederick Parker-Rhodes Memorial Lecture

John Amson

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9th ANNUAL INTERNATIONAL MEETING
ALTERNATIVE NATURAL PHILOSOPHY ASSOCIATION

Department of History and Philosophy of Science
University of Cambridge 23-28 September 1987

Frederick Parker-Rhodes, a founder member of our Association, regularly brought to our annual meetings his extraordinarily imaginative contributions for our delight, puzzlement, irritation and, above all else, the advancement of our understanding of the essences of the problems that lie at the heart of a natural philosophy which - by mutual intent - is devoted to seeking an alternative explanation of the physical world to that currently provided by orthodox, but flawed, quantum mechanical approaches. With his death this March at the age of 72 we are all the poorer, those who knew him but recently as much as those who have searched and argued alongside him through the last thirty years.

Of the many provocative contributions made by Parker-Rhodes, which include, as we are all well aware, the original conception of the self-stopping construction that leads to the four-level combinatorial hierarchy, there is little doubt that the one that could have the greatest effect on the overall directions of the kind of basic research pursued in our Association, and at large, is the one in which he imports a fundamental innovation into logic itself : his theory of Indistinguishables. In its turn this is a logical extension of the idea of indistinguishability in quantum theory. Its principal tenet is that any fundamental primitive act of comparison which leads logically or practically to the idea of things being equal or not-equal is inadequate to deal with the subtleties at the edge of our physical understanding since it is constrained to be precisely what it is declared to be, bi-paritous: things are either equal, or they are not-equal. What Parker-Rhodes did was to open, in a rigorous and detailed way, the possibility that between these two conditions there could lie a third. After Parker-Rhodes things need neither be equal nor not-equal but could, instead, be indistinguishable. It is important that this idea should not be confused with those other seemingly related but quite different ideas of three-valued logics, fuzzy logics, fuzzy sets, and multisets or bags. It is none of these. Its novelty lies in quite a different direction, as I will try to show. It is this which makes Parker-Rhodes' invention truly innovative.

In order to give even a short overview of the nature of Parker-Rhodes' theory of Indistinguishables I propose to describe in some detail the nature of the shore from which we set out on our voyage into the new. My reason for this twofold. First, to appreciate what makes indistinguishability different ('distinguishable' ?) from unequalness it is reasonable to ask that we be fairly confident about what we already understand by things 'being equal'. Second, to make it easier to read what Parker-Rhodes actually wrote in his monograph 'The Theory of Indistinguishables' [1] it will, I feel, be helpful to provide some guide posts as to the formal kind of language and notations in which he expresses what - from any viewpoint - are difficult and unfamiliar ideas. I trust that I may meet my first point to your satisfaction by offering a brief but succinct commentary on those essential parts of 'orthodox set theory' which bear on the topic of equality. Regarding my second point, I am reminded of the gentle criticisms [2] by the son of Robert Stevenson the builder of the Bell Rock Lighthouse, of the description by Smeaton of the latter's truly innovative concept for the construction of the third Eddystone Lighthouse: "One is tempted to conclude that Smeaton had, in the first place, reasoned quite soundly, and arrived, by a perfectly legitimate process, at his true conclusion; and that it was only in the vain attempt to justify these conclusions to others, and convey to them conceptions which a large class of minds can never receive, that he has misrepresented his own mode of reasoning." Did Parker-Rhodes misrepresent his own mode of reasoning by adopting such a rigourous but unforgivingly formal presentation of his concept ? In the light of his other glosses on his work (in particular his eminently readable Apologia 'AGNOSIA' presented at ANPA 7 [3]) the answer is definite: he did not. But what is also without doubt is that his need to develop unfamiliar (not to say idiosyncratic) notations to handle his unfamiliar ideas - ideas which had no prior existence in our formal thinking and writing - make his formal expressions difficult to scan, and hence unfairly obtuse. I hope that by providing a few 'cribs' and 'free translations' to some of his more exotic expressions I may be able to render them less unapproachable. Perhaps then we shall find that at our second attempt his theory is more easily readable and hence understandable.

The material for my talk is written on transparencies; I propose to maintain the format that this imposes in the printed versions offered here. To attempt to transcribe them into a more discursive form would inevitably increase the overall length of this record beyond the page limits allowed. I trust that the few connective asides I have introduced will offset their otherwise somewhat 'telegraphic' style. I have added a short annotated bibliography.

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ORTHODOX SET THEORY

Z F ZERMELO - FRAENKEL [4]

1 . USES 1st-ORDER PREDICATE CALCULUS
AS ITS UNDERLYING DISCIPLINE.

PRIMITIVE SYMBOLS (taken from Logic) are

CONNECTIVES	NEGATION	$\neg$	NOT
	CONJUNCTION	$\wedge$	AND
	DISJUNCTION	$\vee$	OR
	CONDITIONAL	$\rightarrow$	IF...THEN...
	BI-CONDITIONAL	$\leftrightarrow$	...IF AND ONLY IF...
QUANTIFIERS	UNIVERSAL	$\forall$	FOR ALL...
	EXISTENTIAL	$\exists$	THERE EXISTS...

IF **A** AND **B** ARE ARBITRARY STATEMENTS, AND
IF **x** IS AN ARBITRARY VARIABLE
THEN WE CAN FORM EXPRESSIONS SUCH AS

$$\neg A \qquad A \wedge B \qquad A \vee B$$

$$A \rightarrow B \qquad A \leftrightarrow B$$

$$\forall x\, A \qquad \exists x\, B$$

2 . METALANGUAGE – LANGUAGE IN WHICH
WE DEAL WITH EXPRESSIONS IN THE THEORY

THIS WILL BE A NATURAL LANGUAGE (e.g. ordinary English)
SUPPLEMENTED WITH A FEW SYMBOLS,

3 . OBJECT LANGUAGE – LANGUAGE IN WHICH
THE THEORY ITSELF IS FORMULATED

THIS WILL BE AN EXTREMELY RESTRICTED SUB-LANGUAGE
OF e.g. ordinary English
SUPPLEMENTED WITH A FEW SYMBOLS, AND THEIR RULES
= " AN ARTIFICIAL SYMBOLIC LANGUAGE ".

4 . IN 1st-ORDER PREDICATE CALCULUS WE HAVE A POTENTIAL
INFINITY OF *INDIVIDUAL* VARIABLES $x, y, z, x', y', \ldots$

[NB. Does this pre-judge the question of *INDISTINGUISHABLES* ?]

5 . PRIMITIVE RELATIONS

? PRIMITIVE SET-THEORETIC SYMBOLS FOR **ZF**

— ONLY ONE, THE BINARY PREDICATE SYMBOL $\in$
DENOTING *THE* MEMBERSHIP RELATION.

6 . ATOMIC FORMULAS

THE ATOMIC FORMULAS OF **ZF** ARE FORMULAS OF THE FORM $x \in y$

ALL OTHER FORMULAS ARE BUILT UP FROM THESE BY USING *CONNECTIVES* AND *QUANTIFIERS.*

7 . UNIVERSE OF DISCOURSE

THE RANGE OF INDIVIDUALS IN THE THEORY CONSISTS OF OBJECTS (synonym: ELEMENTS)

EACH *OBJECT* IS A *MEMBER* OF SOME *SET*
(e.g. a set which contains just this one object).

ALL OTHER FORMULAS ARE BUILT UP FROM THESE BY USING *CONNECTIVES* AND *QUANTIFIERS.*

8 . SETS and INDIVIDUALS

ELEMENTS WHICH HAVE MEMBERS ARE CALLED *SETS*
ELEMENTS WHICH HAVE NO MEMBERS ARE CALLED *INDIVIDUALS*

— WITH ONE EXCEPTION:-
THE *NULL-SET*, WHICH HAS NO MEMBERS,
IS BOTH A SET AND AN INDIVIDUAL.

9 . A S S U M P T I O N : THERE IS *AT LEAST ONE* INDIVIDUAL

[Needed for both philosophical and practical reasons (*)]

IT TURNS OUT THAT THERE IS ONLY *ONE* (**) INDIVIDUAL —
— *THE* NULL-SET

HENCE IN **ZF** ALL ELEMENTS ARE EITHER SETS WITH ELEMENTS, OR THE NULL-SET

(*) One individual is needed to serve as the FOUNDATION of the universe:-
$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \ldots$

(**) How would *many indistinguishable* null-sets be interpreted ? ...

A FUNDAMENTAL NOTION IN IN SET THEORY IS *EQUALITY* "="

THERE ARE THREE POSSIBLE ATTITUDES TO THIS – (A), (B), (C)

(A) " = " IS UNDERSTOOD TO DENOTE *IDENTITY* AND REGARDED AS PART OF THE UNDERLYING LOGIC
[In **ZF** this is " 1st.Order Predicate Calculus WITH IDENTITY "]

ITS BASIC PROPERTIES ARE THEN *LOGICAL TRUTHS* :-

(i) REFLEXIVITY $(\forall x)\ \ x = x$
(ii) SYMMETRY $x = y \ \rightarrow\ y = x$
(iii) TRANSITIVITY $x = y \wedge y = z \rightarrow x = z$
(iv) SUBSTITUTIVITY For very statement $P(x)$, if $P(x)$ holds and $x = x'$ then $P(x)$ holds also.(*)

NB Substitutivity in FPR's Indistinguishables will become much more complicated than Substitutivity in **ZF**.

(*) Enough to use just two special $P(x)$'s :

$$(iv)'\ x \in y \wedge x = x' \rightarrow x' \in y$$
$$x \in y \wedge y = y' \rightarrow x \in y'$$

NB (i), (ii), (iii) and (iv)' $\rightsquigarrow$ (iv)

NB This comes down to saying (after Zermelo):-

" x and y are said to be *equal* when they are *the same thing* "

| ? What happens if they are "indistinguishable" ? |

(B) " = " IS REGARDED AS ONE OF THE *PRIMITIVE RELATIONS* IN THE SYSTEM, ON A PAR WITH THE OTHERS. *I.e.* WE NOW HAVE *TWO* BINARY PREDICATE RELATIONS $\in$ and $=$.

- THEN (i) (ii) (ii), (iv)' ARE *AXIOMS* IN OUR SYSTEM.
- ALL INSTANCES OF (iv) FOR ALL DIFFERENT STATEMENTS $P(x)$ ARE THEN *THEOREMS* IN OUR SYSTEM.

(C) "$=$" IS INTRODUCED AS A *DEFINITION*, IN SUCH A WAY THAT (i) (ii) (iii) (iv) BECOME *PROVABLE*, EITHER BY ARGUMENTS OF LOGIC ALONE, OR BY ARGUMENTS USING OTHER AXIOMS OF SET THEORY.

[Fraenkel (see Introduction in Bernays [5]) suggests that Attitude (C) is SUPERIOR to Attitude (B) because it needs only a *single* Primitive Relation $\in$, and is SUPERIOR to Attitude (A) because the resulting system is constructed on a *weaker* discipline.]

Notation: In **ZF**, instead of $\neg x = y$ and $\neg x \in y$
we write $x \neq y$ and $x \notin y$

When $x \neq y$ we say that x is *DIFFERENT FROM* y.

["$=$" is therefore a *BI-PARITOUS* relation in FPR terms]

DEF.1 RELATION OF INCLUSION

If y and z are sets such that, for all x, $x \in y \rightarrow x \in z$,
then we write $y \subseteq z$,
y is a *SUBSET* of z (y is *INCLUDED* in z).

Also, if there is at least one w such that $w \in z \wedge w \notin y$,
then we write $y \subset z$,
y is a *PROPER SUBSET* of z (y is *PROPERLY INCLUDED* in z).

THM.1 Every set is a subset of itself :- $x \subseteq x$: $\subseteq$ is *REFLEXIVE*

$\subseteq$ is *NOT SYMMETRIC* : $x \subseteq y$ does not imply $y \subseteq x$;

$\subseteq$ is *TRANSITIVE* : $x \subseteq y \wedge y \subseteq z \rightarrow x \subseteq z$.

$\subset$ is transitive, but not reflexive, not symmetric.

DEF.2 x is *MEMBERSHIP CONGRUENT* to y : $x \equiv_M y$

If, for all z, $x \in z \leftrightarrow y \in z$ and, dually,
for all u, $u \in x \leftrightarrow u \in y$ (*i.e.* $x \subseteq y \wedge y \subseteq x$).

In other words: two sets x and y are MEMBERSHIP CONGRUENT if and only if :-
x , y BELONG TO exactly the SAME SETS — and, dually, —
x , y CONTAIN exactly the SAME ELEMENTS.

NB "$\equiv_M$" is reflexive, symmetric, transitive, substitutive with respect to the atomic statement $x \in y$; i.e. (i) (ii) (iii), (iv)′ hold for $\equiv_M$.

NB Adopting 'EQUALITY (C)' [i.e. taking '$=$' to be a DEFINITION]
it can be shown that, for all x, y, $x = y \leftrightarrow x \equiv_M y$; so that
••• WE LOSE NOTHING BY *DEFINING* $x = y$ *TO BE* $x \equiv_M y$ •••

[? What happens to '$x \equiv_M y$' for "indistinguishables" ?]

(Some) AXIOMS in ZF

AX.1 — EXTENSIONALITY

"*SETS containing the SAME* † *MEMBERS* are *EQUAL* ‡

i.e. $x \subseteq y \ \wedge\ y \subseteq x \ \rightarrow\ x = z$

i.e. $\forall z\, [\, z \in x \ \leftrightarrow\ z \in y\,] \ \rightarrow\ x = y$ $\qquad (\star)$

i.e. "Sets with the *SAME EXTENSION* are equal"

i.e. "Each set is completely determined by its members"

NB Adopting 'EQUALITY (A)' [*i.e.* '=' is a LOGIC IDENTITY]
or 'EQUALITY (B)' [*i.e.* '=' is a PRIMITIVE RELATION like '$\in$']
gives us : if $x = y$ then $x \equiv_M y$.
Then AX.1—EXTENSIONALITY gives us the converse :
if $x \equiv_M y$ then $x = y$.
So, even if EQUALITY is a PRIMITIVE, rather than DEFINED,
then MEMBERSHIP CONGRUENCE *COINCIDES* WITH EQUALITY.

AX.1.dual — "*ELEMENTS contained in the SAME* † *SETS* are *EQUAL*

i.e. $\forall z\, [\, x \in z \ \leftrightarrow\ y \in z\,] \ \rightarrow\ x = y$ $\qquad (\star\star)$

† ? What happens if 'the same' is replaced by 'indistinguishable' ?

NB. Either of these two [$\star, \star\star$] characteristic properties of EQUALITY
can be used as a DEFINITION OF EQUALITY '='
instead of using MEMBERSHIP CONGRUENCE '$\equiv_M$ '.

- If we DEFINE $x = y$ to mean that x , y are *members of exactly the same sets*[$\star\star$]
then we need AX.2—PAIRING and AX.5—SUBSETS to show that
if $x = y$ [$\star\star$] then x , y have the same EXTENSION [$\star$]
and hence, by AX.1—EXTENSIONALITY, $x = y \ \leftrightarrow\ x \equiv_M y$.

- On the other hand,
if we DEFINE $x = y$ to mean that x , y *have the same extensions*[$\star$]
then AX.1—EXTENSIONALITY is a tautology, and to *prove* that
$x = y \ \rightarrow\ x \equiv_M y$, we have to ASSUME it as an AXIOM :

AX.1′ $\qquad x \in z \ \wedge\ y = x \ \rightarrow\ y \in z$

- And so on ...
[See Fraenkel *et al.*[4], Chapter II, §2. *Some basic notions, equality and extension.*]

‡ AX.1 is the *Axiomatic Definition of Equality* in **NBG** (von Neumann, Bernays, Gödel) Set Theory. [See Bernays [5], Chapter I, §2. *Equality and Extensionality.*]

SUMMARY

WHICHEVER METHOD IS USED FOR INTRODUCING *EQUALITY* INTO SET THEORY, THE *INTENDED* INTERPRETATION OF "$x = y$" IS THAT THE OBJECTS DENOTED BY "x" AND "y" ARE *IDENTICAL*.

For example, in **ZF** Set Theory, the direct way to say that

"a set z has exactly one member"

is to say

"There is a member x of z such that every member y of z is equal to x ".

IF *EQUALITY* IS *NOT INTENDED* TO BE NECESSARILY *IDENTITY*, THEN SUCH A 'SINGLETON' SET CAN CONTAIN *TWO OR MORE* MEMBERS *ALL EQUAL TO EACH OTHER* .

It is this concept which leads us into the notion of a MULTISET ...

MULTISETS

[Earlier references to Multisets can be found in Knuth [6], Manna & Waldinger [7] (where they are also called 'Bags'). Wayne Blizard has carried out a formal axiomatisation of Multiset Theory '**MST**' in [8] and has provided a short, succinct account in [9] in these Proceedings.]

MULTISET THEORY *'CONTAINS'* ORTHODOX SET THEORY.
A NAIVE DESCRIPTION OF **MST** :

1. A MULTISET IS A COLLECTION OF ELEMENTS IN WHICH CERTAIN ELEMENTS MAY OCCUR MORE THAN ONCE.
2. REPEATED OCCURRENCES OF A PARTICULAR ELEMENT ARE *'INDISTINGUISHABLE'* [in Blizard's, not FPR's sense !!!].
3. EACH OCCURENCE OF AN ELEMENT CONTRIBUTES TO THE CARDINALITY OF THE MULTISET.
4. EACH ELEMENT CAN OCCUR ONLY A FINITE WHOLE NUMBER OF TIMES, 1,2,... [But see Final Summary below !].
5. THE NUMBER OF DISTINCT ELEMENTS IN A MULTISET NEED NOT BE FINITE.
6. A MULTISET IS COMPLETELY DETERMINED IF WE KNOW WHICH ELEMENTS BELONG TO IT *AND* THE NUMBER OF TIMES EACH ELEMENT BELONGS TO IT.

THE UNDERLYING DISCIPLINE ('Language') for **MST** is 1st.Order Predicate Calculus *WITH EQUALITY.* [NB we have to use two kinds of equality in that calculus : '$=_M$' for Multisets, and '$=_N$' for Numbers. Don't confuse the former with our earlier Membership Congruence symbol $\equiv_M$.]

THERE ARE TWO KINDS OF *PRIMITIVE SYMBOLS* IN **MST** :

ternary predicate " $\in^n$ "
binary predicate " $\in$ "

where

$x \in^n y$ means x is an element belonging n times to y .

$x \in y$ means x is an element belonging an unspecified number of times to y .

(Some) AXIOMS in MST

AX.I — EXACT MULTIPLICITY [ZF irrelevant]

The multiplicity with which an element belongs to a MULTISET is unique :

$$x \in^n y \;\wedge\; x \in^m y \;\rightarrow\; n = m$$

I.e. if x belongs to y both n and m times, then $n = m$.

AX.II — EXTENSIONALITY [ZF enhancement]

Two MULTISETS are *EQUAL* if they have exactly the *SAME ELEMENTS* occurring with exactly the *SAME MULTIPLICITIES* .

AX.III — EMPTY SET [ZF identical]

There is at least one MULTISET *WITHOUT ELEMENTS* .
[It turns out that there is *ONLY ONE* , *i.e.THE* unique empty set $\emptyset$]

AX.IV — ELEMENTARY MULTISETS (*)

(i) For any MULTISET x , for any number n, there is a (unique) MULTISET y containing exactly n copies of x and nothing else.

(ii) For any two *DISTINCT*† MULTISETS x , y , for any numbers m, n, there is a (unique) MULTISET z containing exactly m copies of x , n copies of y ,

(*) Important to note that this is *NOT* an exact analogy of the ZF AXIOM of EXTENSIONALITY when restricted to SETS, since in **MST** 'sets' need not themselves be 'sets'. [For such a 'set', $x \in^n y \rightarrow n = 1$.]

[We also need the idea of *Transitive Closure* TC(x), and *Hereditary Set* HSET; see [9].]

(*) ? How do elements of Elementary Multisets compare with FPR 'indistinguishables' ?

(†) ? What happens if the Multisets are '*indistinguishable*' ?

These questions lead us naturally into the concept of *INDISTINGUISHABLES* ...

FPR'S CONCEPT OF INDISTINGUISHABILITY

FIRST : two objects are *EQUIVALENT* if they can consistently be treated in one context as *IDENTICAL*, and in another context as *distinct*.
Generally, EQUIVALENTS can be distinguished from one another *ON CLOSER SCRUTINY* — even if this is forbidden in a mathematical context, or is technically impossible in an observational context.
Indeed, "equivalence" is a technique, not a theory !

SECOND : There is the question of whether there is a class of things which are *FUNDAMENTALLY INDISTINGUISHABLE* ?

FPR's novel concept is that of a *TRI-PARITOUS RELATION* which uses *THREE* Binary Predicate Relations (SYMBOLS) : =, Δ, !

$x = y$	means	x	and	y	are IDENTICAL
$x \Delta y$		x		y	are TWIN
$x\,!\,y$		x		y	are DISTINCT

and which has *THREE NEGATIONS* :

$x \neq y$	$\leftrightarrow$	$x \Delta y$	OR	$x\,!\,y$	; x and y are NON-IDENTICAL
$x \underline{\uparrow} y$	$\leftrightarrow$	$x = y$	OR	$x\,!\,y$	; x and y are BIPAR(itous)
$x \mid y$	$\leftrightarrow$	$x = y$	OR	$x \Delta y$	; x and y are INDISTINCT

ZF SET THEORY can be modified into (at least!) two new theories :-

MULTISET THEORY *I.e.* **MST** (as discussed above).

SORT THEORY *E.g.* **SORT** (as invented by FPR).

? What is the UNDERLYING LOGIC DISCIPLINE (Language) IN **SORT** ?
FPR is (apparently) using 1st. Predicate Calculus *WITHOUT EQUALITY*.

? What are the PRIMITIVE RELATIONS (Symbols) IN **SORT** ?
SORT has *FIVE* :-

THREE PARITY RELATIONS : =, Δ, ! (as seen above)
(*Cf.* '=' under 'EQUALITY **(B)**' in **ZF** above)

ONE [*CLASS*] MEMBERSHIP RELATION $\in$
used as *e.g.* $x \in C$ where C is a *class* [not necessarily a *set*, note!]
Note also that in his 'RPN' notation, FPR writes $\in Cx$
[Is FPR assuming *CLASS THEORY* as his background discipline, possibly **NBG** ?]

ONE 'CATEGORISER' RELATION $\mathbf{C}$

used *e.g.* as $x \in \mathbf{C}$ to mean 'x is a CLASS'
— or perhaps as "For all classes x". FPR writes $\in \mathbf{C}\, x$
Cf. the standard categoriser "CLASS(x)".

? What is the DOMAIN OF DISCOURSE IN **SORTT** ?

SORT THEORY consists of OBJECTS (ELEMENTS) and each element is a member of some CLASS (NB not some SET !).

? What is the rôle of INDIVIDUALS in **SORTT** ?

From the very start, FPR separates SORT THEORY from SET THEORY by *EXCLUDING THE EXISTENCE OF INDIVIDUALS* in **SORTT**.
Recall that, in **ZF** , a SET is an element which itself HAS MEMBERS, and an INDIVIDUAL is an element with NO MEMBERS;
also (AXIOM) that there is ONE UNIQUE INDIVIDUAL — the NULL-SET, $\emptyset$;
also, that WITHOUT ONE INDIVIDUAL, **ZF** may be VACUOUS.

FPR claims that this exclusion follows immediately from his First Axiom:

AX.1 (4.1 in [1]):

$$\in \mathbf{C}x, \quad \in \mathbf{C}y, \quad \in xy, \quad \in yx, \quad \rightarrow \quad = xy$$

$$[\ \textit{I.e.}\quad \forall x \quad \forall y \quad [y \in x \quad \wedge \quad x \in y] \quad \rightarrow \quad x = y\ ]$$

FPR notes that the converse is false : $x = y$ does not imply that $y \in x \quad \wedge \quad x \in y$
i.e. identical classes are NOT necessarily mutual members.

Since "=" is introduced as a Primitive Relation of the kind **(B)** its properties have to be introduced via AXIOMS. In particular, *SUBSTITUTIVITY* , (iv), needs VERY SPECIAL TREATMENT :

AX.2 (4.2 in [1]):

IF $\mathbf{P} \rightarrow \mathbf{A}$ IS A STATEMENT, THEN $\mathbf{P} \rightarrow \mathbf{A}'$ HAS THE SAME TRUTH VALUE AS $\mathbf{P} \rightarrow \mathbf{A}$, WHERE $\mathbf{A}'$ IS FORMED IN ONE OF THE FOLLOWING WAYS :

2.1 if $\mathbf{P} \rightarrow x = y$: replace every occurrence in $\mathbf{A}$ of x by y , OR ELSE of y by x.

2.2 if $\mathbf{P} \rightarrow x \triangle y$: replace every occurrence in $\mathbf{A}$ of x by y , AND of y by x throughout any *NEXUS* [*] in $\mathbf{A}$ in which any one such replacement is made.

2.3 if $\mathbf{P} \rightarrow x\,!\,y$: replace every occurrence in $\mathbf{A}$ of x by y , AND of y by x throughout $\mathbf{P}$ and every relation INFERRED from $\mathbf{P}$.

2.4 if none of these : replace NO x by y , AND NO y by x .

[[*] A NEXUS is a set of symbols in a statement, of a special kind, and which are related in a special ("CONCURRENT") way; *cf.* page 32 in [1].]

THE CENTRAL CONCEPT OF A 'SORT'

To avoid any confusion between the idea of a SET in orthodox Set Theory (such as **ZF** or **NBG**) and the idea of collections of things which might contain indistinguishables in the tri-paritous context of his novel theory, FPR calls such a latter collection a "*SORT*". The resulting body of his theory he denotes by **T** — *i.e.* our **SORTT**.

FPR then gives the following FORMAL DEFINITION of a SORT :

DEF.1 (4.14 in [1])

$$() \Rightarrow \in \mathbf{S},\, C\left(\neq ()\updownarrow,\ \mathrm{x}(\in C\mathrm{x}),\ \mathrm{y}(\in C\mathrm{y})\right)$$

To interpret this, we need to know the following conventions in FPR's notations :

$() \Rightarrow A$ means "it is TRUE that A holds" [$(/)$ would give "FALSE"]
$\in \mathbf{S}, \mathrm{b}$ means $\mathrm{b} \in S$, *i.e.* "b belongs to the class of SORTS"
$\mathrm{P}, \mathrm{x}(\mathrm{Q}\mathrm{x})$ means "P is TRUE for all x for which $\mathrm{Q}\mathrm{x}$ holds (is TRUE)"
C denotes a Class.

Hence

$() \Rightarrow$	It is true that
$\in \mathbf{S},$	a SORT, is
the Class $C\left(\neq ()\updownarrow,\ \mathrm{x}(\in C\mathrm{x}),\ \mathrm{y}(\in C\mathrm{y})\right)$	
*i.e.*the Class C such that	
$\neq ()\updownarrow$	it is NOT TRUE that BIPAR are
$\mathrm{x}(\in C\mathrm{x})$	all x such that $\mathrm{x} \in \mathrm{C}$, and
$\mathrm{y}(\in C\mathrm{y})$	all y such that $\mathrm{y} \in \mathrm{C}$

in other words :-

SORT contains all those Classes C for which not every PAIR $\mathrm{xy} \in \mathrm{C}$ are BIPAR [*i.e.*IDENTICAL or DISTINCT] — some pairs MAY BE bipar, some WILL BE twin.

FPR also defines PERFECT SORTS : these have all their members either IDENTICAL or TWIN.

... and so on ...

COMMENTS ON THE DEFINITION OF A SORT

1. The definition asserts that SORT is a sub-class of the class of all classes. The EXISTENCE of CLASSES is a consequence of FPR's initial assumption that one of the five Primitive Relations is the "Class Membership" relation $\in$. Compare with Set Theory where the Membership relation $\in$ makes no prior assumption as to what things can be members of.

2. The definition makes the tacit (unstated) assumption that there exist CLASSES WITH PAIRS. Compare with Set Theory where this needs the explicit AXIOM OF PAIRING.

FPR also defines various special Sorts :

ELEMENTARY SORT : SORTS which belong to themselves [$S \in S$].
SINGULAR SORT : "singleton" SORTS, those for which $\mathbf{x} \in \mathrm{S} \wedge \mathbf{y} \in \mathrm{S} \rightarrow \mathbf{x} = \mathbf{y}$.
EMPTY SORT : SORTS which have NO MEMBERS. [But *cf.* the apparently contradictory assertion (page 57, [1]) that there "is nothing in **C** which does not have members" ?]

FPR also notes that if a SORT HAS MEMBERS, then these members ARE ALSO SORTS. Compare with Set Theory where SETS may have MEMBERS WHICH ARE NOT SETS.

SUMMARY — and OUTLOOK

There appear to be three places in theories of a General Set-like Character in which orthodox SET THEORY is susceptible to modifications which can completely alter the character of the theory :-

(i) The Underlying (Logic) Discipline

(ii) The Primitive Bi-Paritous Relation '=' in the Object Language

(iii) The Primitive Binary Relation '$\in$' in the Object Language

We have seen that

modifying (iii) to a ternary relation '$\in^n$' (multiple membership) introduces MULTISET THEORY;

modifying (ii) to a Tri-Paritous Relation $=$, Δ, $!$ introduces SORT THEORY.

But still further possibilities exist :

Wayne Blizard [8,9] has studied the introduction of INTEGER and NON-INTEGER multiple membership, where the index n in $\in^n$ can be a positive, zero, or NEGATIVE integer, — or RATIONAL, — or REAL, — or even COMPLEX !; whilst Eilenberg [10] allows n to belong to a very general algebraic system, a SEMI-RING, which contains all of the above algebraic systems, and many others besides ...

And further still — we may ask —

? What happens if we modify (i), the Underlying Logic,
  (a) from a 2–Valued Logic to a k–Valued Logic ?
  (b) from using a bi-paritous relation to a tri-paritous relation ?

? What happens in SORT THEORY if, as would be quite natural now that Frederick Parker-Rhodes has shown us the way, we were to use not merely a Tri–Paritous relation $=, \quad \Delta, \quad !$, but an n–Paritous Relation

$$\overset{(1)}{==}\ ,\ \overset{(2)}{==}\ ,\ \overset{(3)}{==}\ ,\ \ldots\ ,\ \overset{(n)}{==}$$

where $\overset{(1)}{==}$ is 'ordinary equality '=' ' ?

? Or we use a **HIERARCHY OF SORT THEORIES**

$$\mathbf{S}_2\ ,\ \mathbf{S}_3\ ,\ \mathbf{S}_4\ ,\ \ldots$$

in which operate 2–Paritous, 3–Paritous, 4–Paritous, ... relations, perhaps interconnected in some "Combinatorial Hierarchical" way ?

? Could it be that Frederick Parker-Rhodes' innovative introduction of a 3-Paritous relation opens the door to previously unimagined realms of

"INDISTINGUISHABILITY"

— which, in turn, could lead to an

INCHOATIVE PLANE*

of extraordinary subtlety ?

[* See [1], Chapter 3, page 41]

ANNOTATED BIBLIOGRAPHY

1. A. F. PARKER-RHODES, *The Theory of Indistinguishables*, Synthese Libray/Volume 150, Dordrecht, Holland, 1981.

The subtitle, *A Search for Explanatory Principles Below the Level of Physics*, succinctly decribes the main purpose behind Parker-Rhodes' invention of "Indistinguishables". Chapters I (*Introduction*) and IV (*Sort Theory—Axioms and Definitions*) should first be read together before attempting any deeper study of the theory. (There is an unfortunate ommission of the ǀ symbol for the Conditional Quantification Functor in the 1st line of the last paragraph on page 64 which should begin 'The functor ǀ was introduced in (2.28) [....]'). At this early stage in his theory of indistinguishables it is not clear whether the complexities of Parker-Rhodes' notations are irretrievably essential to the development and use of his theory. If not, then a parallel, alternative account using more orthodox notations would be a important contribution to its wider acceptance. For the present, the intending user will need to pay particular attention to the proper rules for the new notations; these are worked out from first principles in Chapter II. An indirect benefit is fresh insight into the ways in which the habitual notations of logic and mathematics (allbeit developed in a bi-paritous context) severely impede any ready extension of our ways of thought to the new context of tri-paritous relations.

2. Alan STEVENSON, C.E., *A Rudimentary Treatise on the History, Construction and Illumination of Lighthouses*, London 1850.
(A.S. was the son of Robert Stevenson, the engineer, and the uncle of Robert Louis Stevenson the writer.)

3. A. F. PARKER-RHODES, *AGNOSIA, a Philosophical Aplogia for INDISTINGUISHABLES*, *Proceedings, ANPA-7*, 1986.

This extended essay provides an eminently accessible overview of the background, development, rôle and general concept of Parker-Rhodes' Indistinguishables and Sorts. It could very usefully be read in conjunction with this present article, and should certainly be read before launching into the main text (Ref.1 above) itself.

4. A. A. FRAENKEL, Y. BAR-HILLEL, A. LEVY, *Foundations of Set Theory*, 2nd Revised Edition, North-Holland Pub.Co., Amsterdam 1973.
The first part of Ch.II (*Axiomatic Foundations of Set Theory*) — §1. *Introduction*, ¶2. *Some Basic Notions, Equality and Extensionality* — is recommended background reading for anyone wishing to approach Parker-Rhodes' Theory of Indistinguishables.
(See also A.A.FRAENKEL, *Abstract Set Theory*, 3rd Revised Edition, North-Holland, 1966, Ch.I, ¶2, p.13, and Ex.6, p.33, for (i) Fraenkel's adaption of Leibniz' "*identitas indiscernibilium* : that is to say, two things are equal if they cannot be distinguished *within the system*", and (ii) a very brief discussion of equality and substitutivity.)

5. Paul BERNAYS, *Axiomatic Set Theory, with a Historical Introduction by Abraham A Fraenkel*, North-Holland Pub.Co., Amsterdam, 1968.
In Ch.I, *The Frame of Logic and Class Theory*, there are four sections §1. *Predicate Calculus; Class Terms and Descriptions, Explicit Definitions*, §2. *Equality and Extensionality*.

Application to Descriptions, §3. *Class Formalism. Class Operations*, §4. *Functionality and Mappings*; together they provide a useful account of the notions intrinsic to the "other" orthodox ("NBG"—von Neumann-Bernays-Gödel) Set Theory and Class Theory. These sections, read together with the orthodox "ZF" Set Theory referred to in the present article, form a useful framework to match against that developed in Parker-Rhodes' Theory of Indistinguishables in Ref.1 above. The treatment of 'Classes' in NBG theory is also relevant.
(See also Patrick SUPPES, *Introduction to Logic*, Van Nostrand Co., Princeton, New Jersey,1957,..,1960, for a helpful, introductory account of Set-Theoretic Predicates, in Ch.12. *Set-Theoretical Foundations of the Axiomatic Method.*)

6. Donald E. KNUTH, *The Art of Computer Programming, Volume 2/Seminumerical Algorithms*, Addison-Wesley Pub.Co., Reading, Massachusetts, 2nd.Edition, 1981.
There are three interesting but brief references to "Multisets" (or "Bags"): pages 454, 464 (Ex.19), 636 (Ex.19,Notes). The last discusses the (then) lack of a suitable notation and terminology for multisets. (But see Blizard, Refs.8,9 below.)

7. Z. MANNAR and R. WALDINGER, *The Logical Basis for Computer Programming, Volume I, Deductive Reasoning*, Addison-Wesley Pub.Co., Reading, Massachusetts, 1985.
Ch.11 *"BAGS"* (= *multisets*) gives an orderly acount of the essential features of multisets; the treatment parallels that in Ch.10 *"SETS"*, and so makes their differences and similarities easier to comprehend. But note that since the context of this text is one of actual computing, all sets and multisets used here are FINITE so that the subjects cannot be developed in their full generality. Despite this the text forms a useful and well organised introduction.

8. Wayne D. BLIZARD, *MULTISET THEORY*, (to appear in *Notre Dame Journal of Formal Logics.*
This paper presents the first formal axiomatisation of Multiset Theory and contains an important list of references. The first section, *A Survey of the Literature* offers a brief but effective overview of the subject and confirms that the *notion* (but not the effective *treatment*) of multisets has actually been around for a very long time! Section II establishes the Formal Theory (MST), using first-order predicate calculus *with equality* (please note!), using conventional logical symbols. Section III provides a Model of MST.
The efficacy of Blizard's formal treatment is laudable—the reader is immediately aware of what is being introduced, what is different from orthodox set theory, and what the implications of the innovations are. The use and adaptation of conventional symbols is an important aid in this respect. (It is conceivable that a new treatment of Parker-Rhodes' Indistinguishables and Sorts along the same lines would provide an easier access to his theory than is at present available from the main text in Ref.1 above.)

9. Wayne D. BLIZARD, *Multisets : Collections Containing Indistinguishable Elements*, these Proceedings.
This short, useful account emphasises the essential features of formal Multiset Theory (MST) and the fact that it is 'classical' in the sense (i) that it is formulated in the first-order predicate calculus with equality, and (ii) contains an exact copy of 'orthodox' ZF Set

Theory. It also notes where it differs in essence from Parker-Rhodes' Theory of Sorts. It concludes by drawing attention to the very significant fact that the multiplicity 'numbers' need not be drawn from Peano Arithmetic (PA) (*e.g.* non-negative integers). Different axiom systems can be substituted for PA, for example, axiom systems for a semi-ring (see Ref.10), a ring, a field, a lattice, a boolean algebra or a Heyting algebra—each giving rise to their corresponding multiset theories—and each containing a classical copy of ZF Set Theory.

10. Samuel EILENBERG, ***Automata, Languages and Machines, Volume A***, Academic Press, New York, 1976.
Chapter VI is devoted to an exposition of "Multiplicity"; the context is that of paths in automata but the notion is essentially that of a multiset in which membership is generalized from the idea of n–fold membership (integer n) to k–fold membership where k is an element of a semi-ring K. The latter is an (ordinary) set equipped with two operations—addition and multiplication; the rules relating them are sufficiently general to include a very wide range of algebraic systems (of the kind adumbrated by Blizard in Ref.9 above).

DISCRIMINATION SYSTEMS AND PROJECTIVE GEOMETRIES

Version 87.B

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[Work in progress]

Abstract

The idea of discriminating between objects with just two exclusive attributes is introduced by way of a 'Sequential-Act-Of-Comparison'. Fundamental to this idea is that the result of this act of comparison is itself a new object in the system: if the objects are 'equal' the result is a copy of one kind, if 'unequal' the result is a copy of the other kind. The system is thus self-organising, but in two possible modes, only one of which leads to diversity.

The discrimination rule used turns out to be equivalent to boolean logic XOR acting on pairs of objects.

In order to provide more structure than is available in a primitive discrimination system, the idea of pairs and then strings of objects has to be introduced. Discrimination between strings is then a compound XOR constructed term-wise. Two special strings exist which have all their terms equal to one of the basic objects, one of one kind, one of the other kind, respectively; they are called the neutral and the anti-neutral strings.

In a system which has strings all with the same number of terms the neutral string plays the rôle of the neutral in a group in which the discrimination of strings *via* the compound XOR comparison acts as group addition (+). Then two strings are equal if and only if their sum is the neutral string. Such a system is a cyclic group which can be identified in a natural way with a particularly simple kind of vector space over the field of two elements.

This vector space becomes our discrimination system. In it, independence of two strings (vectors) reduces to the two being unequal, *i.e.* discriminated. The fundamental property of a collection of strings being discriminately closed is identified with the collection being a vector subspace with the neutral string (the origin) removed; such a collection is called a dc-subset. Numerous combinatoric results about dc-subsets are given.

The fact that dc-subsets are vector subspaces with the origin removed allows them to be identified in a very natural way with the objects (points, lines, planes,...) in a projective geometry over the finite field of two elements.

The possibility of moving freely between a discrimination system and its associated projective geometry allows us to draw upon the very many combinatoric results already established for projective geometries. In particular the counting results for collineations, etc., are made available to us in terms of counting various kinds of automorphisms in our discrimination system. In conclusion, the possibility of exploiting many other properties familiar in the projective geometry context, such as polarities and quadrics, is sketched in.

The intention is to suggest further ways of exploring the complicated and rich structure that is now seen to be inherent in our discrimination systems—especially in the interconnected discrimination systems of increasing dimension which constitute the Combinatoric Hierarchy.

[1] — A PRIMITIVE DISCRIMINATION SYSTEM

1. We start with a 'collection' **E** (for "Enigma"...) of 'objects'.

2. 'Objects' are generally undefined except that each object possesses one of a pair of mutually exclusive "attributes" denoted by |e| and |u|. Thus an object can be either an 'e' object or a 'u' object. An 'e' (resp. 'u') object is usually denoted by the same symbol e (resp, u), but may be written 'e-object' (resp. 'u-object'). (An alternative notation might use '0','1' instead of 'e','u', but at this stage the numerical symbols might prejudice the interpretations to be placed on what the rôles of 'e' and 'u' might turn out to be.)

3. There is at least one object in **E** .

4. Since **E** may contain more than one e-object or more than one u-object, **E** may be a multiset. (The question of it being an FPR Sort is left for a later study.)

5. Each object in **E** has the special property of being able to perform a "SAOC" on objects in **E** (including itself). "SAOC" stands for "Sequential-Act-Of-Comparison". Thus each object in **E** can act as an "IO" (Interactive-Observer (*cf.* "operator")) on the collection to which it belongs.

6. Because of this property, (**E** , SAOC) is a Self-Organising-System.

7. An object in **E** can be thought of as carrying out a SAOC in 3×3 sequential stages (the following description uses 'metalanguage') :

- 1S Select an object (object-1) from **E** (could be itself!)
- 1M Memorize the attribute (A1) of object-1
- 1R Replace object-1 in **E**

- 2S Select an object (object-2) from **E** (could be itself again!)
- 2M Memorize the attribute (A2) of object-2
- 2R Replace object-2 in **E**

- 3C Compare attributes A1 and A2
- 3P Produce a "CRO" (Comparison-Result-Object) (see below)
- 3I Insert this CRO into **E** .

8. A CRO, by definition, can belong to **E** and hence possess one or other of the attributes |e|,|u|, (referred to below as A3) and once inserted into **E** it can function as an IO, just like any "previous" member of **E** .

9. To be meaningful, SAOC must have RULES for producing CROs.
We adopt these '**discrimination**' rules :

If attributes A1 and A2 are 'equal' then A3 = $\lfloor e \rfloor$
otherwise (*i.e.* A1 and A2 are 'unequal') A3 = $\lfloor u \rfloor$

10. EXAMPLE .'A'

Here $\mathbf{E}$ consists of just one e-object : $\mathbf{E} = \{e\}$

• By Axiom 5, this single e can perform SAOCs — which it does as follows :

1S ≡ e; $A1 = \lfloor e \rfloor$
2S ≡ e; $A2 = \lfloor e \rfloor$

our e-object now compares A1 $\bowtie$ A2 = $\lfloor e \rfloor \bowtie \lfloor e \rfloor$ = 'equal' (hence A3 = $\lfloor e \rfloor$), and produces an e-object as its CRO.
Thus $\mathbf{E}$ now has two members : $\mathbf{E} = \{e, e\}$.

• By Axiom 5, either or both of the e members in $\mathbf{E}$ can now perform SAOCs. Suppose, for the sake of argument, that both do. Then a situation similar to what we have just described will produce two more e-objects, and results in $\mathbf{E} = \{e, e, e, e\}$.

• This process can be continued with each of the 4 e members performing SAOCs. Indeed, it can be repeated indefinitely ...

• Notice that in this example, $\mathbf{E}$ exhibits *No Diversity* : no matter for how long the SOAC + CRO activities persist, all members of $\mathbf{E}$ continue to reproduce exact copies of themselves.

11. EXAMPLE 'B'

Here $\mathbf{E}$ consists of just one u-object : $\mathbf{E} = \{u\}$

• By Axiom 5, this single u can perform SAOCs — which it does as follows :

1S ≡ u; A1 = $\lfloor u \rfloor$
2S ≡ u; A2 = $\lfloor u \rfloor$

our u-object now compares A1 $\bowtie$ A2 = $\lfloor u \rfloor \bowtie \lfloor u \rfloor$ = 'equal' (hence A3 = $\lfloor e \rfloor$), and produces an e-object as its CRO.
Thus $\mathbf{E}$ again has two members, one a u-object, the other an e-object : $\mathbf{E} = \{u, e\}$.

• By Axiom 5, either or both of the e and u members in $\mathbf{E}$ can now perform SAOCs. Suppose first, for the sake of argument, that only one does—it doesn't matter which. Here SAOC has more selection choice : its two selected members, 1S and 2S, and its output CRO member can take one of four forms :

$(e, e) \longrightarrow e;$ $(e, u) \longrightarrow u$
$(u, u) \longrightarrow e;$ $(u, e) \longrightarrow u$

Thus $\mathbf{E}$ now has 3 members : $\mathbf{E} = \{u, e, e\}$ or $\{u, e, u\}$
If both members in $\{u, e\}$ performed SAOCs then the resulting state of $\mathbf{E}$ would be one of these four possibles : $\mathbf{E} = \{e, u, e, e\}$ or $\{e, u, e, u\}$ or $\{e, u, u, e\}$ or $\{e, u, u, u\}$

• This process can be continued with each of the 4 members performing SAOCs. Again it can be repeated indefinitely ...

• Notice that in this example, **E** now exhibits *DIVERSITY* : the continuing SOAC + CRO activities create new e and u members in **E** , but the precise composition of **E** in terms of the number of e's and u's is indeterminate. It depends on the Selections made by a member at each stage in the growth of **E** . In this 'Primitive' Discrimination System no assumptions have been made about any possible 'self-regulating' process for the production of new members.

12. Because Example 'B' (unlike Example 'A') showed Diversity, we prefer to use Example 'B' as an illustration of what we mean by a Discrimination System (or, to be precise, a Primitive Discrimination System).

13. The discrimination rule introduced in no.9 above can be written :

(A1) = INPUT-1	e	e	u	u
(A2) = INPUT-2	e	u	e	u
OUTPUT =(A3)	e	u	u	e

This table reveals that the discrimination rule is plainly equivalent to the boolean logic XOR operation on the two valued set $\{e, u\}$.

14. It follows that our Primitive Discrimination System is an example of an unusual mathematical construction : It is a sequence of multisets $M_1, M_2, M_3, \ldots$, each multiset having just two elements (e , u), with the multiplicity of the e's and u's in the multisets increasing along the sequence.
Acting on each multiset in the sequence is a binary XOR relation; the output from the action(s) of this relation is one or more copies of e and/or u which are adjoined to the current multiset to create the next multiset in the sequence. Since the generation of the multisets in the sequence $M_1, M_2, M_3, \ldots$, from the initial $M_1 = \{u\}$, is an indeterminate process, the multiplicities of the e's and u's in the multisets has a probability distribution. The details of this distribution need not concern us for the present.

15. Apart from the statistical distribution of the multiple appearances of the e's and u's in our **E** there is little else in the way of "significant and useful structure" in our Primitive Discrimination System. One way of introducing extra organisation is first to assume that the process has "been running" for a "very long time". We may then regard the "current state" of **E** as consisting of a "very large" number of e's and u's. The details of the additional structure and organisation needed in **E** to lead us to our Structured Discrimination System are the topic for our next Section.

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[2] — A STRUCTURED DISCRIMINATION SYSTEM

1. Our Primitive Discrimination System introduced in chapter 1 failed to possess any significant and useful structure. One way in which this might be corrected is by introducing the idea of 'pairs' of elements. Later on we shall also need more general 'strings' of elements, but we start with pairs. We use notations such as (a, b) for a pair of elements a, b in $\mathbf{E}$.

2. Pairs have to be ordered because we want to be able to 'compare' two pairs in an unambiguous way: two pairs (a, b) and (x, y) can be 'compared' — written $(a, b) \bowtie (x, y)$ by performing the 'term-wise' comparisons $a \bowtie x$ and $b \bowtie y$.

3. In particular, with ordered pairs we can then unambiguously say what we mean by two pairs being 'equal' : (a, b) and (x, y) are 'equal' iff $a = x$ and $b = y$, *i.e.* two pairs are equal iff they are 'term-wise equal'. We then write $(a, b) = (x, y)$ to denote this situation. Otherwise the two pairs are 'unequal' and we write $(a, b) \neq (x, y)$. Here the meaning of 'equal' and 'unequal' for terms is the same as that for elements in our Primitive Discrimination System $\mathbf{E}$. There we used a 'comparison' rule (which we called 'discrimination') which turned out to be the boolean XOR function (see Ch.1, no.9). (In Appendix 1 we give more background to why the choice of boolean XOR is an appropriate comparison-rule.)

4. We use the same comparison rule for our pairs — but now applied separately to their terms. This produces a compound XOR function which we can express as follows :

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be three pairs of elements in $\mathbf{E}$; thus $\mathbf{x} = (x_1, x_2)$ where the elements x_1 and x_2 belong to $\mathbf{E}$, so that $\mathbf{x}$ belongs to $\mathbf{E}\times\mathbf{E}$. ($\mathbf{E}\times\mathbf{E}$ is the cartesian product of $\mathbf{E}$ and $\mathbf{E}$.) The pairs $\mathbf{y}$ and $\mathbf{z}$ are formed analogously.

Now let C denote the comparison rule for pairs of elements. Written formally, we have: $C : \mathbf{E}\times\mathbf{E} \rightarrow \mathbf{E}$ with $C(\mathbf{x}, \mathbf{y}) = \mathbf{z}$, where the first and second members of $\mathbf{z}$ are constructed by the pair of rules $z_1 = x_1 \,\mathrm{XOR}\, y_1$ and $z_2 = x_2 \,\mathrm{XOR}\, y_2$.

For an example, suppose $\mathbf{x} = (e, e)$ and $\mathbf{y} = (u, e)$; then $\mathbf{z} = (u, e)$ since comparing the first members of $\mathbf{x}$ and $\mathbf{y}$ we have $e \,\mathrm{XOR}\, u = u$, and comparing the second members we have $e \,\mathrm{XOR}\, e = e$.

5. To simplify writing, we introduce these special notations :

- $\mathbf{e} = (e, e)$ – the pair with e-elements for both its members;
- $\mathbf{u} = (u, u)$ – the pair with u-elements for both its members;
- $+$ for the comparison-rule XOR.

The example at the end of no.4 can now be written : $(e, e) + (u, e) = (u, e)$.

6. Notice the simple—but very important—result :

If $\mathbf{x}$ and $\mathbf{y}$ are any pairs in $\mathbf{E}\times\mathbf{E}$ then

$$\mathbf{x} = \mathbf{y} \quad iff \quad \mathbf{x} + \mathbf{y} = \mathbf{e}.$$

To emphasise this we can express it in its contra-positive form :

$$\mathbf{x} + \mathbf{y} \neq \mathbf{e} \quad iff \quad \mathbf{x} \neq \mathbf{y}.$$

7. From pairs we now move on to 'strings' of elements in **E**. A string is list of elements from **E**, consecutively ordered by the natural numbers $1, 2, \ldots$. In any instance of a string the largest order number used is called the 'length' of the string: a string has length n *iff* there are exactly n elements from **E** in the string. We use notations such as (x_1, x_2, x_3, x_4) or $(x_1, \ldots, x_n)$ or $(x_k)_{k=1..n}$. Two strings are 'equal' *iff* they have the same length (n) and are 'term-wise equal', *i.e.* for each 'index' $k = 1..n$, the k-th terms of both strings are equal; otherwise the strings are 'unequal'. It is convenient to *fix* a natural number $n = 1, 2, \ldots$ and then to work only with strings of this fixed length n. Such a string is then a member of the n-fold cartesian product $\mathbf{E}\times\mathbf{E}\times, \ldots, \times\mathbf{E}$; another shorter notation for the latter is $\mathbf{E}^n$.

8. At this point we have to draw attention to an "abuse of language". We have just introduced the notation $\mathbf{E}^n$ to denote the collection of all strings of length n with members drawn from the collection **E**. From chapter 1 we know that **E** is a multiset consisting of (possibly very many) copies of the two primordial elements 'e' and 'u'. So $\mathbf{E}^n$ is an n-fold cartesian product of *multisets*. However, the notation $\mathbf{E}^n$ suggests that it is the usual familar cartesian product of n *sets* **E** where each *'set'* $\mathbf{E} = \{e, u\}$. From the point of view of the mathematical structures which we are about to study it will not do any harm if we think of **E** and $\mathbf{E}^n$ as notations for standard *set* theoretical objects: *i.e.* we can think of $\mathbf{E}^n$ as the *set* formed as the n-fold cartesian product of n copies of the *set* $\{e, u\}$; the members of $\mathbf{E}^n$ are strings of length n; the members of these strings are the e-objects and u-objects 'e' and 'u'. But we must always bear in mind that we should really be talking about *multisets*.

9. Just as for pairs, we give special notations for two special strings in $\mathbf{E}^n$:

- $\mathbf{e} = (e, e, \ldots\ e)$ – the string with e's for all its n members;
- $\mathbf{u} = (u, u, \ldots\ u)$ – the string with u's for all its n members;

In anticipation of the later rôles of these two special strings we give them these names: **e** is the 'neutral string' and **u** is the 'anti-neutral string', in our discrimination system $\mathbf{E}^n$.

10. Just as we did for pairs, we can introduce a notion of 'comparison' into $\mathbf{E}^n$ through an algebraic structure. We let '+' denote the binary operation $\mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$, in which $\mathbf{x} + \mathbf{y} = \mathbf{z}$, where $\mathbf{x} = \{x_1, \ldots, x_n\}$, $\mathbf{y} = \{y_1, \ldots, y_n\}$, $\mathbf{z} = \{z_1, \ldots, z_n\}$ and the operation '+' is the compound boolean XOR defined term-wise by $x_k + y_k = z_k$ for $k = 1..n$.

11. Again, we have the simple—but very important—result :

If $\mathbf{x}$ and $\mathbf{y}$ are any n-strings in $\mathbf{E}^n$ then

$$\mathbf{x} = \mathbf{y} \quad iff \quad \mathbf{x} + \mathbf{y} = \mathbf{e}.$$

In words: *Two strings are* **equal** *if and only if the result of their 'discrimination' is the neutral string.*

To emphasise this we can again express it in its contra-positive form :

$$\mathbf{x} + \mathbf{y} \neq \mathbf{e} \quad iff \quad \mathbf{x} \neq \mathbf{y}.$$

In words: *The result of the 'discrimination' between two strings is not the neutral string if and only if the two strings are* **unequal**.

12. In the next chapter we go on to develop the structural ideas which flow from this construction of a Structured Discrimination System $(\mathbf{E}^n, +)$.

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[3] — MORE STRUCTURE IN A DISCRIMINATION SYSTEM

1. It is easy to see that $(\mathbf{E}^n, +)$ as constructed in the previous chapter is now a group in which the group law is our discrimination operation '+' acting on strings in $\mathbf{E}^n$. In fact (see Appendix NN) $(\mathbf{E}^n, +)$ is an abelian group, a cyclic group of order n, and is the product of n cyclic groups of order 2 — each of which is isomorphic to our primitive cyclic group $(\{e, u\}, +)$. The neutral string $\mathbf{e} = (e, e, \ldots\ e)$ is the group's neutral element since : $\quad \mathbf{e} + \mathbf{x} \;=\; \mathbf{x} \;=\; \mathbf{x} + \mathbf{e} \quad$ for every string $\mathbf{x}$ in $\mathbf{E}^n$.

2. Whilst the group structure of our discrimination system $(\mathbf{E}^n, +)$ is, in principal, sufficient for our subsquent study and exploitation, there is a good practical case for treating it as if it were an n-dimensional vector space over the field of two elements $(\{0,1\}), +.*)$. The justification for this is the fact that with just two "scalars" 0 and 1 the "multiplication" of strings by scalars becomes trivial: multiplication by 0 is equivalent to removing a string from consideration, multiplication by 1 is equivalent to leaving the string unaltered. In this way a "linear combination" of strings of the form $\alpha.\mathbf{x}\ + \beta.\mathbf{y}\ +, \ldots\ , +\varsigma.\mathbf{z}$ reduces to just the sum of those strings with '1' as their scalar multipliers (the '1's may then be dispensed with).

3. In fact we can use $\mathbf{E}$, *i.e.* $\{e, u\}$ itself instead of $\{0,1\}$, provided that we interpret addition and multiplication in $\{e, u\}$ as being defined by the usual tables :

+	e	u
e	e	u
u	u	e

*	e	u
e	e	e
u	e	u

We now have the scalar multiplication laws:

$$\mathrm{e}.\mathbf{x} \;=\; \mathbf{e} \qquad\qquad \mathrm{u}.\mathbf{x} \;=\; \mathbf{x}$$

for all strings $\mathbf{x}$ in $\mathbf{E}^n$.

Plainly, "multiplication by e" is a **neutralisation** operation— which replaces any string in question by the neutral string $\mathbf{e}$ in $\mathbf{E}^n$, whilst "multiplication by u" is an **identity** operation on $\mathbf{E}^n$.

4. Once we know we are working with a vector space (allbeit over the field of two elements) we can investigate the rôle of "linear dependence" and "linear independence".

The strings $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$ are 'dependent' in $\mathbf{E}^n$ *iff* there are scalars $\alpha, \beta, \ldots, \varsigma$ in $\{e, u\}$, *not all* = e, such that

$$\alpha.\mathbf{x}\ +\ \beta\,.\mathbf{y}\ ,\ \ldots\ ,\ +\ \varsigma.\mathbf{z} \qquad = \qquad \mathbf{e}$$

Because of the simple nature of the scalars e and u this condition reduces to the simple statement:

The strings $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$ are 'dependent' in $\mathbf{E}^n$ *iff* either one or more of them are $= \mathbf{e}$, or the sum of some or all of them is $= \mathbf{e}$.

For example, three strings $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are dependent if either $\mathbf{x} = \mathbf{e}$, or $\mathbf{y} = \mathbf{e}$, or $\mathbf{z} = \mathbf{e}$, or $\mathbf{x} + \mathbf{y} = \mathbf{e}$, or $\mathbf{x} + \mathbf{z} = \mathbf{e}$, or $\mathbf{y} + \mathbf{z} = \mathbf{e}$, or $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{e}$.

5. Plainly, our notion of 'discrimination' is just a special case of 'dependence' : if x and y are **equal** then they are **dependent**. (But the converse is false : x and y can be dependent without being equal—*e.g.* one of them might be neutral and the other non-neutral!.)

6. 'Independence' is the converse of 'dependence':

The strings $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$ are 'independent' in $\mathbf{E}^n$ *iff* when scalars $\alpha, \beta, \ldots, \varsigma$ in $\{e, u\}$, are such that

$$\alpha.\mathrm{x} + \beta.\mathrm{y}, \ldots, + \varsigma.\mathbf{z} = \mathbf{e}$$

then the scalars $\alpha, \beta, \ldots, \varsigma$ are all $= \mathbf{e}$.

Again because of the simple nature of the scalars e and u this condition reduces to the simple statement:

The strings $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$ are 'independent' in $\mathbf{E}^n$ *iff* *none* of them are $= \mathbf{e}$, and no matter which of them are added together, the sum is *never* $= \mathbf{e}$.

For example, *three* strings x, y, z are independent if the following *seven* conditions are satisfied :

$\mathrm{x} \neq \mathrm{e}$ and $\mathrm{y} \neq \mathrm{e}$ and $\mathbf{z} \neq \mathrm{e}$,
and $\mathrm{x} + \mathrm{y} \neq \mathrm{e}$ and $\mathrm{x} + \mathbf{z} \neq \mathrm{e}$ and $\mathrm{y} + \mathbf{z} \neq \mathrm{e}$,
and $\mathrm{x} + \mathrm{y} + \mathbf{z} \neq \mathrm{e}$.

7. It is at this stage of the study of our Discrimination System that we encounter the particular concept—*Discriminate Closure*—that gives our system its special characteristics.

We set the scene by recalling the special rôle of 'discrimination' — to test whether two strings x and y are equal or unequal we simply examine their sum : the strings are equal when their sum is the neutral string e; they are unequal when their sum is not e.

The important idea here is we have "discriminated" two strings if they turn out to be unequal, *i.e.* their sum is not the neutral string.

8. Now consider a non-empty subset S of strings in our Discrimination System E^n. If S contains more than one string then, being a set, all such strings are unequal (in pairs): if we select two strings x and y from S then $\mathrm{x} + \mathrm{y} \neq \mathrm{e}$. Of course, there is no reason why the string $\mathrm{x} + \mathrm{y}$ itself should be in S; but if it is — and for every pair x and y in S — then S will be of special interest to us :

The subset $\mathbf{S}$ in $\mathbf{E}^n$ is called **discriminately closed** *iff*
either
$\mathbf{S}$ contains precisely one *non-neutral* string (*i.e.* $\mathbf{S}$ is a non-neutral singleton set);
or
(for all $\mathbf{x}, \mathbf{y} \in \mathbf{S}$) $\quad \mathbf{x} + \mathbf{y} \in \mathbf{S} \quad \Leftrightarrow \quad \mathbf{x} \neq \mathbf{y}$;
or
$\mathbf{S}$ is the empty set (this vacuous case is included to prevent having to deal with exceptional cases).

(Hence $\mathbf{e}$ can *never* belong to a discriminately closed subset (for if $\mathbf{e} \in \mathbf{S}$, then $\mathbf{e} = \mathbf{e} + \mathbf{e} \in \mathbf{S} \quad \Rightarrow \mathbf{e} \neq \mathbf{e}$, a contradiction.)

We may use the obvious abbreviations 'dc' and 'dc-subset'.

9. But any subset $\mathbf{S}$, in the vector space $\mathbf{E}^n$, which contains the sum of every pair of strings from $\mathbf{S}$ is a well known subsystem : it is a *vector subspace* of $\mathbf{E}^n$. So a discriminately closed subset $\mathbf{S}$ is "almost" a vector subspace; it only fails to be a vector subspace through the single fact that the neutral string $\mathbf{e}$ is excluded from it. We can use this fact to characterise our discriminately closed subsets in the following more useful way:

A subset $\mathbf{S} \subseteq \mathbf{E}^n$ is a **Discriminately Closed Subset**
iff $\quad \mathbf{e} \notin \mathbf{S}$ and $\mathbf{S} \cup \{\mathbf{e}\}$ is a vector subspace in $\mathbf{E}^n$.

10. From this characterisation we deduce these useful facts :

(a) The collection of all vector subspaces in $\mathbf{E}^n$ and the collection of all discriminately closed subsets in $\mathbf{E}^n$ are in 1-1 correspondence.

(b) Each dc-subset $\mathbf{S} \subseteq \mathbf{E}^n$ can be got by deleting the neutral string $\mathbf{e}$ from some (unique) vector subspace $\mathbf{T} \subseteq \mathbf{E}^n$; the dc-subset Sis called the 'dc-reduction' $\mathbf{T}^{\setminus e}$ of the vector subspace $\mathbf{T}$, so that $\mathbf{S} = \mathbf{T}^{\setminus e} = \mathbf{T} \setminus \{e\}$.

(c) To each dc-subset $\mathbf{S} \subseteq \mathbf{E}^n$ there corresponds a (unique) vector subspace $\mathbf{S}^e = \mathbf{S} \cup \{\mathbf{e}\}$ called its 'augmentation'.

(d) For each $k = 0..n$ we can say that a dc-subset $\mathbf{S}$ has 'dimension' k *iff* its vector subspace augmentation $\mathbf{S}^e$ has dimension k.

(e) There are as many dc-subsets of dimension k in $\mathbf{E}^n$ as there are vector subspaces of dimension k, namely :

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(2^n - 1)(2^n - 2)\ldots(2^n - 2^{k-1})}{(2^k - 1)(2^k - 2)\ldots(2^k - 2^{k-1})} = \frac{(2^n - 1)(2^{n-1} - 1)\ldots(2^{n-k+1} - 1)}{(2^k - 1)(2^{k-1} - 1)\ldots(2 - 1)}$$

The construction of this formula is explained in no.12 below.
It is easy to see that $\left[{n \atop k}\right] / \left[{n \atop n-k}\right] = 1$ and hence that $\left[{n \atop k}\right] = \left[{n \atop n-k}\right]$, *i.e.* there are as many dc-subsets of dimension k as there are of codimension k (dimension $n - k$).

11. The total number of dc-subsets of all dimensions in the n-dimensional vector space $\mathbf{E}^n$ is frequently given—incorrectly—as $2^n - 1$. The latter number is just the number of

dc-subsets that can be formed from a given *fixed* basis of n independent strings in E^n; it is smaller than the number of all possible dc-subsets that can be formed without restricting oneself to a single fixed basis — it is $66/15 = 4.4$ times smaller when $n = 4$ and about 2.05×10^{16} times smaller (!) when $n = 16$.

12. The correct combinatoric counting of dc-subsets can be understood as follows :

(I) There are many possible choices of n strings to form an (independent) basis for E^n. Suppose we fix on one choice : $b_1, b_2, \ldots, b_n$ say. These first contribute n 1-dimensional dc-subsets $\{b_1\}, \{b_2\}, \ldots, \{b_n\}$; then $\binom{n}{2}$ 2-dimensional dc-subsets such as $\{b_1, b_2, b_1 + b_2\}$, etc., ..., and finally one n-dimensional dc-subset $= E^n$ itself. The total number is the sum

$$\sum_{k=1..n} \binom{n}{k} = 2^n - 1.$$

(II) Now suppose we do not wish to restrict ourselves to just one fixed choice of n basis strings.

(a) First we count the number of 1-dimensional dc-subsets: each one of these is of the form $\{x\}$ where x is a non-neutral string; there are $2^n - 1$ of these.

(b) Next we count the number of 2-dimensional dc-subsets. Each of these is of the form $\{x, y, x + y\}$ where the first string x can be chosen from amongst all the $2^n - 1$ non-neutral strings in E^n, and the second string y must then be chosen from the remaining $2^n - 2$ non-neutral strings in E^n. This ensures that $x \neq y$, $x \neq e$, $y \neq e$, and hence that x and y are independent as required.

However, this way of counting means that we have obviously counted some dc-subsets more than once, for instance in a 2-dimensional dc-subset of the form $\{x, y, z\}$ there are $2^2 - 1 = 3$ different ways of picking the first string from x, y, and z; then there are $2^2 - 2 = 2$ different ways of picking the second string from the remaining two strings, and then there is only 1 way of picking the last string since it is the sum of the previous two. So all together there are $(2^2 - 1)(2^2 - 2) = 6$ different ways of choosing the three strings x,y,z to form the *same* dc-subset $\{x, y, z\}$. Hence the number of *distinct* 2-dimensional dc–subsets is

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{(2^n - 1)(2^n - 2)}{(2^2 - 1)(2^2 - 2)}$$

Here, the numerator counts the number of 2-dimensional dc-subsets—including their repetitions, and the denominator gives the number of different ways each individual dc-subset can be counted. This feature is common to all the $\begin{bmatrix} n \\ k \end{bmatrix}$ formulas.

(c) It is now clear how we should go about counting the number of distinct 3-dimensional dc-subsets in E^n. The result is

$$\begin{aligned}\begin{bmatrix} n \\ 3 \end{bmatrix} &= \frac{(2^n - 1)(2^n - 2)(2^n - 4)}{(2^3 - 1)(2^3 - 2)(2^3 - 4)} \\ &= \frac{(2^n - 1)(2^n - 2)(2^n - 4)}{7.6.4} = \frac{(2^n - 1)(2^n - 2)(2^n - 4)}{168}\end{aligned}$$

(d) The formula in the general case of k-dimensional dc-subsets is now easy to understand. The total number of dc-subsets in $\mathbf{E}^n$ is therefore the sum of these n separate counts :

$$\sum_{k=1..n} \begin{bmatrix} n \\ k \end{bmatrix};$$

where the formulae for the $\left[{n \atop k}\right]$ are given in no.10.(e) above. (It would be desirable to have a closed expression for this sum).

13. On the following page, TABLE 1 gives a partial CENSUS of the number of distinct dc-subsets of dimension k in a Discrimination System $\mathbf{E}^n$ of dimension n.

TABLE 1. CENSUS of dc-subsets in discrimination systems E^n

$\left[{n \atop k} \right]$ = number of k-dimensional dc-subsets = $\prod_{i=0}^{k-1} \frac{(2^{n-i}-1)}{(2^{k-i}-1)}$

n =	1	2	3	4	5	6	7	8
k								
1	1	3	7	15	31	63	127	255
2	-	1	7	35	155	651	2,667	10,795
3	-	-	1	15	155	1,395	11,811	97,155
4	-	-	-	1	31	651	11,811	200,787
5	-	-	-	-	1	63	2,667	97,155
6	-	-	-	-	-	1	127	10,795
7	-	-	-	-	-	-	1	255
8	-	-	-	-	-	-	-	1
all	1	4	15	66	373	2,824	29,211	417,198

— — continuation — —

n =	15	16
k		
1	32,767	65,535
2	178, 940,587	715, 795,115
3	209,386, 049,731	1, 675,267, 338,435
4	57, 162,391, 576,563	914, 807,651, 274,739
5	3,774, 561,792, 168,531	120,843, 139,740, 969,555
6	61,291, 693,863, 308,051	3 ,926,442, 969,043, 883,795
7	246,614, 610,741, 341,843	31, 627,961, 868,755, 063,955
8	246,614, 610,741, 341,843	63, 379,954, 960,524, 853,651
9	61,291, 693,863, 308,051	31, 627,961, 868,755, 063,955
10	3,774, 561,792, 168,531	3, 926,442, 969,043, 883,795
11	57, 162,391, 576,563	120,843, 139,740, 969,555
12	209,386, 049,731	914, 807,651, 274,739
13	178, 940,587	1, 675,267, 338,435
14	32,767	715, 795,115
15	1	65,535
16	-	1
all	623,476, 476,706, 836,147	134, 732,283, 882,873, 635,910

[4] — A PROJECTIVE GEOMETRY ?

1. There are two ways of introducing the idea of a "Projective Geometry" : the synthetic (or axiomatic method) and the analytic (or algebraic coordinate method). The synthetic way postulates undefined abstract concepts called points, lines and planes, etc., and the 'incidence' relationships that have to exist between them. The second way assumes we already know about vector spaces, independence, and linear transformations, and goes on to construct a related system by giving priority to the "rays through the origin" rather than individual vectors themselves.

The fact that in our discrimination system $\mathbf{E}^n$ each "ray through the origin" happens to coincide with a *single* vector (string)—rather than a multiple aggregate of strings, and the fact that the origin (the neutral string **e**) both has no ray and is missing from every dc-subset, make the algebraic method peculiarly attractive (and potentially important !).

2. The algebraic method may be recalled briefly as follows. We start with a vector space **V** of dimension n over a scalar field **K**, with the null vector **0** (the 'origin'). A "ray" in **V** is a 1-dimensional vector subspace, the aggregate of all vectors **x** which are scalar multiples of some non-null vector **a**, *i.e.* $\{\mathbf{x} \mid \mathbf{x} = \alpha.\mathbf{a},\ \alpha \in \mathbf{K}\}$; this ray is "determined" by **a**. The trick is to regard each such **ray** as a single object, namely a **point** in a new system G (of 'dimension' $n-1$). Of course, the origin **0** cannot have any representation in the new G. This new system G will be the $(n-1)$-dimensional Projective Geometry 'associated' with the n-dimensional vector space **V**. We can write $\mathbf{G} = \mathbf{G}(\mathbf{V})$.

- The 1-dimensional **rays** in **V** are in 1-1 correspondence with the 0-dimensional **points** in G.

3. Next consider a 2-dimensional vector subspace in **V**. It is spanned by two independent vectors **a** and **b**, say, and can be written as $<\mathbf{a},\mathbf{b}>$. Any vector **x** in $<\mathbf{a},\mathbf{b}>$ has the linear combination form $\mathbf{x} = \alpha.\mathbf{a} + \beta.\mathbf{b}$ where α and β are any scalars. In the projective geometry **G** the rays through **a** and **b** will be two definite points **A** and **B**; moreover the variable rays through **x** will be variable points **X**. The important thing is that as **x** moves through $<\mathbf{a},\mathbf{b}>$ in **G** the point **X** moves about in **G** along a line **A-B**. [Of course, if **x** happens to move along a ray in $<\mathbf{a},\mathbf{b}>$ then **X** remains fixed on the line **A-B**.]

Thus we see that every 2-dimensional vector subspace in **V** is represented as a 1-dimensional **line** in **G**. To be more precise :

- The 2-dimensional vector subspaces in **V** are in 1-1 correspondence with the 1-dimensional **lines** in **G**.
- Every vector in a 2-dimensional vector subspace in **V** *except the origin* **0** has its corresponding **point** in the corresponding **line** in the projective geometry **G**.

4. Similar notions are available for k-dimensional subspaces in $\mathbf{V}$, with $k > 2$. For $k = 3$, 3-dimensional vector subspaces in $\mathbf{V}$ correspond with **planes** in $\mathbf{G}$. For $k > 3$, k-dimensional vector subspaces correspond to $(k-1)$-**hyperplanes** in $\mathbf{G}$.

5. Intersection and meet; union and join.

(a) Two vector subspaces of dimensions k and k' intersect in a vector subspace of dimension $i \leq \min[k, k']$. The corresponding objects (points, lines, planes, etc.) **meet** in an $(i-1)$-dimensional object. *E.g.*the lines corresponding to two 2-dimensional subspaces meet either in their common line, or in single point, or not at all. (An empty meet is conventionally said to have dimension $i = -1$).

(b) The union of two vector subspaces of dimensions k and k' is a j-dimensional vector subspace, where $\max[k, k'] \leq j \leq k + k'$. The corresponding objects have a $(j-1)$-dimensional **join**. *E.g.*the **join** of two lines corresponding to two 2-dimensional subspaces is either a line, a plane, or a 3-dimensional hyperplane (a solid).

(c) We note that for two vector subspaces of dimensions k and k' with intersection and union of dimensions i and j, we have the fundamental relation $i + j = k + k'$ in $\mathbf{V}$. Since their corresponding projective geometry objects all have dimensions one less than these, the same relation between i, j, k, k' holds in $\mathbf{G}$.

6. From the fundamental i, j, k, k' relation we can at once deduce the so-called Propositions of Incidence in a Plane in $\mathbf{G}$:

- Given two distinct points, there is one and only one line which contains them both.
- Given two distinct lines (lying in the same plane), there is one and only one point common to both.

Analogous propositions of incidence hold for points, lines and planes, etc., in higher dimensional objects in $\mathbf{G}$.

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[5] — A DISCRIMINATION SYSTEM IS A PROJECTIVE GEOMETRY

1. Let us remind ourselves of the important characterisation of a discriminately closed subset in a discrimination system $\mathbf{E}^n$:

> A subset $S \subseteq \mathbf{E}^n$ is a Discriminately Closed Subset *iff*
>
> $e \notin S$ and $S \cup \{e\}$ is a vector subspace in $\mathbf{E}^n$.

2. Let us next observe that since our discrimination system $\mathbf{E}^n$ is a vector space over the field $\{e, u\} \cong \{0,1\}$ with only two elements, its rays are particularly simple. A "ray" a in $\mathbf{E}^n$ contains only two strings; one is the "origin" or neutral e, the other is the determining (non neutral) string a. Thus every non-neutral string a determines a unique ray in $\mathbf{E}^n$ and every ray arises in this way : there is a 1-1 correspondence between the 1-dimensional rays and the non-neutral strings. Let $G^{n-1} = G(\mathbf{E}^n)$ denote the projective geometry associated with our discrimination system $\mathbf{E}^n$. Hence there is a 1-1 correspondence between the non-neutral strings in $\mathbf{E}^n$ and the 0-dimensional points in G^{n-1}. Nothing in G^{n-1} corresponds to the neutral string e in $\mathbf{E}^n$.

3. Now we know the 2-dimensional vector subspaces in $\mathbf{E}^n$ correspond to the 1-dimensional lines in G^{n-1}. But every 2-dimensional vector subspace—with its origin removed—in our discrimination system $\mathbf{E}^n$ is a 2-dimensional discriminately closed subset (and conversely). Thus we arrive at our important identification :

- Every 2-dimensional **dc-subset** in our n-dimensional discrimination system $\mathbf{E}^n$ is represented by a 1-dimensional **line** in the corresponding $(n-1)$-dimensional projective geometry G^{n-1}.

- The 2-dimensional dc-subsets in $\mathbf{E}^n$ are in 1-1 correspondence with the 1-dimensional lines in G^{n-1}.

4. Furthermore, each 2-dimensional dc-subset in $\mathbf{E}^n$ has precisely three unequal strings, *e.g.* a, b, and $c = a + b$. Hence each line in G^{n-1} has precisely three distinct points, *e.g.* A, B, and $C \equiv A + B$.

5. Each 3-dimensional dc-subset in $\mathbf{E}^n$ is spanned by three independent strings a,b,c, say, and can be represented by $< a, b, c >$. It contains $2^3 - 1 = 7$ strings a, b, c, $p = a+b$, $q = b+c$, $r = c+a$, $s = a+b+c$. The corresponding plane in G^{n-1} contains seven points A, B, C, P, Q, R, S.

There are also seven 2-dimensional subspaces in the 3-dimensional dc-subset $< a, b, c >$, each containing 3 strings :

(a,b,p), (b,c,q), (c,a,r), (a,q,s), (b,r,s), (c,p,s), (p,q,r).

The plane containing the three distinct points A, B, C in G^{n-1} therefore also contains the seven 3-point lines :

(A,B,P), (B,C,Q), (C,A,R), (A,Q,S) , (B,R,S) , (C,P,S), (P,Q,R).

These features are illustrated in Figure 1.

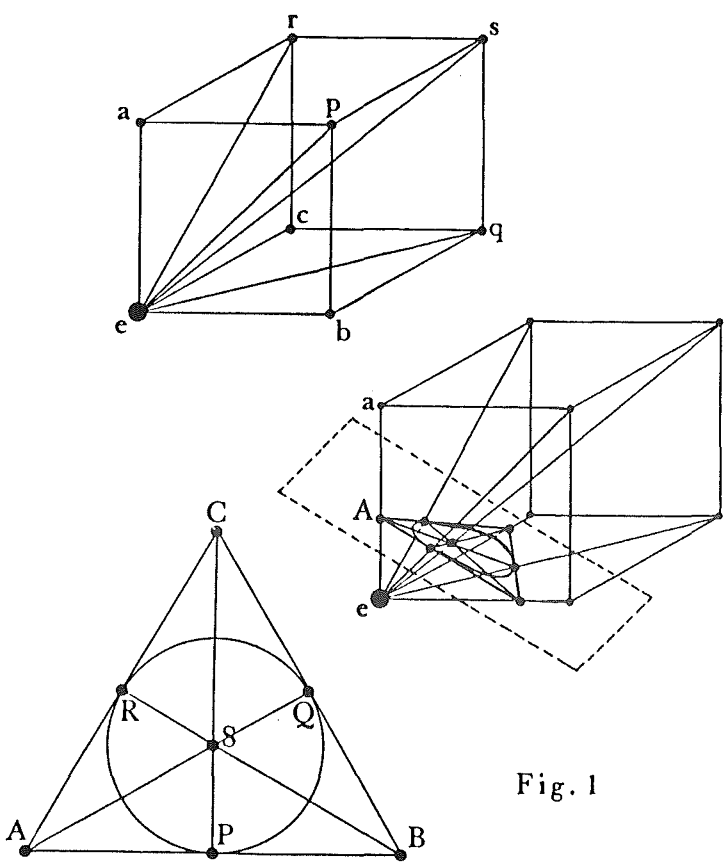

Fig. 1

[6] — MAPPINGS, COLLINEATIONS, AUTOMORPHISMS

1. An important application of Discrimination System theory is in the construction of the Combinatoric Hierarchy. This makes use of a sequence of discrimination systems with increasing dimensions ('levels') :

TABLE 2. Levels of the Combinatoric Hierarchy

Discr.System	dimension n	no. of points $= 2^n$
$\mathbf{E}_0 = \mathbf{E}$	1	2
$\mathbf{E}_1 = \mathbf{E}^2$	2	4
$\mathbf{E}_2 = \mathbf{E}^4$	$2\times2 = 4$	16
$\mathbf{E}_3 = \mathbf{E}^{16}$	$4\times4 = 16$	65,536
$\mathbf{E}_4 = \mathbf{E}^{256}$	$16\times16 = 256$	(*)$1.1579..\times10^{77}$

(*) The full-precision value is :
115792, 089237,316195, 423570,985008,
687907,853269, 984665,640564, 039457,584007, 913129,639936

These levels have algebraic connections between them given in terms of non-singular mappings which preserve the vector subspace (and hence dc-subset) structure in the level in which they operate. Such mappings are plainly linear; indeed they are automorphism, and hence given by non-singular matrices. The matrices for one level are identified with vectors in the next higher level (the details of the construction, which need not concern us here, have been reported elsewhere).

2. Since we can pass freely between a discrimination system $\mathbf{E}^n$ and its associated projective geometry $\mathbf{G}^{n-1}$ it becomes important to consider which are the mappings in $\mathbf{G}^{n-1}$ which preserve the structures corresponding to the dc-subsets in $\mathbf{E}^n$. Such mappings will plainly correspond to the automorphisms in $\mathbf{E}^n$.

3. Let $\mathbf{G}$ be a projective geometry corresponding to a vector space $\mathbf{V}$. By definition, a collineation in $\mathbf{G}$ is a permutation which maps lines to lines, *i.e.* which preserve the 'line structure' in $\mathbf{G}$.

4. Being a permutation, a collineation necessarily maps distinct points into distinct points (*i.e.* is 'non-singular' in an obvious sense). Since a line in $\mathbf{G}$ corresponds to a unique 2-dimensional dc-subset the corresponding mapping in $\mathbf{V}$ also preserves the 2-dimensional dc-subset structure in $\mathbf{V} \setminus \{e\}$. If we extend this mapping to the neutral string e by mapping it to itself, then it is now a permutation on all of $\mathbf{V}$, and one which preserves the vector space structure in $\mathbf{V}$. Hence it is therefore a non-singular linear mapping, *i.e.* an automorphism of $\mathbf{V}$.

5. Conversely, each automorphism of **V** corresponds to a (unique) collineation of G. The argument for this is as follows : An automorphism maps the neutral to itself and permutes all the non-neutral strings; hence the corresponding mapping in **G** is also a permutation. An automorphism is a 1-1 linear mapping and hence maps vector subspaces onto vector subspaces; hence, in particular, it maps 2-dimensional dc-subsets onto 2-dimensional dc-subsets; hence the corresponding mapping in **G** maps lines to lines. Thus the corresponding mapping is a collineation in **G**, as asserted.

6. From nos.5 and 6 we have the fundamental result:

• The automorphisms on our discrimination system $\mathbf{E}^n$ are in 1-1 correspondence with the collineations on our associated projective geometry $\mathbf{G}^{n-1}$. (N.b. In more general projective geometries there are usually more collineations than automorphisms; they happen to have the same numbers here because our scalar field happens to have only 2 elements.)

7. Since an automorphism will transform one basis (a set of n independent strings) in $\mathbf{E}^n$ into another, the number of automorphisms on $\mathbf{E}^n$ will be the same as the number of bases. We can choose the first string in a basis from $2^n - 1$ different non-neutral strings, the second from the remaining $2^n - 2$, the third from the $2^n - 4$ non-neutrals which are independent from those in the subspace generated by the first two choices, and so on. Hence, gathering together the common power-of-2 factors, we see that the number of automorphisms on $\mathbf{E}^n$, and hence the number of collineations on $\mathbf{G}^{n-1}$, are given by the formula :

$$|\,\mathrm{AUT}_n\,| \;=\; |\,\mathrm{COLL}_{n-1}\,| \;=\; 2^{n(n-1)/2}.\prod_{k=2}^{n}(2^k - 1)$$

A short CENSUS of automorphisms in $\mathbf{E}^n$ is given in TABLE 3. at the end of this chapter.

8. A 'subspace' of dimension $k = 0..(n-1)$ in the projective geometry **G** is an aggregate of points which corresponds to a dc-subset of dimension k and hence a vector subspace of dimension $k+1$ in the vector space **V**. A **hyperplane** in **G** corresponds to a **maximal subspace** *i.e.* a vector subspace of codimension 1 in **V**(one whose dimension is 1 short of the full dimension n of **V**). *E.g.* the lines in **G** are 1-dimensional subspaces; the 3-dimensional solids are hyperplanes in $\mathbf{G}^4$.

Since there are as many hyperplanes as there are points (see 3.9.(e)), there are $\left[{n \atop n-1}\right] = \left[{n \atop 1}\right] = 2^n - 1$ hyperplanes in $\mathbf{G}^{n-1}$.

A subspace is 'fixed' by a collineation in **G** *iff* every one of its points is mapped into itself ('fixed') by the collineation. It is 'invariant' if each of its points is mapped into some point (not necessarily the same one) in the subspace.

9. Plainly every hyperplane in **G** is fixed by the identity collineation (the "trivial" collineation). But some collineations may never fix any hyperplane. If it should happen that some hyperplane is fixed by a collineation, the hyperplane is called an **axis** of the

collineation (the transformation "revolves around it") and the collineation is said to be an **axial** collineation.

Hence the automorphism on **V** corresponding to an axial collineation fixes some maximal subspace (its 'axis space').

10. Since a linear mapping is determined by what it does on a basis, an automorphism will fix a maximal subspace M if it fixes a basis set of $n-1$ independent strings in M. By the same kind of argument as in no.7 above there are

$$|\,\mathrm{AUT}_{n-1}\,| = 2^{(n-1)(n-2)/2}.\prod_{k=2}^{n-1}(2^k-1)$$

such bases for M. Hence the ratio of the total number of automorphisms to to the number that fix a maximal subspace M is

$$|\,\mathrm{AUT}_n\,| \div |\,\mathrm{AUT}_{n-1}\,| \quad = \quad 2^{n-1}.(2^n-1).$$

These values are also shown in TABLE 3. below. Since there are 2^n-1 maximal subspaces, the total number of automorphisms that fix maximal subspaces ($\equiv$ the total number of axial collineations) is $|\,\mathrm{AUT}_{n-1}\,| \times 2^n-1$ which is easily seen to be $|\,\mathrm{AUT}_n\,| / 2^{n-1}$.

I.e. the fraction $1/2^{n-1}$ of all automorphisms on $\mathbf{E}^n$ fix maximal susbspaces in $\mathbf{E}^n$ ($\equiv$ the fraction of all collineations that are axial in $\mathbf{G}^{n-1}$).

11. If a point **P** in **G** has every subspace containing it fixed by a collineation (it is enough that every line through **P** be fixed), then the point **P** is called a **centre** of the collineation, and the collineation is said to be a **centred** collineation.

Hence the automorphism on **V** corresponding to a centred collineation fixes every subspace through the corresponding 'centre' string **p**.

12. It turns out that a non-trivial collineation has at most one axis and at most one centre. And if it has either an axis or a centre then it has both.

Correspondingly, a non-trivial automorphism on **V** has both an axis and a centre or neither.

13. If a collineation has an axis—and therefore a centre—then the centre need not lie in the axis. If it does then the collineation is called an **elation**; if it doesn't then the collineation is called a **homology**. In our geometry (with its field of just 2 scalars) the only homology is the trvial one, the identity. So every axial ($\equiv$ centred) collineation is an elation, *i.e.* the centre is always in the axis. The important feature of centred collineations is that they generate the group of all collineations on **G**. In our case this means that every collineation is the result of applying a sequence of axial ($\equiv$ centred) collineations (*i.e.* a sequence of elations).

Correspondingly, every automorphism on **V** is the product of axial ($\equiv$ centred) automorphisms. From the counting result in no.10 above, it follows that the group of automorphisms on $\mathbf{E}^n$ is generated by the fraction $1/2^{n-1}$ of its total number. (To simplify the study of automorphisms on our discrimination systems, it would be desirable to construct canonical forms for these generating automorphisms.)

TABLE 3. CENSUS of automorphims in discrimination systems E^n

dimension n	$\mid AUT_n \mid$	$\mid AUT_n \mid \div \mid AUT_{n-1} \mid$
2	6	6
3	168	28
4	20,160	120
5	9,999,360	496
6	20,158,709,760	2,016
7	$1.638,499..\times 10^{14}$	8,128
8	$5.348,063..\times 10^{18}$	32,640
9	$6.996,123..\times 10^{23}$	130,816
10	$3.664,401..\times 10^{29}$	523,776
11	$7.681,054..\times 10^{35}$	2,096,128
12	$6.441,762..\times 10^{42}$	8,386,560
13	$2.161,232..\times 10^{50}$	33,550,336
14	$2.900,580..\times 10^{58}$	134,209,536
15	$1.557,189..\times 10^{67}$	536,854,528
16	(*) $3.343,988..\times 10^{76}$	2,147,450,880

$$\mid AUT_n \mid \;=\; 2^{n(n-1)/2} . \prod_{k=2}^{n} (2^k - 1)$$

(*) The full-precision value is :
33439, 887126,531088, 671831,929227,
837976,590084, 758712,242507, 868731,544889, 972490,240000

1. In chapter 6 we discussed collineations : mappings which permuted the points of our projective geometry G^{n-1} and which mapped lines to lines (and hence subspaces to subspaces). In this chapter we will be interested in mappings which permute the subspaces in G^{n-1} and which have the extra property of 'inverting inclusions': *i.e.* if P is such a mapping and S, T are any two subspaces in G^{n-1} with $S \subseteq T$, then $P(T) \subseteq P(S)$. Such a mapping is called a **correlation**.

2. In 3.10.(e) we saw that in G^{n-1} the number of subspaces of dimension k is the same as those of dimension $n-k$: $\left[{n \atop k}\right] = \left[{n \atop n-k}\right]$. *E.g.* there are as many points as hyperplanes. Because of this we can identify each subspace of dimension k with a unique "dual" subspace of dimension $n-k$, and *vice-versa*; *e.g.* we can make points and hyperplanes correspond in a 1-1 way. Any mapping which carries out such an identification is, in fact, a correlation: every correlation interchanges the set of all subspaces of dimension k with the set of all those of dimension $n-k$.

3. Correlations can be constructed by making use of the analogue of the "orthogonal subspace" idea familiar to euclidean spaces. For this we need the idea of a "bilinear form" on our vector space $\mathbf{E}^n$.

A bilinear form is a function $B : \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{K}$, which is 'bilinear':

$$B(\mathrm{x}+\mathrm{y},\mathrm{x}'+\mathrm{y}') \;=\; B(\mathrm{x},\mathrm{x}') + B(\mathrm{x},\mathrm{y}') + B(\mathrm{y},\mathrm{x}') + B(\mathrm{y},\mathrm{y}')$$

for all x, y, x', y' in $\mathbf{E}^n$. Thus B assigns 0 or 1 to each pair x, y in $\mathbf{E}^n$ (recall that $\mathbf{K}$ is the two element field $\{e,u\} \equiv \{0,1\}$).

4. As a simple illustration, we see that a bilinear B has

$$B(\mathrm{x},\mathrm{z}) = B(\mathrm{y},\mathrm{z}) \Leftrightarrow B(\mathrm{x},\mathrm{z}) + B(\mathrm{y},\mathrm{z}) = 0 \Leftrightarrow B(\mathrm{x}+\mathrm{y},\mathrm{z}) = 0.$$

5. The prototype bilinear form in a vector space is, of course, a 'scalar or dot product'. If $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (y_1, \ldots, y_n)$ in G^{n-1}, then we can define

$$\mathrm{x} \bullet \mathrm{y} = (\mathrm{x}_1 * \mathrm{y}_1) + \cdots + (\mathrm{x}_n * \mathrm{y}_n)$$

where $+$ is addition (discrimination, XOR) in the field $\{e,u\}$, and the operation * is field multiplication (see the multiplication table in 3.3, where it is plain that * is just the boolean logic AND operation).

6. It is useful to borrow the 'perpendicular' symbol $\perp$ from geometry to write any bilinear function expression $B(\mathrm{x},\mathrm{y}) = 0$ as $\mathrm{x} \perp \mathrm{y}$, when it is clear which particular B is intended. (Otherwise we would use $\perp_B$.)

7. Just as "orthogonal complements" are formed in euclidean space we can form a "bilinear complement" for any subspace W in $\mathbf{E}^n$:

$$W^\perp \ = \ \{\mathrm{x} \in \mathrm{G}^n \mid \mathrm{x} \perp \mathrm{W}\}$$

Note that $W^\perp$ is a vector subspace in $\mathbf{E}^n$ even if the subset W itself isn't a subspace (use no.4 above). So it is plain that 'forming the perp' is an mapping from subspaces W to other subspaces $W^\perp$. ('perp' $\sim$ perpendicular $\sim$ orthogonal.)

8. We also need the idea of a bilinear B being *non-degenerate*; this is the case *iff* $B(\mathrm{x},\mathrm{y}) = 0$ for all $\mathrm{y} \in \mathbf{E}^n \Leftrightarrow \mathrm{x} = 0$, or, equivalently, *iff* $(\mathbf{E}^n)^\perp = \{\mathrm{e}\}$. *I.e.* only the neutral string is orthogonal to every string (if B were degenerate then there will be at least one non-neutral string orthogonal to every string in $\mathbf{E}^n$). This will ensure that the 'forming the perp' mapping is non-singular, *i.e.* is in fact a permutation of the subspaces in $\mathbf{E}^n$.

9. Then we have the useful fact that any non-degenerate bilinear B induces a permutation $(W) \longrightarrow (W)^\perp$ of the subspaces (W) in the projective geometry G^{n-1}.
(Here (W) is just a temporary notation for the subspace in G^{n-1} associated with the vector subspace W in the discrimination system $\mathbf{E}^n$).

10. Because 'forming the perp' is itself plainly an 'inclusion inverting' operation, it follows that such a permutation is also inclusion inverting and is hence a correlation on G^{n-1}. It is also the case that every correlation on G^{n-1} can be formed in this way *via* a non-degenerate bilinear form on $\mathbf{E}^n$.

Because bilinear forms on $\mathbf{E}^n$ can always be represented by an $n \times n$ matrix with entries from $\mathbf{K}$, using the familiar transposed-vector$\times$matrix$\times$vector multiplication $B(\mathrm{x},\mathrm{y}) = \mathrm{x}^{tr}.\mathrm{MATRIX}.\mathrm{y}$, it also follows that every correlation on G^{n-1} is represented by a non-singular $n \times n$ matrix.

As an example, consider the "trivial" bilinear form given by the dot- product in no.5 above. Its matrix is just the $n \times n$ identity matrix. For any non-neutral string x in $\mathbf{E}^n$, the correlation image $\{\mathrm{e},\mathrm{x}\}^\perp$ of the 1-dimensional vector subspace $S = \{\mathrm{e},\mathrm{x}\}$ is the $(n-1)$-dimensional vector subspace of those y such that $\mathrm{x} \bullet \mathrm{y} = 0$, *i.e.* such that $\mathrm{x} \perp \mathrm{y}$, *i.e.* the maximal subspace $S^\perp$ orthogonal to $\{\mathrm{e},\mathrm{x}\}$. Now pass to the projective geometry G^{n-1}, and let $\mathbf{X}$ and $\mathbf{Y}$ be the points corresponding to the 1-dimensional subspaces with non-neutral strings x and y. The image under the "$\bullet$"-correlation of the corresponding point $\mathbf{X}$ is the hyperplane corresponding to the maximal subspace $S^\perp$, *i.e.* the hyperplane $\mathbf{X}^\perp$ orthogonal to $\mathbf{X}$. Similarly it can be shown that the "$\bullet$"-correlation of a 2-dimensional line through two points $\mathbf{X}$ and $\mathbf{Y}$ is the $(n-2)$-dimensional intersection of the two $(n-1)$-dimensional hyperplanes $\mathbf{X}^\perp$ and $\mathbf{Y}^\perp$, and so on.

11. Of course, there are precisely the same number of such 'correlation' matrices as there are automorphisms on $\mathbf{E}^n$, so we already know how many correlations there are

on G^{n-1} (see TABLE 3 above). For brevity—when there is no danger of confusion—we frequently use the same symbol such as B for a correlation, its bilinear form and its representing matrix.

12. The next geometrical situation of interest to us is a **polarity**; this is any correlation B which, when applied a second time, leaves every point un-transformed, *i.e.* $\mathbf{x} \longmapsto B(\mathbf{x}) \longmapsto B((B(\mathbf{x}) = \mathbf{x}$, *i.e.* $B^2 = \mathbf{I}$, the identity transformation. (Another way of saying this is that a polarity is an 'involutory' correlation.)

Since every correlation is inclusion inverting (see no.1 above), it is not difficult to see that for a subspace S we have $S \subseteq B(B(S))$ for any correlation B. Using 'perp' notation this becomes $S \subseteq S^{\perp\perp}$. The polarities are then precisely those correlations which, for all subspaces S in $\mathbf{E}^n$ we have $S = S^{\perp\perp}$.

13. It also turns out that an equivalent condition for a correlation to be a polarity is that its bilinear form shall be *symmetric*, *i.e.* $B(\mathbf{x},\mathbf{y}) = B(\mathbf{y},\mathbf{x})$ ($\mathbf{x} \perp \mathbf{y} = \mathbf{y} \perp \mathbf{x}$) for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{E}^n$. (*N.b.* We take this for granted in a familiar euclidean space, but we must not automatically assume in our discrimination systems that if $\mathbf{x}$ is perpendicular to $\mathbf{y}$ then $\mathbf{y}$ is perpendicular to $\mathbf{x}$; it depends on which particular 'perpendicularity'—*i.e.* correlation—we are using.)

Thus to count the polarities on our projective geometry G^{n-1} we need to count the *symmetric* non-singular $n \times n$ matrices over the field $\{0, 1\}$. The result is .... (unfinished: 4 out of 6 in AUT_2, 28 out of 168 in AUT_3, 448 out of 20,160 in AUT_4,......., general formula ?).

14. Another geometrical situation of interest to us is a **null polarity**; this is any polarity B which has $B(\mathbf{x},\mathbf{x}) = 0$, *i.e.* $\mathbf{x} \perp \mathbf{x}$, for all $\mathbf{x}$ in $\mathbf{E}^n$ — "every string is perpendicular to itself". (A null polarity is sometimes called a symplectic polarity; we will use the shorter name.) A simple example is easy to find in $\mathbf{E}^2$: Take for B the symmetric 2×2 matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and any string $\mathbf{x} = \{\xi, \eta\}$. Then

$$B(\mathbf{x},\mathbf{x}) = [\xi \ \eta].\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \xi * \eta + \eta * \xi = \varsigma + \varsigma = 0.$$

In our projective geometry this means that every point belongs to the hyperplane which is perpendicular to it under a null polarity. (In fact this is a necessary and sufficient condition for a polarity to be null.) Any point is said to be **absolute** if it belongs to the hyperplane perpendicular to it under some polarity; so a polarity is null *iff* every point is absolute.

A polarity can be null only if the corresponding vector space has even dimension. Since the discrimination systems in the combinatorial hierarchy each have even dimension (2,4,16,256), it follows that we will always have null polarities present.

Their number (for even dimension n) is then given by the formula:

$$|\mathrm{NULPOL}_n| \quad = \quad 2^{a(n)} . \prod_{k=1}^{b(n)} (2^{2k+1} - 1).$$

where $a(n) = \frac{(n-1)^2-1}{4}$ and $b(n) = \frac{n-2}{2}$.

A short CENSUS of null polarities is given in TABLE 4. at the end of this chapter.

All these null polarities are equivalent in the sense that if B and B' are any two, then an automoprhism A can be found such that $B' = A^{-1}.B.A$. *I.e.* their matrices are similar.

15. The final geometrical situation of interest to us here is an extension of the idea of an absolute point (see no.14 above) to all of a vector subspace. A vector subspace S in E^n is called **totally isotropic** *iff* every one of its strings is perpendicular to some string in the same subspace under some polarity, *i.e.* $S \subseteq S^{\perp}$. Clearly the dimension of S cannot be greater than $n/2$, since if S has dimension k then $S^{\perp}$ has dimension $n-k$, so $k \leq n-k$ and $2k \leq n$. If the polarity is a null polarity then $\mathbf{x} \perp \mathbf{x}$ for every $\mathbf{x}$; so if $\mathbf{x} \in \mathrm{S}$ then $\mathbf{x} \in \mathrm{S}^{\perp}$; so $S \subseteq S^{\perp}$, *i.e.* S is totally isotropic and in fact its dimension could take its maximal value $n/2$.

For any one of these (equivalent) null polarities, the number of totally isotropic subspaces of dimension k ($\leq n/2$) in $\mathbf{E}^n$ is given by the formula:

$$|\mathrm{TOTISO}_k^n| = \prod_{i=0}^{k} \frac{(2^{n-2i}-1)}{(2^{i+1}-1)}$$

A short CENSUS of totally isotropic subspaces in $\mathbf{E}^n$ is given in TABLE 5. at the end of this chapter.

TABLE 4. CENSUS of Null Polarities in $\mathbf{E}^n$ (see no.14 above)

n	$\lvert \text{NULPOL}_n \rvert$
2	1
4	28
6	13,888
8	112, 881,664
10	$1.476,672..\times 10^{13}$
12	$3.095,295..\times 10^{19}$
14	$1.038,481..\times 10^{27}$
16	$5.575,137..\times 10^{36}$

The full-precision values of the last four counts are:

14, 766,727, 757,824
30, 952,951, 521,552, 105,472
1,038, 481,923, 739,784, 380,093, 038,592
557,513, 723,005, 853,938, 196,620, 829,393, 944,576

TABLE 5. CENSUS of Totally Isotropic vector subspaces of dimension k in discrimination system $\mathbf{E}^n$ (see no.15 above)

$n =$	2	4	6	8	..	16
k						
1	3	15	63	255	..	65,535
2	0	15	315	5,355	..	357, 886,635
3	-	0	135	11,475	..	209,363, 681,475
4	-	0	0	2,295	..	14, 278,603, 076,595
5	-	-	0	0	..	117, 453,025, 307,475
6	-	-	0	0	..	117, 453,025, 307,475
7	-	-	-	0	..	13, 872,404, 563,875
8	-	-	-	0	..	163,204, 759,575
9..16	-	-	-	-	..	0
totals	3	30	513	19,380	..	263, 429,984, 648,640

1. A quadric (*alias* quadric variety) in our discrimination system $\mathbf{E}^n$ is the analogue of a conic in euclidean space. As in that familiar situation, quadrics here are determined by quadratic forms—functions intimately connected with the symmetric bilinear forms studied in the preceding chapter. They are the generalisations of the elementary $x \mapsto x^2$ function which has the fundamental quadratic property $(x+y)^2 = x^2 + y^2 + b(x,y)$ where $b(x,y)$ is the bilinear function $2xy$, so that $2xy = (x+y)^2 - x^2 - y^2$.

2. A function $Q : \mathbf{E}^n \longrightarrow \mathbf{K}$ (recall that $\mathbf{K}$ is the scalar field $\{\mathrm{e},\mathrm{u}\} \equiv \{0,1\}$, so that '$-$' $=$ '$+$') is called **quadratic** when $Q(\mathrm{e}) = 0$, and

$$B(\mathrm{x},\mathrm{y}) \quad \stackrel{\mathrm{def}}{=} \quad Q(\mathrm{x}+\mathrm{y}) + Q(\mathrm{x}) + Q(\mathrm{y})$$

is bilinear (it is plainly symmetric). Plainly B is null, since for all $\mathrm{x} \in \mathbf{E}^n$,

$$B(\mathrm{x},\mathrm{x}) \;=\; Q(\mathrm{e}) + Q(\mathrm{x}) + Q(\mathrm{x}) \;=\; 0 + \varsigma + \varsigma \;=\; 0$$

Two cases can arise, depending on whether the symmetric bilinear form B is degenerate or not.

(a) If B is non-degenerate (*i.e.*$\mathbf{E}^{n\perp} = \{\mathrm{e}\}$) then the quadratic form Q is also called non-degenerate. In this case, the quadratic Q induces *via* the non-degenerate symmetric B a polarity (see 7.13); and because we have just seen that $B(\mathrm{x},\mathrm{x})$ is identically 0, it is a null polarity. This also means that the dimension n of the discrimination system $\mathbf{E}^n$ must be even (see 7.14).

(b) If B is degenerate (*i.e.*$\mathbf{E}^{n\perp}$ contains at least one $\mathrm{z} \neq \mathrm{e}$) then we define Q to be non-degenerate *iff* $Q(\mathrm{z}) \neq \mathrm{e}$ for all such non-neutral z. In this case the dimension n of $\mathbf{E}^n$ can be odd.

As usual, we can think of Q and B as acting either in the discrimination system $\mathbf{E}^n$ or the corresponding projective geometry $\mathbf{G}^{n-1}$.

3. A **quadric** is then defined to be the set of all strings x in $\mathbf{E}^n$ such that $Q(\mathrm{x}) = 0$, for some quadric Q.

[Again, by abuse of language, we use the same symbol Q to refer to the quadratic form, the null polarity induced by it, or to the quadric determined by it. And, passing to the projective geometry $\mathbf{G}^{n-1}$ we can identify the corresponding points which 'belong' to the 'quadric' Q in $\mathbf{G}^{n-1}$.]

4. If a vector subspace lies in a non-degenerate quadric then it must be totally isotropic (see 7.15), and hence its dimension cannot be more than $n/2$. The maximal dimension of such a vector subspace is denoted by $m(Q)$ and is called the 'index' of the quadric Q.

Just as for null polarities in spaces of even dimension n, two quadrics Q and Q' are called equivalent (see 7.14) *iff* there is an automorphism A such that $Q' = A^{-1}.Q\,A$. (This is the same as saying there is a change of basis for $\mathbf{E}^n$ under which the symmetric matrix for Q will be transformed into the matrix for Q' and *vice versa*.)

5. If Q is non-degenerate quadric in $\mathbf{E}^n$ with symmetric non-singular matric $B = [B_{ij}]$ then the 'points' on the Q are those strings $\mathbf{x} = [x_i]$ which satisfy the 'binary quadratic form' $\mathbf{x}^{tr}\, B\, \mathbf{x} = 0$, *i.e.*

$$0 \quad = \quad \sum_{i,j} B_{ij} x_i x_j \quad = \quad \sum_i B_{ii} x_i x_i \quad + \quad \sum_{i \neq j} B_{ij} x_i x_j$$

The terms in the last summation can be grouped together in pairs $B_{ij} x_i x_j + B_{ji} x_i x_j$, each of which vanish since by symmetry $B_{ij} x_i x_j = B_{ji} x_i x_j$ and $1 + 1 = 0$ in our field $0, 1$. We also have $x_i x_i = x_i$ in that field. It follows that the binary quadratic form for a quadric in a discrimination system $\mathbf{E}^n$ becomes very simple :

$$0 \quad = \quad B_{11} x_1 + \ldots + B_{nn} x_n$$

If we write $diagB$ for the string whose n elements are those of the diagonal of the $n \times n$ matrix B, then we can also write this as :

$$0 \quad = \quad diagB \bullet \mathbf{x}$$

In other words, every string $\mathbf{x}$ in a quadric $Q(\mathbf{x}) = 0$ has a zero scalar- product with the diagonal $diagB$ of the quadric's symmetric non-singular matric B, and hence the set of such strings form a subspace $(diagB)^{\perp}$. (This is in contrast to the situation in geometries over fields with more than 2 elements, where quadrics may contain subspaces but are not in general a subspace themselves.)

Using this relationship makes it very much easier to calculate the strings which belong to a given quadric in a discrimination system. Because of the potential importance of the rôle of quadrics, a fuller report on "Quadrics in a Discrimination System" is under preparation, and will be presented at a future meeting of ANPA.

[9] — Further Outlook, Bibliography

The topics discussed in this paper all deal with the structure that occurs naturally in a discrimination system when it is regarded as a vector space over the field of two elements, or equivalently — when the rôle of its discriminately closed subsets is emphasised — as its associated projective geometry. The vector space structure and the permutations which preserve that that structure — its automorphisms — could all be studied without ever referring to the projective geometry. But this would be to tie our hands needlessly since there is already such a wealth of information about these projective geometries.

The next topic to which priority should be given to its study is the action of the groups of automorphisms on the discrimination systems. Many of the significant facts about which points and which dc-subsets are left fixed, or are un-fixed, by which automorphisms, are known, but there are other aspects which are quite possibly of further significance for our understanding of discrimination systems.

This is especially true when we remember that each automorphism on the discrimination system which forms one level of the Combinatorial Hierarchy is itself represented as a member of the next level. The particular way in which certain automorphisms are selected in the higher level according to their action of fixing and un-fixing certain objects in the lower level can be regarded as a means of transferring "information" between levels. It is crucially important to be fully aware of the structures that are available in the groups of these automorphisms.

Equally important is the need to have the fullest amount of information about the combinatorial aspects of the internal structures in and between each level. Projective geometries over the finite field of two elements (the field of characteristic 2) give rise to special occurrences of geometrical and algebraic structures, such as quadrics and ovoids, which are familiar elsewhere but are less so in the present context of discrimination systems. These ought to be explored in detail since they can clearly be regarded as relevant to the task of recognising sub-systems which might be seen to be identifiable with primitive objects in a discrete physics.

Many standard studies of projective geometries and finite groups of automorphisms have placed their emphasis on the systems with fields of more than two elements. Indeed, many of the 'nice' properties of those systems fail to hold in our systems over the field of two elements, and the latter are therefore often accorded less detailed treatment or have to have their results established by non-general methods. In that respect it would be a useful and practical service to extend the study of our present systems by treating in more detail these less frequently worked areas.

There is yet a further aspect of our systems over the field of two elements which must have far-reaching significance for the development of a discrete physics that is ultimately founded on a discrimination system approach. From this point of view, each of our discrimination systems is not only a vector space and a projective geometry, but also a boolean algebra. In addition to this, the boolean algebra can be given the structures of a (finite) metric space, a normed space, and an inner-product space, by adopting the compound boolean logic AND operation between two strings as the inner-product of those strings. It will be recalled that this 'multiplication' was not introduced gratuitously into our study, but presented itself in an essential and natural way in the guise of structure preserving mappings—the automorphisms. These are of course represented by matrix×vector multiplications which themselves are built up from row×column multiplications, which are essentially the inner-product of two strings.

The extension of the present study to include the extra structures brought in by this normed boolean algebra approach must surely lead to some very interesting interpretations. This will be even more the case when the study is yet further extended from the present "static" systems to the essentially "dynamic" ones discussed by Noyes and Manthey in their Program Universe. Since the inherent geometrical and algebraic structure of a growing system is not strictly in evidence until certain stages of its development have been consolidated, it becomes important to study the novel idea of the evolution of such 'traditional' structures within the growing system.

One final point has to be raised. At the beginning of this paper discrimination systems were introduced as extendable multisets with diversity. This aspect was quietly set aside as soon as it became necessary to concentrate on the application of known algebraic and geometrical theories. Most of these theories are founded inexorably on set theoretic concepts and little is known about what becomes of them when their foundational basis is shifted onto multiset theoretic concepts. Since the essence of a discrimination system is 'discrimination' between objects to assess the extent to which they are the 'same' or 'different, the present study must sooner or later take a turn back towards its beginnings and re-examine its conclusions—and its combinatorics—in the face of multiple memberships of multisets. The rôle of 'indistinguishables' must move into the forefront of this kind of study if it is to make any meaningful contact with a physical world in which 'indistinguishability' is an inherent feature.

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A useful selection of essential results, with the emphasis on the group structures. Chapter 2 (Geometry of finite spaces) is a useful preface to this paper.

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Many of the combinatoric formulas given in this paper are taken from here: see chapter 1, section 4, pages 23–55 (Geometry of finite vector spaces). They have to be carefully checked and modified to use them in our 'field of two elements' context. Uses 'symplectic' where we use 'null'.

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A very detailed and extended source of technical information. Useful sections include the introductory material in 2.1 and the elementary properties and examples of small systems in chapter 7 (First properties of $PG(2,q)$).

NOTES

(i) The notations in these texts are not standardised, and have to be carefully verified against each other to ensure they refer to the same things.

(ii) Since our field only has two elements (characteristic 2), all notions which are important with fields with more than two elements such as semilinear automorphisms, companion automorphisms, etc., become trivial and irrelevant, and all our non-zero determinants are unimodular (value = +1). Also the three fundamental groups—semilinear, linear, and unimodular—are one and the same group. So are the corresponding projective groups: the Full Projective and the Little Projective.

MULTISETS: COLLECTIONS CONTAINING INDISTINGUISHABLE ELEMENTS

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Classical ZF (Zermelo-Fraenkel) set theory is based on the primitive concept of set membership. This primitive concept manifests itself formally in the atomic formula $x \in y$ which is to be interpreted as stating "x is an element of the set y". Classical ZF set theory requires that elements of sets be *distinct*; that is, no element may occur more than once in a given set.

A *multiset* is a collection of objects (called *elements*) in which elements may occur more than once. In [1], a first -order theory MST for multisets is developed in which the intended interpretation of the atomic formula $x \in^n y$ is "x is an element of the multiset y with multiplicity n" where n is some positive integer. If $x \in^n y$ holds, we say that "x has an n-fold presence in y" or "there are n indistinguishable copies of x in y". The number of times an element occurs in a multiset is called its *multiplicity* in the multiset. The *cardinality* of a multiset is the sum of the multiplicities of its elements. A *set* in MST is a multiset in which all elements have multiplicity exactly one. Although the multiplicity of an element is a (finite) positive integer, the number of distinct elements in a multiset need not be finite.

Repeated elements in multisets conform to the Parker-Rhodes principle ([4],p.7) for indistinguishables: they behave as identicals in isolation, but they behave as a plurality when elements of the same multiset (they contribute 'severally' to the cardinality of the multiset). The Theory of Sorts developed by A.F. Parker - Rhodes ([4], Chapter IV) differs radically from classical mathematical theories.

The theory MST is 'classical' in the sense that it is formulated in the first-order predicate calculus with equality and it contains an exact copy of ZF set theory. Classical sets become *hereditary sets* in MST (sets whose elements are sets, whose elements of elements are sets, ... and so on). In short, MST contains ZF. Therefore, MST is not an

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alternative to classical set theory, but a *generalization* of classical set theory: nothing is lost, but something is gained (namely, one is able to make formal assertions about collections containing indistinguishable elements). However, the theory MST is 'non-classical' in the sense that there are many theorems of MST for which there are no classical counterparts in ZF. For example, the ZF set containing exactly the element x is unique (denoted by $\{x\}$). In MST, there are infinitely many distinct multisets whose only element is x; namely, $\{x\}$, $[x,x]$, $[x,x,x]$, ...

A hierarchy of positive integer-valued functions constructed in ZF can be shown to be a *model* of MST (every axiom of MST holds in the hierarchy). Therefore, MST is *relatively consistent*; that is, one is no more likely to encounter a contradiction in MST than one is in ZF.

The technical details of MST need not concern us here. However, one feature of MST is striking: it is formulated in a *two-sorted language* (that is, there are two sorts of variable symbols: *numeric* variable symbols (like $k,l,m,n,...$) and *multiset* variable symbols (like $x,y,z,...$)). [The word "sort" as used here is not to be confused with the word "Sort" as used in [4].] In the atomic formula $x \in^n y$, the numeric variable symbol n denotes a multiplicity; whereas the multiset variable symbols y and x denote a multiset and an element of a multiset.

There is nothing perverse or unusual about two-sorted formal languages. One need only consider the axioms for a vector space: there are scalar variable symbols and vector variable symbols. The axioms for a vector space include the axioms for a field (stated in scalar symbols only) together with the remaining axioms (stated in both scalar and vector symbols). In exactly the same way, the axioms of MST include the axioms for Peano Arithmetic (PA) stated in numeric variable symbols. The other axioms of MST are generalizations of the classical axioms of ZF, stated in both sorts of variable symbols. Thus, MST is constructed by 'grafting' PA into ZF. The numeric variable symbols of MST are intended to range over everyday positive integers. There are, however, *non-standard models* of MST (structures in which the numeric variables range over non-standard 'natural' numbers).

The two-sort strategy employed in MST need not be restricted to PA axioms. Any algebraic structure can be substituted instead of PA. In this way, one obtains, for example, *ZF*-like axiom systems for multisets with integer, rational real, or complex multiplicities. Most notable among these is the theory MSTR for multisets with positive real-valued multiplicites (developed in [2]) which when restricted to the real interval $(0,1]$ gives a *ZF*-like theory MSTF for fuzzy sets.

If multiplicities are allowed to take *any* integer value (positive or negative) one obtains the theory MSTZ (developed in [3]) in which elements may have negative multiplicity. In MSTZ, for every multiset y there exists a unique *shadow* multiset y^- such that the "additive union" of y and y^- equals the empty set. Thus, a multiset and its shadow are said to *annihilate* each other. In MSTZ, one has *unrestricted complementation*; that is, for *any* multisets x and y, the multiset $x - y$ is always defined. In MSTZ, the *hereditary shadow* (the shadow, the shadow of elements, the shadow of elements of elements, ... and so on) operation allows one to define negative cardinal numbers. If the numeric sum of the multiplicities of elements of a multiset is negative, then the cardinality of that multiset is the corresponding negative cardinal number (the multiset that is the hereditary shadow of the corresponding classical cardinal number).

The axiom systems substituted for PA need not be number systems. If one uses axioms for a semi-ring, a ring, a field, a lattice, a boolean algebra or a Heyting algebra, for example, one obtains the corresponding multi-set theory (each containing a copy of classical ZF set theory).

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SHAPES AND SIZES

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1. INTRODUCTION:

In the classical approach, it is possible to assign shapes and sizes to the things we want to describe. On the contrary, in the framework of quantum mechanics the central concept is that of state, and some people think that the notions of shape and size are alien to quantum systems. R. G. Wooley points out that "the molecular structure makes no appearance in a quantum treatment of molecules starting from first principles[1]. This is why, in his opinion, a molecule does not necessarily have a shape in every circumstance.

In a first approach, the traditional method to assign a geometrical structure to such kinds of quantum systems rules like this: we consider the nuclei as mass-points at rest and we calculate the electron distribution $\rho(\bar{r})$ for every electron. The condition (i) $\rho(\bar{r}) = \kappa$ (a constant) defines a level curve Σ and a region R inside Σ. Then the condition (ii)

$$\int_R d^3r\rho(\bar{r}) = P ,$$

where P is close to one, allows us to chose a particular Σ_j corresponding to the j–electron. The envelope of the Σ_j's is the boundary of the molecule; let us call it Σ_0. Then we assign to the molecule the shape and size of R_0 inside Σ_0[2].

This method presents two inconveniences: (a) In order to evaluate the boundary of the system it is necessary to fix an arbitrary chosen value of P. It is true that, in many cases, the shape of the system does not change very much with different and "reasonable" choices of P, but the size of the system depends on this choice. (b) The fact that the size of a free system is not stable, but grows, has not properly been taken into account. These are our first motivations to face the problem of shapes and sizes of quantum systems.

On the other hand, in our approach to quantum mechanics[4] we do not establish any difference between micro and macrosystems for, in our view, every system must be described by quantum mechanical laws. Nevertheless, in many circumstances macro objects have well-defined geometries that should be related, we think, to the corresponding states of these objects.

We guess that the well-defined shapes and sizes of macrosystems are emergent properties of many-body quantum systems. So we want definitions of shapes and sizes of a quantum system such that, whether they are meaningful for small atoms and molecules or not, they become more and more significant for larger systems, and are compatible with our intuitive ideas on these matters in the limit of macro objects. We also want to avoid the objections to the traditional method pointed out before.

2. THE BOUNDARY OF AN ELEMENTARY ENTITY:

Let us assume that elementary entities (E) exist in nature and that they are well described by quantum mechanical laws. If E is in a potential $V(\bar{r}, t)$, then the state of E is $|\psi(t)>$, and

$$\psi(\bar{r}, t) = \langle \bar{r}|\psi(t)\rangle .$$

Let us consider the point $\bar{r}$ for which

$$\left|\left|\{\overline{\nabla}|\psi(\bar{r}, t)|\}\right|\right| ,$$

in the direction of $\overline{\nabla}|\psi(\bar{r}, t)|$ is a relative maximum. In such a case,

Definition 1:

If there exists a closed surface σ_E such that every point fulfilling the previous condition is on σ_E or inside σ_E, we shall say that σ_E is the boundary of E.

Definition 2:

The region R_E of E is the region inside σ_E.

Definition 3:

The shape and the size of E are, respectively, the shape and the size of R_E.

Let us point out that these definitions do not imply that E has a boundary (nor a shape and size) for every $|\psi(t)\rangle$. Cases may appear where the boundary of E does not exist.

If $V(\bar{r}, t) = 0$, the elementary entity is free and $\psi(\bar{r}, t)$ spreads out. As a consequence, the size of R_E grows with time. This effect is more important for small masses, and would indicate that the concept of mass-point is not useful to represent these kinds of things[5].

On the other hand, for an attractive and time-independent potential $V(\bar{r})$, it is easy to see that E has a stable shape and size:

(a) If E is in the stationary state

$$\psi(\bar{r}, t) = \phi_n(\bar{r})\, e^{-iH_n t/\hbar} ,$$

where $\phi_n(\bar{r})$ is the eigenstate of the Hamiltonian corresponding to the eigenvalue H_n, then

$$|\psi(\bar{r}, t)| = |\phi_n(\bar{r})| ,$$

and the shape and size of E are stable.

(b) If the state of E is

$$|\psi(t)> = \Sigma_{n,j} c_{n,j}(0) e^{-iH_n t/\hbar}\, |\phi_{n,j}\rangle ,$$

where $|\phi_{n,j}\rangle$ is an eigenvector of the Hamiltonian corresponding to H_n and j, is introduced in order to distinguish between the different eigenvectors that may correspond to one eigenvalue H_n, then

$$|\psi(\bar{r}, t)| = \left[\Sigma_{n,j}\, \Sigma_{m,k}\; c_{n,j}(0)\, c^{*}_{m,k}(0)\; e^{i(H_m - H_n)t/\hbar}\; \phi_{n,j}(\bar{r})\, \phi^{*}_{m,k}(\bar{r})\right]^{1/2} ,$$

where $\phi_{n,j}(\bar{r}) = \langle \bar{r}|\phi_{n,j}\rangle$. In such a case, the size of E is also bounded.

3. THE BOUNDARY OF A SYSTEM OF N ELEMENTARY ENTITIES:

Let S be a system of N entities E. If the state vector of S at time t is $|\psi_S(t)\rangle$, then

$$\psi_S(\bar{r}_k,t) \equiv \psi_S(\bar{r}_1,\ldots,\bar{r}_N,t) = \langle \bar{r}_1,\ldots,\bar{r}_N|\psi_S(t)\rangle ,$$

where $\bar{r}_j$ corresponds to the j-component E.

Let us consider the points where

$$\left|\{\overline{\nabla}|\psi(\bar{r}_k,t)_S|\}\right|$$

is a relative maximum; here $\overline{\nabla} = \Sigma_{j=1}^{N} \overline{\nabla}_j$. In such a case,

Definition 4:

If there exists a closed surface σ_S such that every point fulfilling the previous condition is on σ_S or inside σ_S, we shall say that σ_S is the boundary of S.

Definition 5:

The region R_s of S is the region inside σ_S.

Definition 6:

The shape and size of S are, respectively, the shape and size of R_S.

We have seen that the elementary entities do not have an intrinsic geometry, since it depends on the external potential V_{ext} and on t. On the contrary, in the case of a system S, the geometry depends also on the potential of interaction V_{int}. If $\Delta V_{ext}/\Delta V_{int} \simeq 0$ (where ΔV is the variation of V in R_S), the geometry depends only on the internal parameters of S and on time. In the words of Garcia–Sucre and Bunge, "elementary entities are indeed structureless ... [but] complex entities do possess a geometry which is determined by their intrinsic properties, although influenced by the environment. In particular, although a free electron may be shapeless, a molecule is not." [6].

According to Definition 4, in order to evaluate the boundary of S, we must know the state $|\psi_S(t)\rangle$. Nevertheless, it is easy to see that, if S consists of two subsystems S_a and S_b and if

$$|\psi_S(t)\rangle = |\psi_a(t)\rangle \otimes |\psi_b(t)\rangle ,$$

then the boundary of S is close to the envelope of σ_a and σ_b. In particular, if σ_a is inside σ_b, the boundary σ_S coincides with σ_b; in other words, the boundary of a system depends, basically, on the state of its external (and more energetic) components. This is why, in many cases, it is not necessary to know $|\psi_S(t) >$ in order to evaluate the boundary of S.

4. THE SIZE OF A FREE SYSTEM:

We shall start this section by considering the geometry of a free hydrogen atom. This system consists of a proton of mass m_p and an electron of mass m_e. The problem of the free hydrogen atom is equivalent to that of two fictitious particles: a free particle g of mass $m_g = m_p + m_e$ and state $|\psi_g(t)\rangle$, and a particle F of mass $m_F = m_p m_e/(m_p + m_e)$ that is in a Coulombian potential and has the state $|\psi_F(t)\rangle$. Then the state of the atom is

$$|\psi_S(t)\rangle = |\psi_g(t)\rangle \otimes |\psi_F(t)\rangle .$$

Taking into account the definitions previously stated, it is easy to show that if the region of F exists, then its size is bounded. Moreover, if the atom is in its ground level, F has a radius $b_F \simeq 1$ A (constant in time). On the contrary, the size of g grows with time. For a state initially symmetric with radius $b_g(0)$, the region R_g stays symmetric, but its radius follows the law

$$b_g(t) \;=\; b_g(0)\;\left[1+\frac{9\hbar^2 t^2}{m_g^2\, b_g^4(0)}\right]^{1/2} .$$

If $b_g(0) \ll b_F$, we can say that at $t = 0$ the size of the hydrogen atom is $b_S(0) = b_F = 1$ A. Let t_0 be the time for which $b_g(t_0) = b_F$. At $t \ll t_0$ the boundary of the atom practically coincides with the boundary of g, and we can say that the atom, as a whole, grows with time.

In order to assign a shape and a size to any other free complex system, we can use the same method as we did for the hydrogen atom. For instance, in the case of a molecule, the state in the laboratory frame is

$$|\psi_S(t)\rangle \;=\; |\psi_g(t)\rangle \;\otimes\; |\psi_F(t)\rangle \;,$$

where $|\psi_F(t)\rangle$ is the state of the molecule F in a frame fixed to its center-of-mass (that we treat as a mass point), and $|\psi_g(t)\rangle$ is the state of a fictitious free particle g (the molecule as a whole).

We shall say that the shape and size of F are the intrinsic shape and size of S. In the case of some systems with a stable σ_F we have calculated, for an initial $b_g(0) = 10^{-2} b_F$, the time t_0 necessary for S to acquire a radius $b_S(t_0) > b_F$. The results are:

System	m_g	$b_S(0)$	t_0
H in the ground state	1.6×10^{-27} Kg	10^{-10} m	10^{-16} sec
Particle of dust	10^{-15} Kg	10^{-6} m	9 hours
Piece of metal	10^{-5} Kg	10^{-3} m	10^{+20} sec $> T_{Univ.}$

In some cases this simple calculation also helps us to decide whether a specific classical model is adequate to represent a real thing or not. For instance, if we look at a particle of dust with a resolution 10^{-4} m for some minutes, we can treat it as if it was a mass-point, while, on the contrary, it would be nonsense to claim that a free hydrogen atom can be represented as a mass-point for one *microsecond* when the resolution is *ten Angstroms.*

5. CONCLUDING REMARKS:

Let us recall the aim of this article: we want definitions of shapes and sizes of quantum systems such that, whether they are meaningful for small atoms or not, they become more and more significant for larger systems and are compatible with our intuitive idea on these matters in the limit of macro objects. We also want to avoid the objections to the traditional method for assigning shapes and sizes to these systems, pointed out in the introduction.

What have we done until now? We have proposed some definitions of the boundary, the shape and the size of a quantum system that are, in principle, valid for every system that has a state $|\psi(t)\rangle$, no matter how many components it has. We do not claim that they are the only definitions that fulfill the conditions we have imposed and, moreover, we are not sure that our definitions are completely satisfactory.

Three advantages of our definitions are the following:

(*a*) The shape and size of a quantum system does not depend on the arbitrary choice of any parameter.

(b) We have found that, in some cases, complex systems have an intrinsic structure that remains stable in time, and that the intrinsic shape is similar to that obtained with the traditional method.

(c) We can think of a macrosystem as made of molecules that are made of atoms that are made of elementary entities. The size of every subsystem, if it was free, would grow with time; but when a subsystem interacts with other subsystems, the whole system acquires a stable intrinsic size, and the growth of the free whole system is very slow for a large mass and a large intrinsic size.

Can we conclude, as Garcia–Sucre and Bunge do, that every complex system has a shape? According to our definitions, we cannot do it. Garcia–Sucre and Bunge take as valid Born's postulate, and follow a slightly different version of the traditional method that allows them to assign a shape to every complex quantum system. On the contrary, our method does not warrant that every complex quantum system has a boundary. A simple calculation allows us to say that a hydrogen atom in the state $n = 2$, $\ell = m = 1$ does not have such a boundary. This is an example of a complex microsystem for which our definitions of shape and size are not meaningful.

On the other hand, if we define the density as $\delta = m/v$ where m is the mass and v is the volume of the system, then we should conclude that δ is going down with time. This is very difficult to accept and, in our view, would indicate that our definitions are not meaningful in the case of small quantum systems.

Now, in which way could we test our guess that shapes and sizes of macrosystems are emergent properties of many-body quantum systems? If R_0 is the intrinsic region of a system S, then

$$\int_{R_0} D^3 r_j \, |\psi(\bar{r}_j, t)|^2 = P_j \leq 1 ,$$

where $\bar{r}_j$ is the coordinate of the j–component of S. In a naive interpretation of this formula (based in Born's postulate), the number P_j measures how much the j-component is inside R_0. In the case of the hydrogen atom we obtained the following results:

$$n = 1 ; \quad \ell = m = 0 \rightarrow P_1 = 0.76 ,$$

$$n = 2 ; \quad \ell = m = 0 \rightarrow P_1 = 0.81 ,$$

$$n = 3 ; \quad \ell = m = 0 \rightarrow P_1 = 0.84 .$$

A quantity such as

$$P = \frac{\Sigma_{j=1}^{N} m_j P_j}{\Sigma_{j=1}^{N} m_j} ,$$

where m_j is the mass of the j–component, could be interpreted as the neatness of the boundary of S. Then, if for similar systems P grows with N and approaches one in the limit $N \rightarrow \infty$, we would claim that the concepts of shape and size become more and more significant for macrosystems.

So, if our guess is right, in the case of atoms in the same row of the periodic table, P should increase with the atomic number. A similar calculation in the case of some molecules should also be carried on, before deciding about the significance of our definitions of shape and size.

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ON A DIMENSIONAL APPROACH TO A UNIFIED THEORY

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Summary : Dimensional Relationships of Common Physical Concepts

It is possible to reduce the conventional dimensional expressions in terms of Length, Time, and Mass (Inertia) or Force, to terms containing Length and Time only. Certain advantages occur, and relationships previously obscured by the complexity of a three dimensional format are clarified by tabulating the resulting expressions as shown on the attached chart (TABLE 1).

Not all of the many scientific concepts now in use are shown in the chart, which is primarily intended as introductory and indicative; but it is hoped that it will be of interest to those familiar with the dimensional techniques now being increasingly used.

It is obvious that all the non-dimensional parameters and numbers, such as Reynolds, Froude, Densimetric Froude, Richardson, etc., are merely ratios of permutations and combinations of the values tabulated, and that their number is finite for all practical purposes. For although the tabulation can extend to infinity, the concepts susceptible to measurement appear to be confined to the first six powers on each axis.

What is perhaps not so obvious is that the tabulation of these concepts also shows the *LAWS* or *RULES* of Nature and their relationships as interlinked by Newtonian calculus.

There are certain philosophical and metaphysical deductions to be made from the final array, but it is perhaps sufficient for the purpose of this note to observe that it might have been preferable for scientists to have used a *natural* unit of measurement based on Naperian 'e', 'π', and a *representative velocity*—such as 'c' the *velocity* of light or, more accurately, the constant for the ratio of *length* to *time* to the particular case being examined; and generally, in the study and measurement of natural phenomena.

The Michelson-Morley experiment, and subsequent endeavours to establish the velocity of light and other forms of electromagnetic radiation, is open to another interpretation – that of establishing a ratio of length to time in this universe. In this talk the meaning of these experiments is to be taken as establishing the connection of the dimension length to that of time in an absolute sense, instead of it being measured in units familiar to the scientist. Those units are arbitrary, despite the attempts of Wren and the Abbé Piccard in 1670 AD to relate the unit of length to the unit of time by the beating of a pendulum in a gravitational field. That these experiments, such as the Michelson-Morley, have also established that the value so obtained is a physical constant of the universe is accepted.

The expressions for Newton's Law $P = mf$ and Einstein's Law $E = mc^2$ can be expressed as $P = m\mathrm{L}/\mathrm{T}^2$ and $E = m\mathrm{L}^2/\mathrm{T}^2$, and substituting $m = \mathrm{L}^3/\mathrm{T}^2$ we obtain for these two laws :

$$P = \frac{\mathrm{L}^3}{\mathrm{T}^2} \times \frac{\mathrm{L}}{\mathrm{T}^2} = \frac{\mathrm{L}^4}{\mathrm{T}^4}$$

and

$$E = \frac{\mathrm{L}^3}{\mathrm{T}^2} \times \frac{\mathrm{L}^2}{\mathrm{T}^2} = \frac{\mathrm{L}^5}{\mathrm{T}^4}.$$

The expression for $m = \mathrm{L}^3/\mathrm{T}^2$ is derived by Clerk Maxwell [1] as follows. If (as in the metric system [2]) the unit of mass is defined by its attractive power, the acceleration of a mass m at a distance r is by Newton's Law m/r^2. Suppose this attraction to act for a very small interval of time t on a body originally at rest, and to cause it to describe an interval of space s, then by the formula of Galileo

$$s = \frac{1}{2} f t^2 = \frac{1}{2}\frac{m}{r^2} t^2$$

whence

$$m = \frac{2r^2 s}{t^2}, \quad \text{i.e.} \quad \mathrm{M} = \frac{\mathrm{L}^3}{\mathrm{T}^2},$$

that is, Volume (L^3) and Density ($1/\mathrm{T}^2$) conjointly, which is in line with Newton's Definition for Mass (*"Quantitas materiae est mesora ejuisdam orta ex ilius densitae et magnitudine conjunctem"*) in his Proposition 1 in ref.[3] (see also p.5 in ref.[1]). In practical units Heisenberg [4] confirms this when he takes the measured values of Planck's constant and the velocity of light, assumes a length of roughly 10^{-13}cm, and forms an expression which in its dimension corresponds to a mass, its value being of the order of magnitude of the masses of the elementary particles. In dimensional terms this becomes :

Velocity of Light	*cm / sec*	L/T
Planck's Constant	*erg / sec*	$\mathrm{L}^6/\mathrm{T}^3$
Length	*cm*	L

whence

$$\left(\frac{\mathrm{L}^6}{\mathrm{T}^3}\right) \Big/ \left(\frac{\mathrm{L}}{\mathrm{T}} \times \mathrm{L}\right) = \frac{\mathrm{L}^3}{\mathrm{T}^2}.$$

In these circumstances where mass has the dimensions of $\mathrm{L}^3 / \mathrm{T}^2$ the Gravitational Constant G is non-dimensional, since ([5]) :

$$P = G\frac{m\,m'}{r^2} = G\left(\frac{\mathrm{L}^3}{\mathrm{T}^2}\times\frac{\mathrm{L}^3}{\mathrm{T}^2}\right)\Big/\mathrm{L}^2 = G\left(\frac{\mathrm{L}^6}{\mathrm{T}^4}\right)\Big/\mathrm{L}^2 = G\frac{\mathrm{L}^4}{\mathrm{T}^4};$$

but

$$P = \mathrm{M}\times\frac{\mathrm{L}}{\mathrm{T}^2} = \left(\frac{\mathrm{L}^3}{\mathrm{T}^2}\right)\times\left(\frac{\mathrm{L}}{\mathrm{T}^2}\right) = \frac{\mathrm{L}^4}{\mathrm{T}^4},$$

hence $G = 1$ (non-dimensional).
Alternatively, $G\,\mathrm{M} = \mathrm{cm}^3 / \mathrm{sec}^2$, but $\mathrm{M} = \mathrm{m} = \mathrm{L}^3/\mathrm{T}^2$, so again $G = 1$.

Although Heisenberg believed at least three fundamental constants were necessary to determine the masses and the other properties of elementary particles (e.g. atomic nuclei of the lighter elements) it is only necessary, as will be shown later, to have two – namely Length and Time, and their ratio.

So far I have dealt with the dimensions of mechanical quantities, and it must be stressed that only those properties susceptible to measurement are being investigated. Other meanings of the word '*dimension*' are not applicable to this enquiry.

The dimensions of *Entropy*, the heat weight of Zeuner (defined as $\partial Q / \theta$ [6]), are derived from the equation $(\partial Q / \theta)\times\theta =$ *Work Done* or *Energy*, where ∂Q is the quantity of *Heat*, whereas those of *Temperature* θ are derived from $P\,V = m\,R\,\theta$ where P is *Pressure*, V *Volume*, and R is the *Universal Gas Constant* (a pure number). Thus using $P = (\mathrm{L}^4 / \mathrm{T}^4)\times 1 / \mathrm{L}^2$, $V = \mathrm{L}^3$, $m = \mathrm{L}^3 / \mathrm{T}^2$, $R = 1$, gives $\theta = \mathrm{L}^2 / \mathrm{T}^2$, and *Entropy* $= \mathrm{L}^3 / \mathrm{T}^2$. These are then substituted in the dimensions for thermodynamic concepts.

Since *electrostatic permittivity* μ and *magnetic permeability* ϵ are essentially of the same kind or quality, and $\mu\,\epsilon = 1 / c^2$, it is reasonable to substitute $\mu = 1 / c$ and $\epsilon = 1 / c$ in the electrical entities.

There remains one more step before I can establish that TABLE 1 shows the interconnections of the entities by means of the Calculus of Newton. Consider the volume of a sphere $4 / 3\,\pi\,r^3$, and its surface area $4\,\pi\,r^2$. One can see, by extending these two expressions backwards and forwards that we have an infinite sequence

$$8\pi \qquad 8\pi r \qquad 4\pi r^2 \qquad \frac{4}{3}\pi r^3 \qquad \frac{1}{3}\pi r^4 \qquad \dots$$

or

$$\frac{8\pi}{1} \qquad \frac{8\pi r}{1} \qquad \frac{8\pi r^2}{1.2} \qquad \frac{8\pi r^3}{1.2.3} \qquad \frac{8\pi r^4}{1.2.3.4} \quad \dots \quad \frac{8\pi r^n}{n!} \quad \dots$$

of terms found in the expansion of the exponential expression $8\pi e^r$ or $8\pi \exp(r)$, i.e. $8\pi \exp(L)$. Thus on the horizontal axis in TABLE 1 there is shown

$$L \quad L^2 \quad L^3 \quad L^4 \quad \ldots \quad L^\infty$$

as a shorthand for the terms $8\pi L^n / n!$ $n = 1, 2, \ldots + \infty$.
On the vertical axis of TABLE 1 there is similarly shown .

$$\frac{1}{T} \quad \frac{1}{T^2} \quad \frac{1}{T^3} \quad \frac{1}{T^4} \quad \cdots \quad \frac{1}{T^\infty}.$$

The inverses of these L and T terms are also shown and yield TABLE 1 divided into quadrants indexed by L^n, $1/L^n$, T^n, $1/T^n$. Into this Table can be inserted all the modified dimensional expressions in terms of L, T, and their ratio c, and the permutations and combinations of these expressions which for most practical entities seem to be confined to the first six or eight powers.

An additional analogy which bears on the interpretation of the tabulations is the Column Analogy of Professor Hardy Cross [7] where it is the mathematical identity between the moments produced by continuity in a beam, bent or arch, and the fibre stresses in an eccentrically loaded short column; this has the dimension L/T^4.

Similarly, the equations of Poisson and Laplace have applications in several branches of physics due to their dimensional identity in each of the disciplines. They are equations of *Energy* and have dimensionality L^5/T^4.

Thus the *LAWS* (*e.g.*Newton's) or *RULES* (*e.g.*Column Analogy) are shown in generalised dimensional form in TABLE 1 — which is to be understood as ONE ENTITY embracing all others which are manifested as Hilbert Geometries in Space and Time.

From the times of the Pythagoreans, Plato and Democritus, who said only atoms and empty space have existence; to Newton for whom the Book of Nature was written not in mathematical symbols but in the form of a cryptogram; to Boole who believed that all dynamical and other actions seemed not only to be measurable in themselves but also to be connected to each other, even to the extent of mutual convertibility, by numerical relations of a perfectly definite kind, and that the laws of thought are of the same kind as the laws of mathematics, it is perhaps only now that it is possible to display – through the tabulation of the entities of science in terms of ratios of L, T and c – their fundamentally abstract nature in the form of Hilbert spaces of many dimensions (scalar, vector, quaternionic spaces,...), and, as a consequence, to obtain a consistent theory applicable to the very large and the very small – and to unite the many concepts susceptible to measurement – by a dimensional approach to a Unified Theory.

Omitted so far are Planck's large numbers for L, T and M, which relate all three entities through *Planck's Constant*, the *Velocity of Light* and the *Gravitational Constant*, in numerical terms measured in the *c.g.s.* system, as a further confirmation.

There are, I believe, a number of other expressions which are relevant to the understanding of a Unified Theory based on Dimensions, and the ones which spring to mind most readily arise in Heaviside's *Impulse Theory* and his fascination with the special case of *Euler's Integral* :

$$-\int_0^\infty \frac{x^6}{6!} e^{-x}\, dx \quad = \quad e^{-x}\left[\frac{x^6}{6!}+\frac{x^5}{5!}+\frac{x^4}{4!}+\frac{x^3}{3!}+\frac{x^2}{2!}+x+1\right] \quad = \quad -1.$$

Heaviside's final (unpublished) researches [8] led him to formulate a unified field theory in which electromagnetism is co-related with mass properties and in which there is a reciprocal relationship between radiation and matter.

This is a somewhat similar idea to Professor Jennison's model [9] of an electron as a phase-locked cavity: mass equals a volume (L^3) and two frequencies $1/T_{\text{magnetic}}$ and $1/T_{\text{electric}}$.

Of equal interest is Euler's Gamma Integral and its relation to the Logarithmic Spiral, the Golden Section and the Fibonacci Series.

There is one more relationship which is pertinent to this discussion, and which as as follows :

$G = 1$,	$G = K = k\,c^2/8\,\pi$,	and hence	$k = 1/c^2 = \mu\,\epsilon$
NEWTON	EINSTEIN [10]		CLERK MAXWELL

and is the result of G being 1, i.e. non-dimensional. This indicates that instead of writing $E = m\,c^2$ one should now write

$$E \quad = \quad \frac{m}{\mu\,\epsilon}.$$

As $\mu\,\epsilon$ depends on the physical characteristics of the media, it is more accurate for experiments carried out under terrestrial conditions to calculate the value of the masses of the elementary particles using these values rather than the constant c, that is, $m = E\,\mu\,\epsilon$ rather than $m = E/c^2$.

TABLE 1 of the Unified Theory extends to plus and minus infinity. It still has many gaps such as the entities of *chemical, biological and social sciences.* However, provided that these can be expressed in terms of dimensional equations, including, if unbalanced, one or more constants the values of which are determined by dividing one side into the other, it will be found that these constants either turn out to be non-dimensional numbers or to have dimensions in terms of L, T or c, which are then substituted into the original equations to give a dimensionally homogeneous expression. In many problems the time element $(1 / T^n)$ can be disregarded; but it becomes significant in studies of the very large (cosmology) and the very small (nuclear physics), and in problems involving frequency.

As TABLE 1 extends to infinity in each quadrant, and contains differentials and integrals to the Naperian base e, there is room for non-material entities outwith the physical world, which will be of interest to social scientists. It is significant that a consensus of spiritual experience includes such concepts, from the animist to the theosoph, and from the primitive to the sophist, and perhaps justifies the inclusion of this paper at a meeting of Natural Philosophers.

I now express my great appreciation to Dr John Amson and Dr Christine Crow for their help, to St Andrews University for the use of their Libraries, and to the members of ANPA for their courteous and critical appreciation of the views now made public.

SYMBOLS USED

dimensional		*variables*		*standard*	
M	Mass	m	mass	θ	temperature
L	Length	r, s	length	μ	magnetic permeability
T	Time	t	time	ϵ	electrical permittivity
		f	acceleration	G, K	Universal Gravitation Constant
		c, v	velocity	k	Einstein constant
		Q	heat	R	Universal Gas Constant
		P	pressure		
		V	volume		

APPENDIX - A

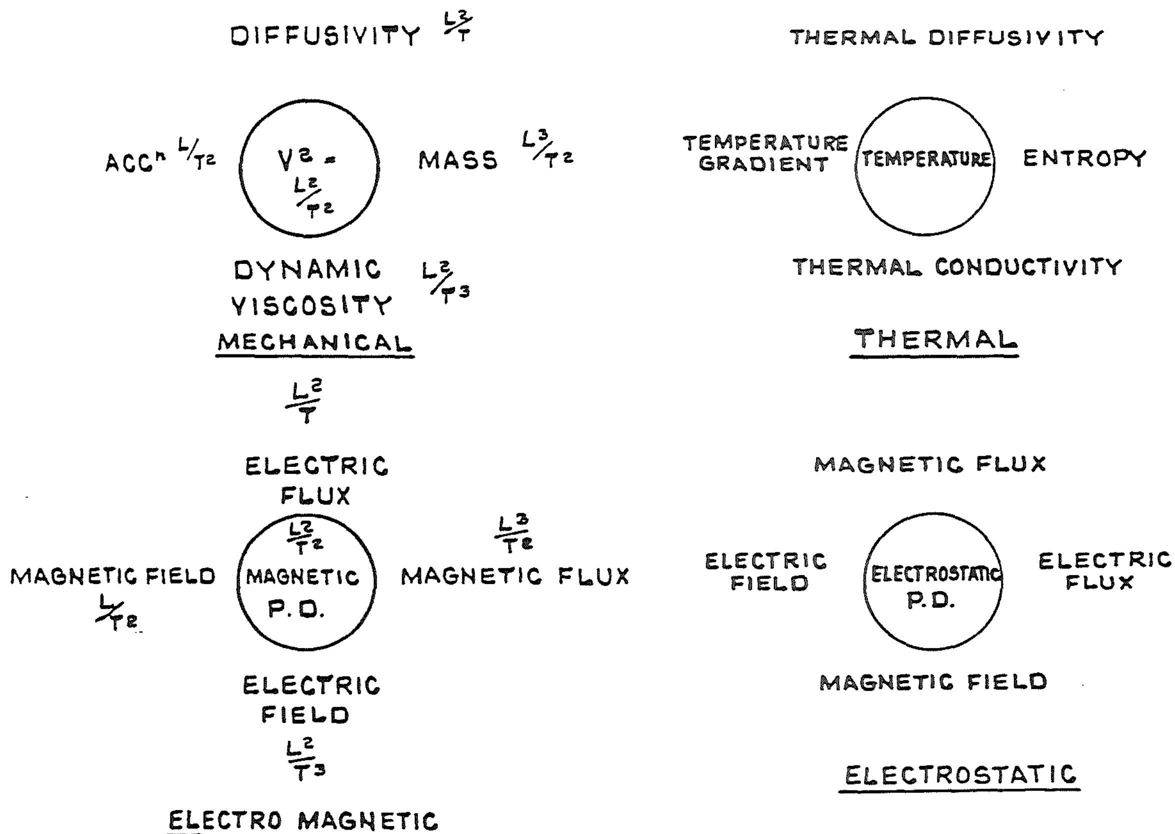
TABLE 1 (see over)

TABLE 2 (see over)

TABLE 3 (see over)

	Mechanical	Thermodynamical	Electrostatic	Electromagnetic	Nuclear, Atomic
$\frac{L^3}{T^2}$	MASS INERTIA	ENTROPY THERMAL CAPACITY BOLZMANN'S CONSTANT	CHARGE ELECTRIC FLUX QUANTITY of ELECTRICITY	POLE STRENGTH MAGNETIC FLUX	
$\frac{L}{T^2}$	ACCELERATION	TEMPERATURE GRADIENT	ELECTRIC INTENSITY ELECTRIC FIELD DISPLACEMENT INDUCTANCE	MAGNETIC FIELD INDUCTION INTENSITY	
$\frac{L^2}{T}$	KINEMATIC VISCOSITY DIFFUSIVITY CIRCULATION	THERMAL DIFFUSIVITY	MAGNETIC FLUX POLE STRENGTH	CHARGE RESISTIVITY ELECTRIC FLUX QUANTITY of ELECTRICITY	DIFFUSION COEFFICIENT ELECTRON MOBILITY ION MOBILITY CHARGED PARTICLE- -MOBILITY
$\frac{1}{T}$	ANGULAR VELOCITY VORTICITY FREQUENCY		ELECTRIC CONDUCTIVITY MAGNETIC INDUCTION	ELECTRIC DISPLACEMENT ELECTRIC INDUCTION SURFACE DENSITY	

TABLE 2.



DIFFUSIVITY L^2/T
ACC^n L/T^2
$V^2 = L^2/T^2$
MASS L^3/T^2
DYNAMIC VISCOSITY L^2/T^3
MECHANICAL
L^2/T
THERMAL DIFFUSIVITY
TEMPERATURE GRADIENT
TEMPERATURE
ENTROPY
THERMAL CONDUCTIVITY
THERMAL
ELECTRIC FLUX
L^2/T^2
MAGNETIC FIELD L/T^2
MAGNETIC P.D.
MAGNETIC FLUX L^3/T^2
ELECTRIC FIELD L^2/T^3
ELECTRO MAGNETIC
MAGNETIC FLUX
ELECTRIC FIELD
ELECTROSTATIC P.D.
ELECTRIC FLUX
MAGNETIC FIELD
ELECTROSTATIC

TABLE 3

APPENDIX - B PROPOSITIONS

1: **THAT** the ratio measured as the *velocity of light*, c, can be interpreted as a ratio of dimensions L = *Length* and T = *Time* in the study of natural phenomena, and that 'c' is THE basic physical constant.

2: **THAT** the dimensions of physical entities can be expressed in terms of a representative length L and/or a representative time T.

3: **THAT** these concepts so expressed can be tabulated as shown in TABLE 1.

4: **THAT** the tabulation indicates that the entities are connected by a calculus (of Newton) and can be split to form entities of a lower order; for example:

$$\frac{L^4}{T^4} = \frac{L^3}{T^2} \times \frac{L}{T^2} \quad \text{or} \quad \frac{L^3}{T^2} \times \frac{L}{T} \times \frac{1}{T}.$$

5: **THAT** each concept so defined exists in a space-time of a multi-dimensional geometry consisting of scalars, vectors, quarternions (Hilbert spaces, in general).
For example: (a) mass exists in a space consisting of three dimensions of length and two dimensions of time (or frequency = 1 / T);
(b) the quantum of Action L^5 / T^3 exists in a space with five dimensions of length and three of time, being the Time Integral of Energy $\int E\,dt$.

6: **THAT** each of the geometries is conservative, e.g. exhibits conservation of mass, or of power, or of energy; but that interchange occurs in physical systems.

7: **THAT** TABLE 1 provides a metric for an Aether in which all the entities operate.

8: **THAT** throughout the tabulation there are Time and Length integrals and differentials connecting the concepts, and that these operate in general from minus infinity to plus infinity.

9: **THAT** if c is a constant, then L / T is constant. (Thus, if L increases, then T must increase to preserve the constant ratio. When T increases, 1 / T decreases in numerical terms. Therefore the greater the distance over which L is measured, 1 / T (i.e. frequency) diminishes. Thus there is a shift to the red in the observed spectrum which cannot be equated to a velocity of recession as in the Doppler Effect.)

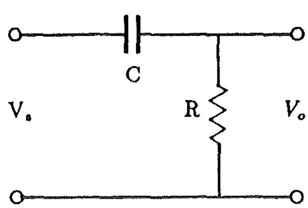
APPENDIX - C SIMILAR OPERATIONS

Operation =	*Integration*	*Differentiation*
Mathematical	$\int \frac{L}{T} dx$	$\frac{L}{T} \frac{\partial}{\partial x}$
Electrical ⋆	$V_o = -\frac{1}{-CR} \int V_s dt$	$V_o = -CR \frac{\partial V_s}{\partial t}$
Mechanical	Babbage	Babbage
Unified Theory	$\times L$ or $\times T$	$\times \frac{1}{T}$ or $\times \frac{1}{L}$

N.B. $\frac{L}{T} \equiv C$ and $x = L$ or T

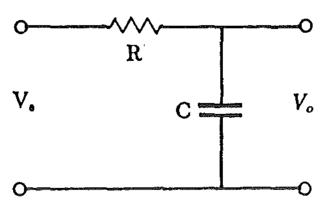
⋆ **Equivalent Electrical Circuits**

INTEGRATION



$V_o = -\frac{1}{-CR} \int V_s dt$

DIFFERENTIATION



$V_o = -CR\, \partial V_s / \partial t$

EXAMPLES

$$\int P\, dL = \frac{L^4}{T^4} \times L = \frac{L^5}{T^4} = \textit{Energy or Workdone.}$$

$$P \frac{\partial}{\partial L} = \frac{L^4}{T^4} \times \frac{1}{L} = \frac{L^4}{T^3} = \textit{Impulse or Momentum.}$$

$$\int E\, dT = \frac{L^6}{T^4} \times T = \frac{L^6}{T^3} = \textit{Planck's constant.}$$

$$E \frac{\partial}{\partial T} = \frac{L^6}{T^4} \times \frac{1}{T} = \frac{L^6}{T^5} = \textit{Power.}$$

N.B. The quantities on the right-hand-sides are constant or conservative in any closed system.

The equation of *Conservation of Mass* (sometimes called the *convection-diffusion equation*) may be written as :

$$\underset{(\alpha)}{\frac{\partial}{\partial t}\Big(AC\Big)} \quad = \quad \underset{(\Delta)}{m} \quad + \quad \underset{(\beta)}{\left[\frac{\partial}{\partial x}DA\frac{\partial C}{\partial x}\right]} \quad - \quad \underset{(\gamma)}{\left[\frac{\partial}{\partial x}(AVC)\right]}$$

where

$$A \equiv L^2 \text{ ......Area;} \qquad C \equiv \frac{L}{T} \equiv V \text{ .....Velocity;}$$

$$m \equiv \frac{L^3}{T^2} \text{ .....Mass;} \qquad D \equiv \frac{L^2}{T} \text{ .....Diffusion } \left(cf\ \frac{L}{T^2}\right).$$

Then:

$$(\alpha) = \frac{\partial}{\partial t}\Big(AC\Big) = \frac{L^2}{1}\times\frac{L}{T}\times\frac{1}{T} = \frac{L^3}{T^2} = m \quad i.e.(\Delta)$$

$$(\beta) = \frac{\partial}{\partial x}\Big(DA\frac{\partial C}{\partial x}\Big) = \frac{1}{L}\times\frac{L^2}{T}\times\frac{L^2}{1}\times\frac{1}{T} = m\,\dagger$$

$$(\gamma) = \frac{\partial}{\partial x}\Big(AVC\Big) = \frac{1}{L}\times\frac{L^2}{1}\times\frac{L}{T}\times\frac{L}{T} = m$$

† *N.B.* $\frac{\partial C}{\partial x} = \frac{\partial \frac{L}{T}}{\partial L} = \frac{1}{T}$

If the substance is decaying at the rate K, proportional to its concentration, then KAC is added to the right-hand-side of the equation, *i.e.*

$$KAC = \frac{L^3}{T^2} = \frac{K}{1}\times\frac{L^2}{1}\times\frac{L}{T}, \qquad i.e.\ K = \frac{1}{T}.$$

If there are two flows in opposite directions then

$$U = U_t \pm \frac{Q}{A},$$

where U_t is the main flow, and A is the *cross-section* under review. The corresponding equation for conservation of *Volume*, *i.e.* L^3, is then

$$\frac{\partial A}{\partial t} \pm \frac{\partial}{\partial x}\Big(AU_t\Big) = 0;$$

i.e. constant *Diffusion / Concentration* $\left(\frac{L^2}{T}\right)$.

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Full size (25 *in.* × 35 *in.*) copies of TABLE 1
may be purchased from the writer.

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ALTERNATIVE NATURAL PHILOSOPHY ASSOCIATION

9th INTERNATIONAL MEETING
KING'S COLLEGE, CAMBRIDGE

September 23rd-28th, 1987

BIOLOGICAL, AND ESPECIALLY NEUROPHYSIOLOGICAL, LIMITATIONS ON HUMAN PERCEPTIONS OF THE UNIVERSE
(Abridged and revised version)

CONTENTS

1. Reductionism applied to mathematics
2. Generalized pattern recognition
3. The holographic brain
4. The origin of the genetic code
5. Biological computation
6. Biochemical communications
7. The travelling salesman problem
8. New foundations of mathematics
9. Energy and matter

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1. REDUCTIONISM APPLIED TO MATHEMATICS

The Concise Oxford Dictionary defines reductionism as "analysis of complex things into simple constituents". The method has, perhaps, been applied more to biology than to other sciences.

The view expressed here is that mathematics is governed by the laws of Nature, rather than the opposite. Moreover, it is asserted that the majority of mathematical methods, e.g., theories of sets and functions leading to differential topology, are inadequate for modelling processes of incremental evolution since the domain and range, for instance, need to be re-defined after each addition, especially where such additions perceptibly alter the behaviour of the structure of interest, the environment in which it exists, and the relationships that may become established between them. In addition, methods for dealing with concurrent events and phase relationships are cumbersome and involve abstruse concepts of combinatorics, multidimensionality, partial differentials and complex variables.

It is also asserted that mathematics is a derivative of thermodynamics which, under certain circumstances, becomes indistinguishable from information theory. The two fundamental assumptions of statistical mechanics (equal *a priori* probabilities and random *a priori* phases) are, however, retained in order first to argue from within the constraints of current dogma, but to them are added premises of a proposed new system:

1. Mathematics is a form of communication

2. All communication is expressible as a Markov process

From this it can be shown that a only a convention, not a theory, of number and exponent is required and, moreover, that number and notation are one and the same. (The expression of this form of mathematics is discernible in natural products and structures and an annotated bibliography of instances is in course of preparation.)

In *The Mathematical Experience* (1), Davis and Hersh point out (p. 391) that definitions of formal systems and rigorous proof rest on intuition and that intuition is visual. It follows that only inductive, and not deductive, logic is admissible in ultimate extensions of mathematics. It appears that the constructionist philosophy of Brouwer is valid, while the logical and axiomatic philosophies of Russell and Hilbert, respectively, are not. Most of the steps in Fig. 1, which follows, involve changes of phase

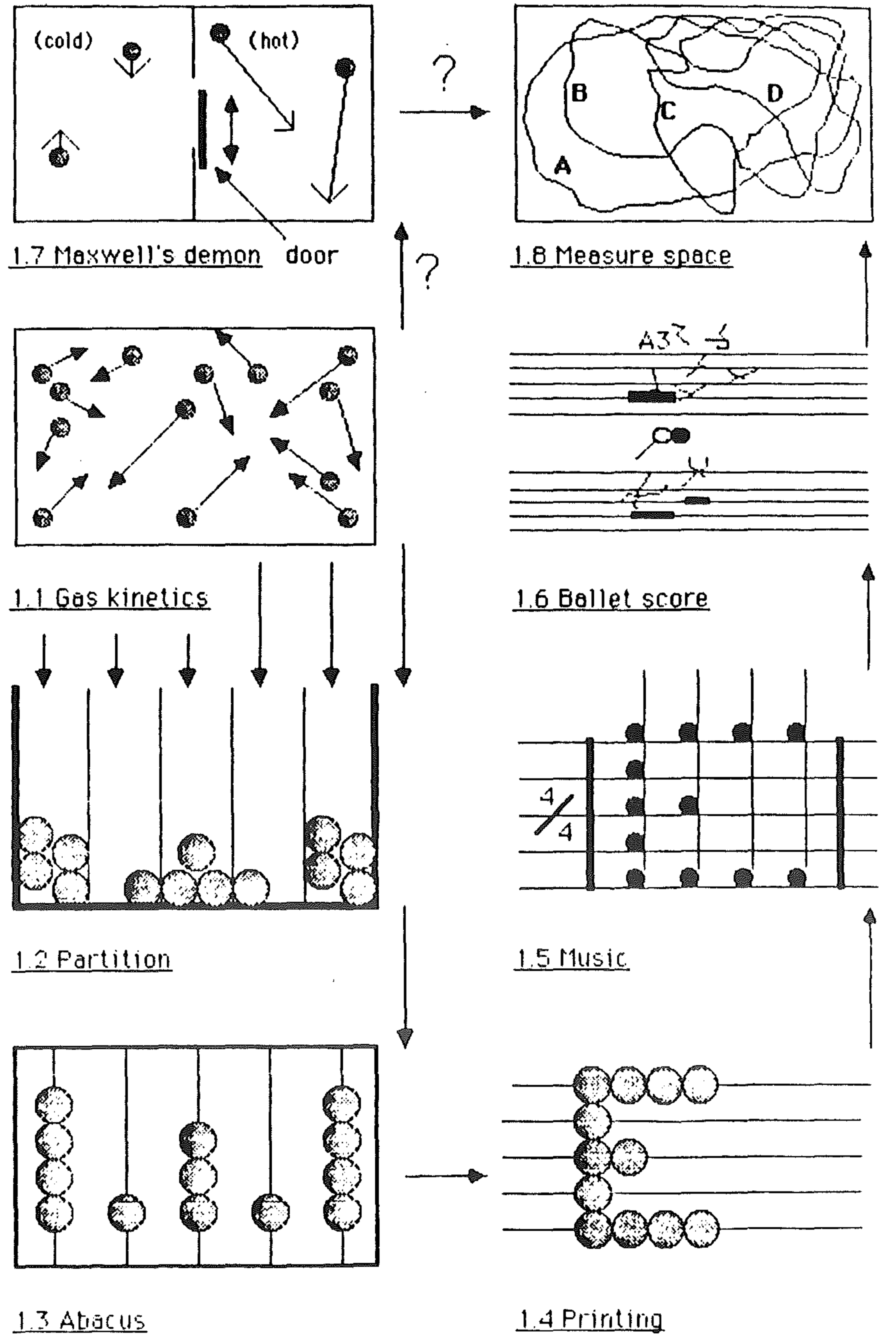
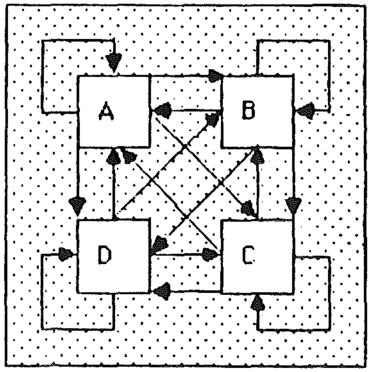
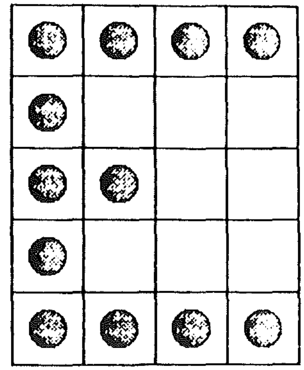

(cold)
(hot)
?
1.7 Maxwell's demon
door
?
A
B
C
D
1.8 Measure space
1.1 Gas kinetics
1.6 Ballet score
4/4
1.2 Partition
1.5 Music
1.3 Abacus
1.4 Printing

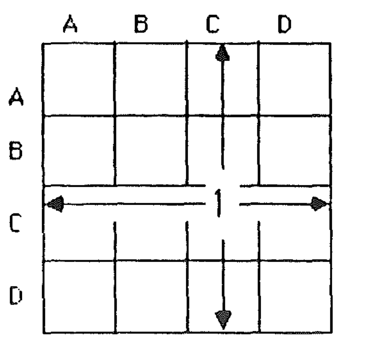
Fig. 1



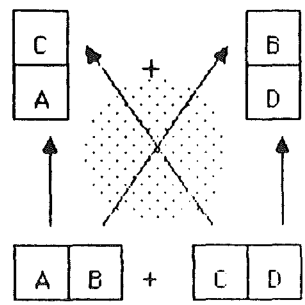
1.9 Closed system



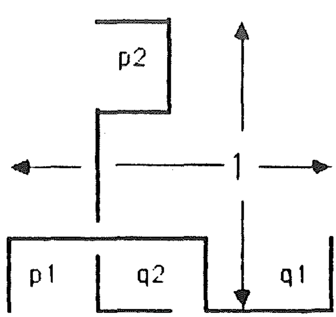
1.14 Joint partitions



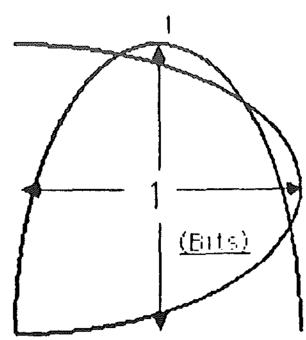
1.10 Transition probabilities



1.13 Transition states



1.11 Phase Rule



1.12 Negentropy

Fig. 1

2. GENERALIZED PATTERN RECOGNITION

Displays in which rows of lamps are lit sequentially in columns to give a 'moving' message are common. For recognition of individual characters, it is not necessary that detectors be set up in exactly the same way, the spatial and temporal components (rows and columns, respectively) can easily be resolved. Suppose that, in Fig. 2.1, lamps are replaced by non-inverting amplifiers which relay constant amplitude voltage input pulses after a short delay, constant for all the amplifiers. In this simple example, rows continuously propagate different tones in the 'audio' range, given as logs base 10 f. It can be seen (vertical arrows) that column elements of the letter 'E' (as joint partition) appear on all columns, but at different times. All that is then necessary to detect the complete character is to provide a circuit board with further delay amplifiers configured such that sampling for 'E' is carried out as four time periods (columns) over five corresponding spatial elements (rows), i.e., as a 5 x 4 matrix in this example.

The central nervous system (CNS) is considered to function in this way, spatiotemporal separation being achieved by dendritic endings of neurons sampling synapses of parallel nerves carrying signals (of differing frequencies) generated by sensory cells. In communications practice, this would be the equivalent of pulse code demodulation over a matrix of matrices, and explains the 'plasticity' of nervous tissue. It is further considered that the postulated delay is the (average) time required for input nervous impulses (action potentials) to cause 'firing' of neurons and is the equivalent of propagation delay times of electronic circuits. It is assumed that these actions are further subject to the influences of neurotransmitters and other substances, and that other forms of signal modulation and demodulation (e.g., gating, inhibition) are also present. Further discrimination and selectivity are available from the known mechanism of the 'all or nothing' principle of nervous action.

It is considered that, by substituting **average frequency** for **probability**, the entropy of received stimuli is progressively reduced in accordance with Shannon's expression $H = -K \Sigma p_i \log f_i$ (2), **executed in parallel**. This then allows concatenation of symbols as in Fig. 2.2.

The action of the central nervous system is thus seen as involving the three phases of (i), quantization (as outputs of sensory cells), (ii), symbol formation (on unchanged (i) signals), and (iii), concatenation of symbols.

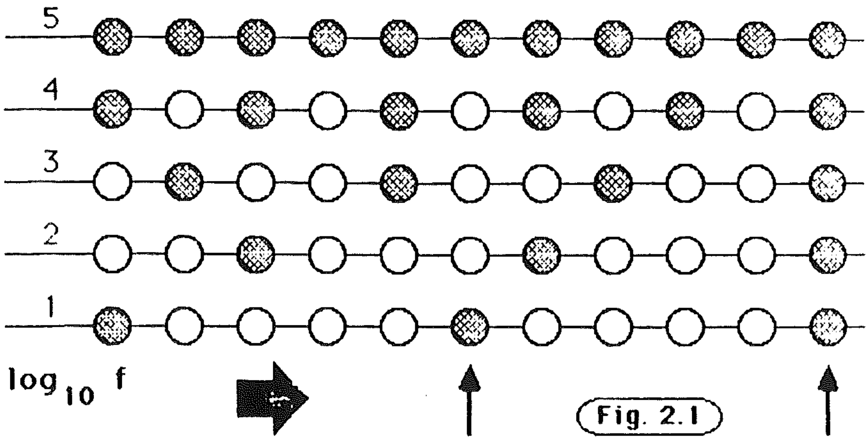

5
4
3
2
1
$\log_{10} f$

Fig. 2.1

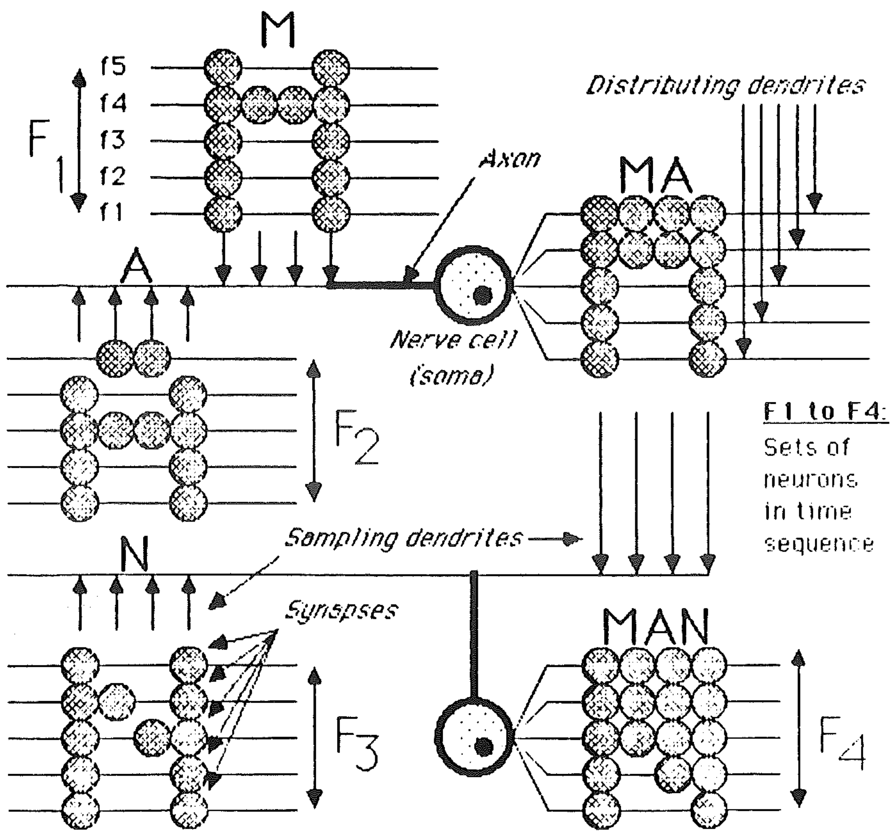

M
f5
f4
f3
f2
f1
F_1
A
Axon
MA
Distributing dendrites
Nerve cell
(soma)
F_2
F1 to F4:
Sets of
neurons
in time
sequence
N
Sampling dendrites
Synapses
MAN
F_3
F_4

Fig. 2.2

3. THE HOLOGRAPHIC BRAIN

Fig. 3.1 is a diagram of an optical phase conjugate hologram (3) expressed in the form of a network of constant delay (i.e., anholonomic) pulse repeater amplifiers (circles) as postulated for synapses of the CNS. The corresponding cellular automaton (4) is discernible in Fig. 1.9. The stream of action potentials of two sensory nerves is compared with those of corresponding motor nerves. This is to say that 'mid-points' in the networks of sensory and motor nerves then act as reflectors such that comparison between inputs and outputs, as coincidences of statistical density functions, takes place by ANDing the signal populations together.

The minimal configuration given explains a wide range of known properties of the CNS. A description will be given in more detail elsewhere.

The configuration is also explanation of memory. It is easy to imagine two feedback loops coming into existence as, say, protein synthesized in replacement of neuronic reflection loops. In Fig. 3.2, blocks C and D represent chemical delay lines (memories) re-cycling 'logic 1 and 0' responses to stimuli received in a manner similar to printing rollers or photographic negatives. The effect would then be to generate multiple planes of holographic images at each intersection of the CNS virtual hologram. The process of learning is then a simple addition to the synaptic plasticity already noted. Further modification of memory is also possible by increase in the number of synaptic vesicles leading to inhibition of the numbers of voltage spikes allowed to pass through a synapse (5). This mechanism is the equivalent of the arithmetical operation of division (and Stirling's approximation). The overall mechanism matches the known characteristics of learning curves.

Simple interaction of two wavefronts of light at right angles produces parallel lines of interference. It is believed that a phase conjugate hologram of the form described is capable of creating cross-hatched interference patterns as illustrated in the inner section of Fig 3.1. In other words, of generating cells, maps, arrays or matrices.

Light from the sun is randomly polarized. That reflected from water is, however, strongly polarized in the vertical and horizontal planes. This means that the entropy of absorbed light is less than that of the incident which, together with the two electromagnetic field vectors, gives four degrees of freedom into which oscillatory relaxation processes could take place as initiating influence (supply of negentropy) in the origin of life

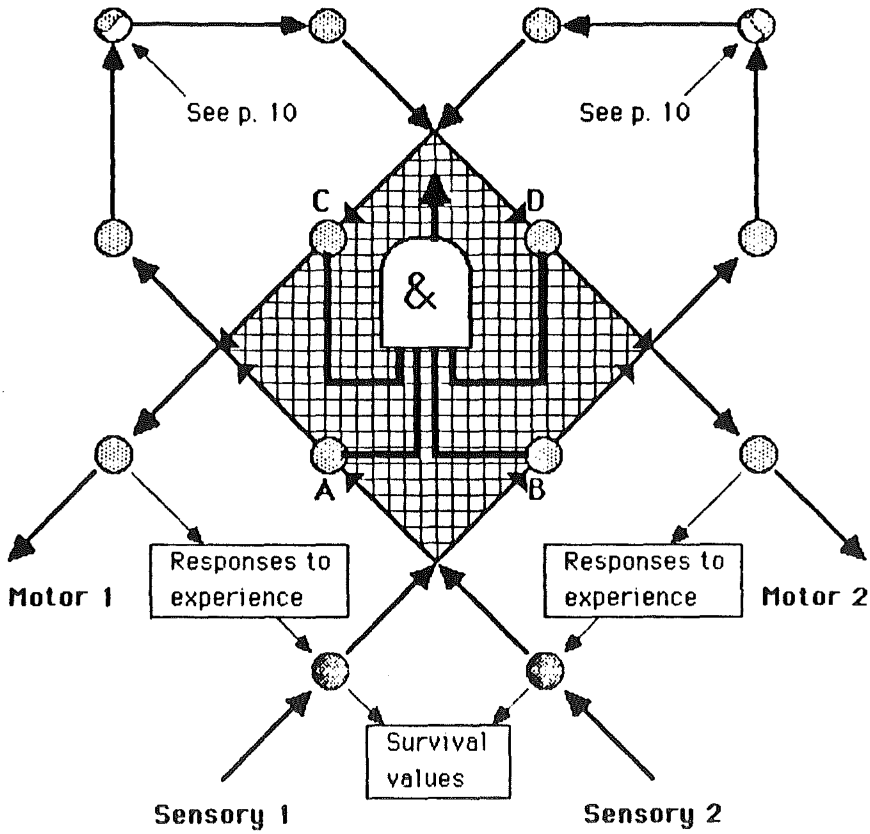

See p. 10
See p. 10
C
D
&
A
B
Responses to experience
Responses to experience
Motor 1
Motor 2
Survival values
Sensory 1
Sensory 2

Fig. 3.1

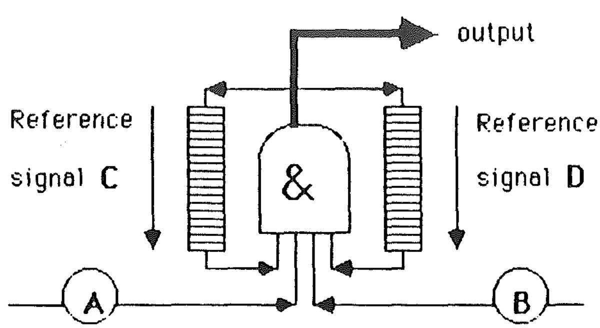

output
Reference signal C
Reference signal D
&
A
B

Fig. 3.2

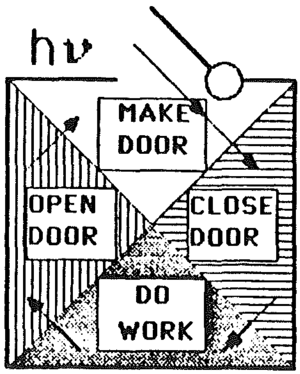
4. THE ORIGIN OF THE GENETIC CODE

It is postulated that life, and the genetic code, originated in nearly-closed physico-chemical systems of the primeval earth subject to the influence of polarized light of differing frequencies, equivalent to Wiener's wave filters (6), and a wide range of other influences. Some of these are listed in Fig. 4.1 and a more detailed desription will be given elsewhere. It is further postulated that biological Maxwell's demons came into existence, the fundamental feature of which was to set up relaxation processes capable of bringing about changes of phase governed by transition state theory. These processes were influenced by a multitude of random events, among which there existed sub-systems expressible as absorbing and reflecting states of Markov processes such that the electrochemical equivalents of phase conjugate holograms were created and allowed the further laying down of various forms of memory in representation of the step-wise reduction of entropy over time.

Fig. 4.2 is a representation of 'four-colour map' platelets forming a mosaic in a 'primordial soup'. Characteristic features of DNA are discernible on the diagonals (trace of the matrix) and an indication is given of the origin of circadian and seasonal rhythms in living organisms; lunar and precessional cycles would be expected if life had originated in, say, estuarine conditions. The configuration is consistent with concepts of cellular automata and Wang tiles, known to have computational and aperiodic features, and is remarkable in that each cell of the mosaic is a reflection of, and is reflected in, each of its nearest neighbours.

The configuration can also be matched to crystal form (Fig. 4.3) of, particularly, clays, and is similar to the structures found in ceramic superconductors. In addition, it can be matched with hypercube symmetry and Hamming distances. Feasible mechanisms of replication with total recursion have been noted in the literature, together with the origins of Fibonacci sequences, frequently found in structures of living organisms.

Finally, the configuration has been linked to the quantum theory of physics (7) and used in construction of a three-dimensional model of DNA. Pattern recognition in the CNS can also be linked to gas kinetics via the Sackur-Tetrode equation for translational energy, giving polynomials similar to those used in cyclic redundancy checks and block, row and column error correction codes of computing.

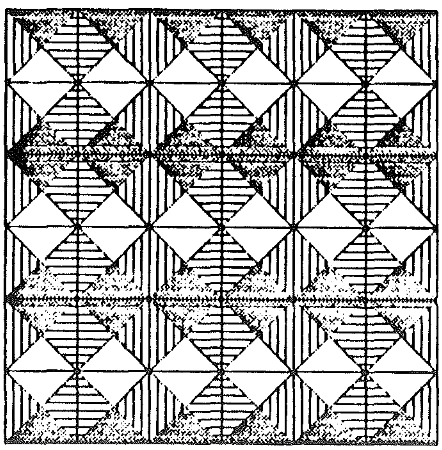


BIOLOGICAL MAXWELL'S DEMON

Phase change factors:

Polarized light
Energy gradient frequencies
Ion exchange
Crystal substrate (clays)
Catalysis
Brownian movement, etc.
Relaxation oscillators
Cyclical matrices
Attractors

Fig. 4.1



PRIMORDIAL SOUP

Key

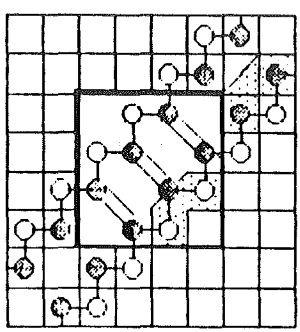
Blue/night/winter

Green/morning/spring

Yellow/noon/summer

Red/evening/autumn

Fig. 4.2



PRECURSOR DNA CRYSTAL FORM

Precursor phosphodiester
Precursor sugar
RNA sub-units

A computational mechanism based on hypercube symmetry

Fig. 4.3

5. BIOLOGICAL COMPUTATION

Fig. 5.1 is a drawing of a three dimensional model of a precursor crystal DNA based on 45/90/45 degree prisms related to anholonomic circular birefringence of light in an optical fibre. There are two right-handed spirals with a central core. The grooves are of the same dimensions as each of the spirals.

The model has, as 'skeleton', the 'butterfly' characteristics of discrete Fourier transforms (Fig. 5.2). It is postulated that DNA executes computation in this way and, moreover, directly in phase space. That is to say, the structure yields two antiparallel directional components, leaving one directional and three momentum components in its rigid form. Fully evolved DNA is, however, more of a polymeric plastic, indicating computation in several orders of phase space. High speeds appear to be likely, since it has been reported, (but not confirmed), that DNA has four broad absorption bands in the 3 to 9 GHz range. Because of the antiparallel (antiphase) construction of DNA, it would appear to be unlikely that the whole of the molecule would oscillate; more likely is oscillation of localised sections, a possible mechanism being given elsewhere.

The structure also conforms to a phase conjugate hologram if it is supposed that a bridging molecule could become attached at either end of DNA. The action of such a molecule would be to provide time delay such that a reflected signal comes into antiphase with its incident signal, together with rotation of the plane of polarization by ninety degrees, putting the two electric field vectors back in phase with each other, and cancelling the magnetic field vectors, say, in antiphase. (See upper corner synapses in Fig. 3.1.) The mechanism might well be the equivalent of superconduction (mathematically, $\sin^2\phi + \cos^2\phi = 1$).

On this basis, the function of the nucleotide bases would be to act as internal mirrors (Abbe image rotator, Fig 5.3) giving, through the hydrogen bonds, the equivalent of changing crystals in a radio set and generating, thereby, Fibonacci and other sequences.

DNA may also have the property of waveguide propagation of energy-conserving solitons (Ref. 8 and Fig. 5.4) and thus act like a 'Replicator'™, a system in which red and green light from two computer-controlled lasers is directed into a tank of solvent containing two chemical reactants sensitive to red and green light, respectively. The result is synthesis of a solid polymer to a fine dimensional tolerance.

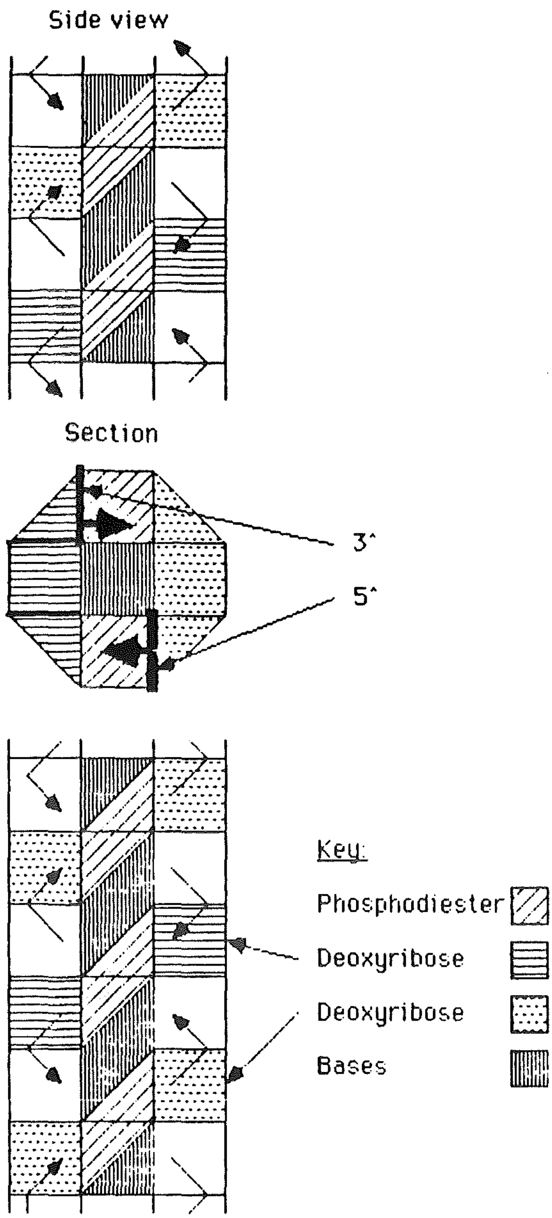


Fig. 5.1

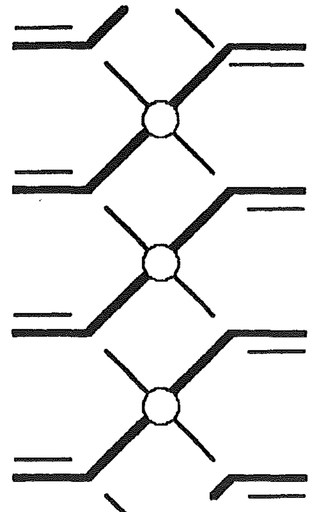


Fig. 5.2

Fig. 5.3

Fig. 5.4

6. BIOCHEMICAL COMMUNICATIONS

Fig. 6.1 is a representation of a generalized biological bilayer membrane, e.g., as found surrounding the nucleus of a cell. Fig. 6.2 is of a section of DNA drawn in linear form. While interatomic spacings may not be exactly the same in the two cases, it can readily be seen that the form of 6.2 could be fitted inside that of 6.1 (arrowed) by relatively simple chemical changes. Fig. 6.3 is of a transmission line as found in control and radio engineering for comparison.

The chemical structure of a biological membrane is very similar to that of a transmission line in that the fatty acid components act as dielectric of capacitors. In addition, it is known that membranes are well supplied with transverse 'ion-pump' batteries polarized in either direction. It is therefore postulated that membranes have the characteristics of two-sided two-dimensional transmission lines and act as multiport networks rather like 'dynamic crossword puzzles' in relation to reception and emission of chemical messages. The literature yields abundant evidence that this is so, the difference from telephony, say, being that information is transmitted on a simultaneous many-to-one <u>and</u> one-to-many basis, rather than on the one-to-one basis of human communications. A strong analogy between the structure of DNA and the 'eye' diagram and equalization filters of data communications practice can be seen, and it is notable that the bilinear function $w = (z - 1)/(z + 1)$ is used in calculation of the scattering matrix formulation of the theory of multiport networks (Smith charts, resembling Laue X-ray diffraction patterns). The expression has the property that when z is replaced with 1/z, it has the effect simply of changing the sign. This may be of significance in explanation of many phenomena, especially 'reversals' of various kinds, e.g., the division of metabolism into 'antiparallel' anabolism and catabolism.

Abundant evidence also exists of oscillators ('biorhythms') and the effects of changes of phase, e.g., from liquid (in solution) to solid (precipitation) and of 'disappearing' phases (effects of hydrogen bonds). A general conclusion, therefore, is that, by suitable application of information theory, the effects of light can be traced as an unbroken connective through all routes and mechanisms leading up to the process of abstract thinking and to social behaviour. The narrative does, however, involve frequent changes of phase equivalent to rotation of the plane of polarization or otherwise expressible as Gibbs' phase rule of chemistry and the Euler/Descartes formula for networks and tilings.

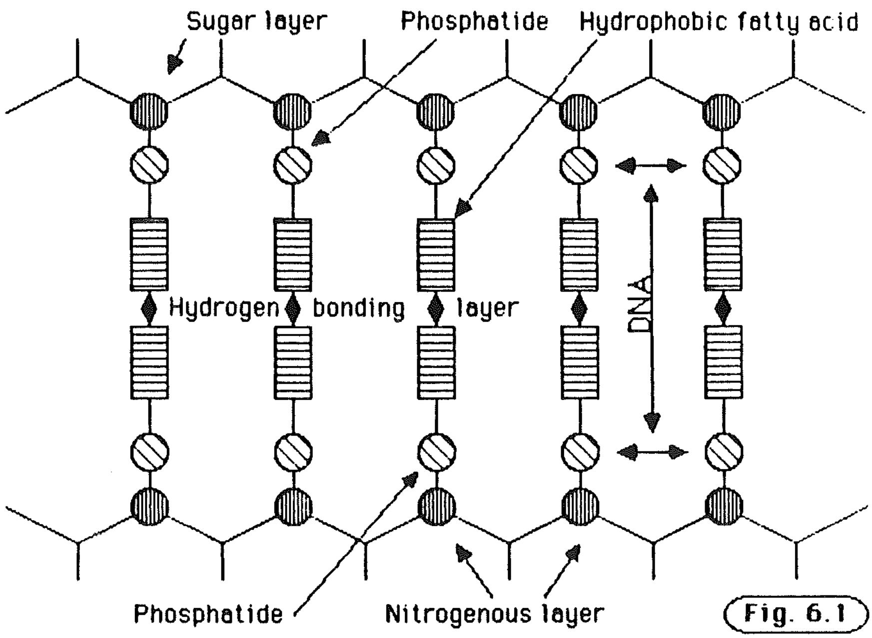


Fig. 6.1

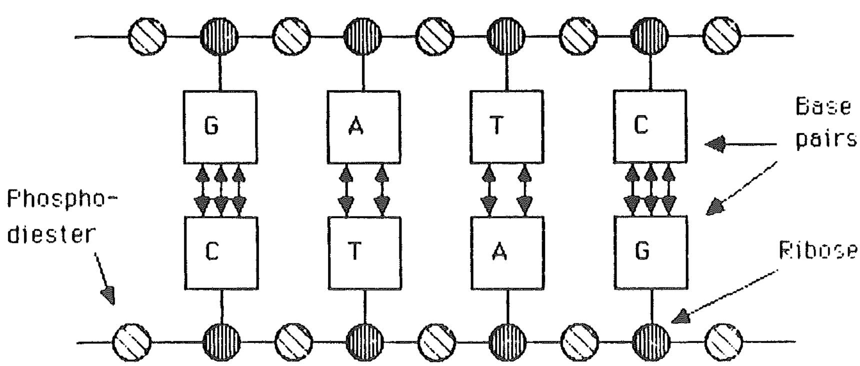


Fig. 6.2

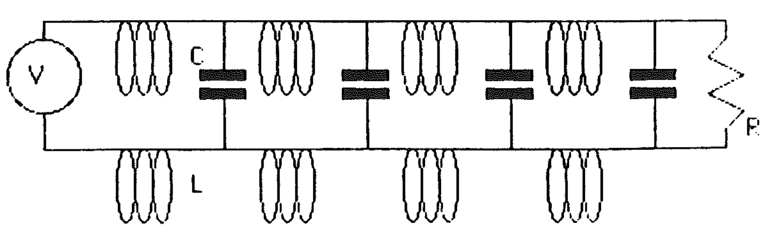


Fig. 6.3

7. THE TRAVELLING SALESMAN PROBLEM

The travelling salesman problem is one in which a salesperson has to visit each of a number of towns once, and once only, returning to a starting point by the shortest possible overall route. It is an example of a 'combinatorial explosion' (NP-complete), since the number of possible routes is (N-1)!/2 and requires specific simulation for its solution.

In Fig. 7.1, a salesperson constructs four circuit boards (A, B, C and D) for each of the towns P, Q, R, S and T in the simple example given Each of the boards is connected to the corresponding board (A to D) in each of the other 'towns' (a quadruple totally connected network topology). All the network wires are interrupted by constant-delay, non-inverting repeater amplifiers, <u>one for each unit of distance</u>. To find the shortest overall route, a constant-amplitude, constant-frequency pulsed signal is continuously fed in parallel to boards A and C of town P, the starting point. Boards B and D in each of the other towns return any signals received in the reverse direction towards town P via corresponding boards B and D.

Clearly, the first signals to return to P at boards B and D will have come from the nearest town (Q, say). These signals are then ANDed with the signal still being input at A and C. The output of the 4-input AND gate is used to enable a counter such that the AND gate then only responds to signals with a periodicity corresponding to that of the signals first returned. The process then applies to the next nearest town (R, say), and then to the next, and so on.

In time, the whole network will follow the same response pattern, with one town acting as a 'mid-point mirror'. If it should be the case that the nearest town is not the first town that should be visited in fulfilment of the shortest route, return signals from this source will not then be enabled until after signals from some other town. In other words, with appropriate circuitry, lamps inserted in the 'outward' network starting from circuit board A will eventually indicate the shortest route overall.

The configuration can be seen to be essentially the same as a phase conjugate hologram and to account for the 'self-sorting' and associative action of human memory. By application of singularity theory, it can also be developed into a general problem solving (GPS) system. Ignoring considerations of circuits and frequencies, it can be seen that the total number of 'attempts' required for solution is dependent on the units of distance specified (density function or 'granularity'). The process is similar to the Trémaux maze-solving algorithm.

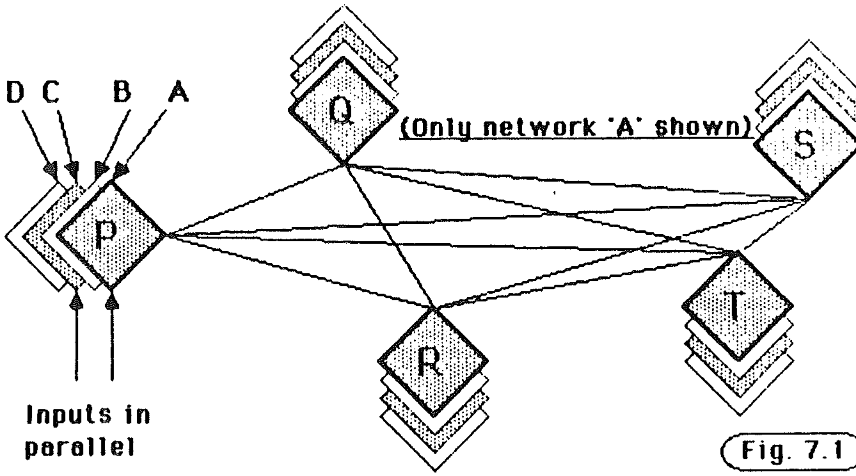

D C B A
Q
(Only network 'A' shown)
S
P
T
R
Inputs in parallel

Fig. 7.1

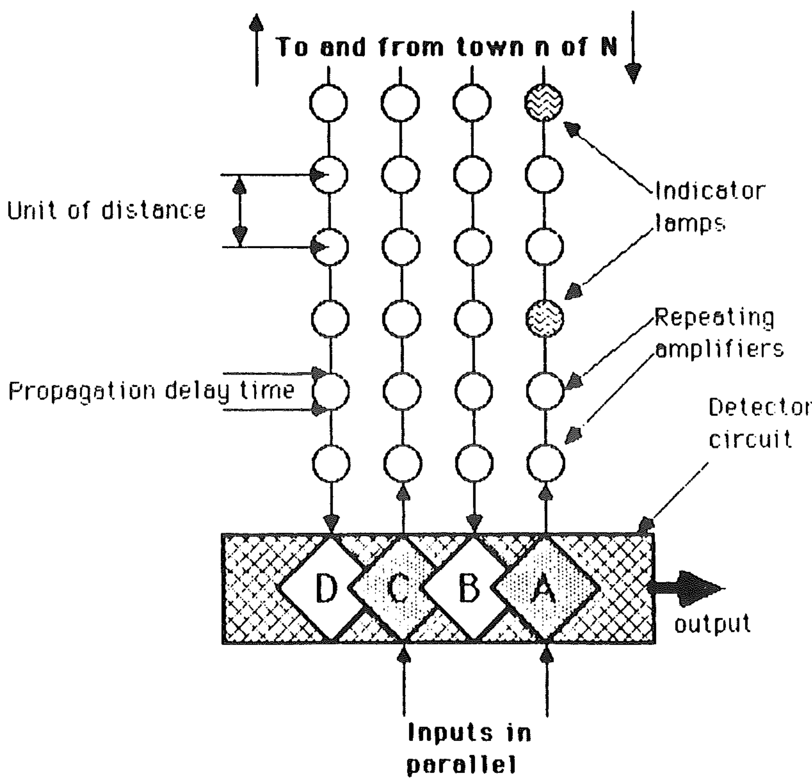

To and from town n of N
Unit of distance
Indicator lamps
Repeating amplifiers
Propagation delay time
Detector circuit
D
C
B
A
output
Inputs in parallel

Fig. 7.2

8. NEW FOUNDATIONS OF MATHEMATICS

Thought is known to be quantized in that discrete voltage pulses ('action potentials') are propagated from neuron to neuron in the brain. It follows that perception is quantized, essentially as counting functions of statistical distributions and densities, and that observed phenomena of the Universe will appear to be quantized, whether they are or not.

It is proposed, therefore, that a new branch of mathematics, based on discrete processes, be established and developed in replacement of current methods. Its foundation would be the concept of phaseless space (9), its development would be a modification of singularity theory (10) governed by Markov processes of local, global and cosmological environments, and its expression would be the material equivalent of a phase conjugate hologram. It is suggested that a quantum of phaseless space could come into existence as a 'gap' due to dissection and rearrangement of arrays (equivalent to gauge field theory), be allowed to develop into a <u>lumen</u>, (as precursor of a cellular automaton maintaining the 'inner product' of other mathematics), and that a lumen would be separated from a <u>numen</u> (external environment) by an active <u>membrane</u>. The symmetry of a small system would be that of a hypercube (with nested tori, 3-D interferograms, planar screens with multiple roller wrap-around and Penning traps as equivalents), extensible to symplectic elliptical and hyperbolic discrete geometries. Non-commutativity would appear to be a necessary condition for maintenance of aperiodicity in structures subject to processes of evolution.

Mathematics of such a form would resemble the structure of, say, a protein, and a situation of 'every problem its own solution' would apply in the sense that structures would be 'conjugate-seeking' in nested environments. They would consist of branching chains of critical paths (thick rule in Fig. 8) equivalent, in simple cases, to linear linkages (like pantographs), but with discrete bending, stretching, torsional and phase components in more complex cases (equivalent to present function theory). The arithmetic of such systems would be related to the appearance of dynamic and deterministic Fibonacci and like sequences as consequence of Markov processes in the environments. Direct correlation with energy processes is proposed, leading to suppositions that some (cardinal) points would be more important than others (ordinal) and that modification by intercalation would also be possible.

The environment in which a procedure is initiated would, of course, be critical in the same sense that the first move becomes critical at some later point in a game of chess. In this sense, board games are holograms.

STRUCTURED ENTROPIC MATHEMATICS

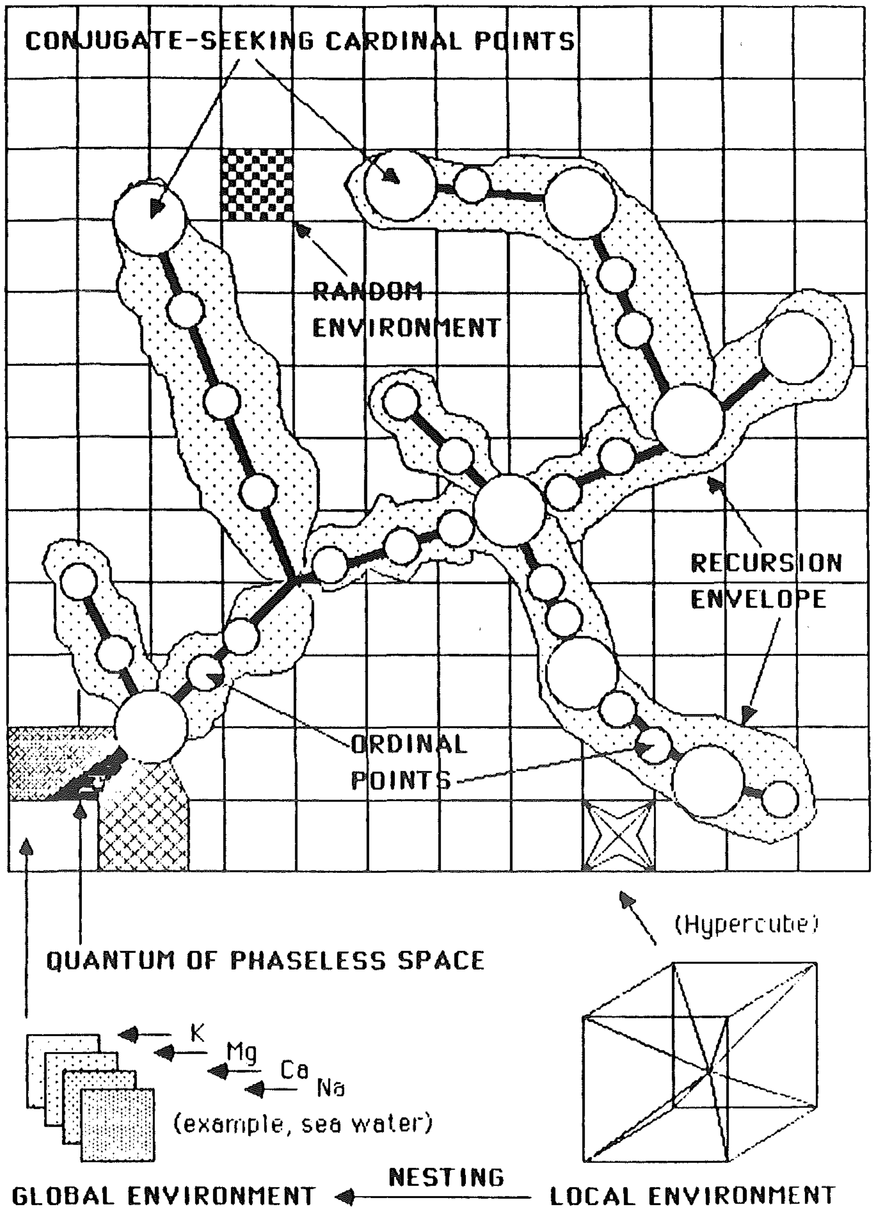


Fig. 8

9. ENERGY AND MATTER

Since so many observable biological phenomena appear to be material expressions of optical phase conjugate holograms derived from reflections of polarized light, it is now postulated that the division between 'energy' and 'matter' is equivalent to the difference between reflection of energy, as in a mirror, and total internal reflection, as in a prism.

Fig. 9 is a generalized representation of a spherical hadron based on combination of the concepts of triple solitons, phase conjugate holograms subject to total internal reflection, and transmission lines (by self-inductance and -capacitance of fields). The three patterns are equivalent to the three colours of quantum chromodynamics and to the individual solitons of a triple soliton. It is supposed that each of the solitons occupies an equatorial orbit of integer wavelengths on one of three mutually-perpendicular axes. Each then has the remaining two solitons, (in nominal antiphase), spiralling about it. Since antiphase would result in cancellation, it is supposed that each of the spiralling solitons is orthogonally redistributed at half-way crossover points. The crossover points then correspond to the six quark flavours and act as quarter-wave matching stubs to the respective equatorial solitons as transmission lines. The diagram indicates phase configurations for rotations (spin) about a horizontal axis (pairs on left) and about a vertical axis (pairs on right). The configuration within a sphere is reminiscent of the 'eight-fold way', the structure of cyclotrons and the relationships of quark charges to that of the electron. Overall, the structure could be imagined as an area of woven gauge field which has rolled up into an elastic ball, and from which energy cannot escape for the simple reason that all perturbations due to 'noise' have been eliminated.

The concept could be represented as the phasing of four 'crossed' relaxation oscillators (multivibrators), the action of two of which combine to control the remaining two. The effect then is to produce a 'strange attractor' and, probably, 'hidden variables'.

So far as living organisms are concerned, the distribution of energy is considered to be such that life is possible between two pairs of upper and lower limits, usually temperature and chemical concentration, consistent with the above concept.

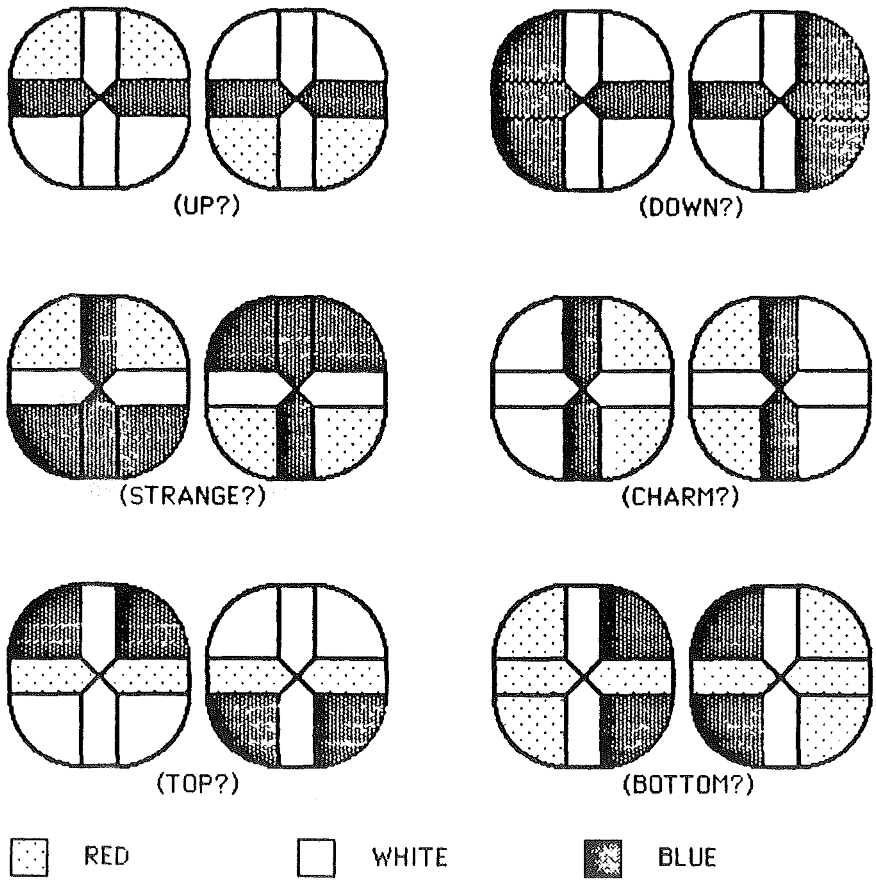


Fig. 9

CONJECTURE

That hadrons have the properties of closed transmission lines, by self-capacitance and self-inductance of solitons, together with those of a phase conjugate hologram, by total internal reflection.

Modelling may be possible as four "crossed" multivibrators, with one pair working in "tandem" as a "strange attractor".

ALTERNATIVE NATURAL PHILOSOPHY - PHYSICS AND THE LIFE SCIENCES*

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"It seems hard to find an acceptable answer to the question of how or why the world conceives a desire, and discovers an ability, to see itself, and appears to suffer the process... Perhaps in view of the form in which we presently take ourselves to exist, the mystery arises from our insistence on framing a question when there is in reality nothing to question."

Spencer Brown, 1972.

* This paper is an abbreviation of the argument presented more fully in Comfort, A., 1984 Reality and Empathy (Albany, SUNY Press) q.v. for reference.

I wish to raise some considerations about alternative natural philosophy which spring not so much from any specific model in physics, but rather from the general area of science in which physics now takes the leading place : "science" includes biology and psychology, where so far neither quantum theory nor the drive to examine alternative philosophies to Democratean-Darwinian-Helmholzian orthodoxy (the "seamless shirt" of XIX century science) have so far penetrated. These sciences have not been exposed to the kind of philosophic urgency which physics has had forced on it, but the models current in physics undermine a great many unexamined assumptions elsewhere. Rather than trying, as a nonmathematical biologist, to contribute to specific model making, I want to look at the implications of discussing an alternative natural philosophy at all.

Philosophy is, to my mind, very much synonymous with a search for acceptable algorithms - that is how the Eleatics saw their own activity : ours, in treating philosophy as a necessary outcome of experimental physics, and rescuing it from linguistics, is similar. We have no problem then in reuniting it with linguistics, by way of computer science. The program universe model is one such algorithm, the idea of geometric and disordinate spaces is a tool in algorithm formation, and so on. How does this activity fit into the grand project of science?

We have to start with a philosophical datum: whereas XIX century science started with middle order reaility and objects, we now recognise that the only substrates of observation are phenomena, i.e. "appearings", meaning inputs from external reality <u>after</u> they have been processed, and processed selectively, by our brains. We have no access to anything else. Our inferences are based on phenomena and processed similarly. This situation is fully conscious in physics, less so in 'life sciences'. A biologist will probably now admit that species are phenomena (arising from our classificatory activity) and perhaps if pressed that trees or crocodiles are phenomena, but the assent carries very little conviction and it is a lot more convenient to treat them as real objects - anything more counterintuitive is misplaced ingenuity. If we go back to Mach's statement concerning phenomena that "there is no cause or effect in Nature: nature simply is. Recurrence of like cases ... exists but in the abstraction which we perform for the purpose of mentally reproducing facts' (Mach 1956)

physicists are acutely aware of what he is expounding: biologists and psychologists tend to see it as an abstraction not specially useful in what they are doing. The same happens if we quote Heisenberg, to the effect that

'the conception of objective reality has evaporated ... into a mathematics that no longer represents the behaviour of particles, but rather our knowledge of that behaviour (1958)'.

These are truisms in quantum physics - if the life sciences have not come to terms with them at all, although particle behaviour is the foundation at the bottom of all hierarchies of observation, physics has not yet gone very far in looking at the processes of abstraction, mentally reproducing facts, and knowing, which are the other half of the phenomenal - at the transforms they involve, or at the limitations, resources and programming of the "obligatory interface", the brain. We have to look at these in order to generate an alternative philosophy, and the more we use computer models and information theory the further we find ourselves pushed into doing so.

The probelem is not insoluble, only difficult. Most of the starting axioms are really self-evident, mere housekeeping.

All observations of nature contain, or rather imply, a "primal division" - not so much between the known and inferential portions of the universe, but between the intact and the observed. The totality of "what is" contains both the potential observer and the potentially observed, and this, so long as it remains a totality, is ex hypothesi unobservable.

The very fact of addressing this totality divides it by introducing a Spencer-Brown "box":

$$T \rightarrow \overline{0}\rceil \sim 0$$

observer and phenomenon, whether 0 is a physicist, a lower form of life, or a machine. In our case it is most usually a physicist, since unlike other organisms physicists are looking for an algorithm descriptive minimally of ~ 0, and perhaps inferentially thereafter of T. This primal division comes close to providing a formal proof by self-evidence of Parker-Rhodes' concept of agnosia. The contents of T in its intact state, which we have to assume, and assume not to be infinite, since we assume T to be patterned or structured, are by the same definition indistinguishable because until we treat 0 as separate there is nobody to observe or distinguish them: this is not quite the sense in which Parker-Rhodes' indistinguishables subsist, but it is a valid form of indistinguishability.

The fact that we perforce work with phenomena, then, introduces a general qualification. We can infer mathematically the conditions existing microseconds after the Big Bang, or infer from fossils the fauna and flora of a Jurassic landscape. What we are inferring, and may well be inferring correctly is not so much what "was there" as what would have been there had a human 0 been present, e.g. a human observer in the Jurassic would hve seen dinosaurs but no larger mammals. This same qualification, of course, applied to what "is there" now, but since we take direct observation for granted, the 0-less examples seem to be more counterintuitive: is Comfort saying that there were no dinosaurs? Only in the sense that there "are no" electrons if we regard electrons naively as objects. The same would apply today to crocodiles or trees, though that is rather harder to empathise.

The nub of this point, which goes back well beyond Berkeley and Whitehead to some of the Eleatics and the whole corpus of Buddhist philosophy, can be illustrated as follows. Imagine a video tape of *Hamlet*. This contains in informational form the whole visual and verbal content of a performance of Hamlet. Where it differs from a performance is simply in not being played. Our visualisation, or rather our inferential description, of the Big Bang is the description of such a video tape. Our Jurassic reconstruction is a videotape which, if it was played, was played only to nonhuman observers, and what they made of it God alone knows, though in certain respects such as dimensionality and sequence the perceptions of all organisms with nervous systems are probably similar (so that if we played them the Hamlet tape dinosaurs would hear and see approximately what we do, without the added significances we attach to drama). What "is there" in the Hamlet tape is a collection of magnetised and demagnetised areas. What "is there" in the array we observe today is potential information, which generates middle-order reality when we play it, but not until we do.

So not only $T \subset 0$, ~ 0, but 'R' ("reality" as we observe it) represents an operation by 0 on ~ 0 : 0 "plays" ~ 0 like the videotape.

That, of course, is where the "objective reality" of machanistic physics has gone, largely because at the subatomic level phenomena behave sufficiently unlike "things" to make us aware of the difference. On a basis of this unthinglike behaviour we are able to construct algorithms which describe the information stored in the "tape" - the reconition of this situation is enshrined in the Copenhagen Solution : not being able to observe either T or the tape itself (~ 0) descriptive algorithms are about the best we can expect. Oddly enough, in this situation, neither physcists, nor those of us whose job it is supposed to be, the neurologists and neuropsychologists, have so far given much attention to finding out exactly what happens when the tape is played, or the origin of the transforms and the transductions which turn magnetic pulses into a dramatic work.

When we do turn our attention to "mind", the system which is doing the observation and transformation, the consequences tend to be even more disconcerting. If we move from the model of the videotape to the model of the video-game, we have an example of phenomena produced from an algorithm. The display constitutes a 2d+t flatland on which "objects" appear - say asteroids and space ships. The quasi-objects follow cause-effect rules (if they collide there is a simulated explosion) and they are manipulable. We recognize (a) that there are in fact no "objects" there, i.e. they are virtual and constituted from superimposed pulses generated from an algorithm hard-wired in a "black box" (so that without the VDT the game could continue, but it would be analogous to the unplayed Hamlet tape, and transcomputable for the players) (b) that the pulses bear no resemblance to the objects generated, are not isomorphic with the screen display and are nonlocal with respect to the symbols appearing there. They are not even "there" all the time. Moreover the display symbols are not objects, and their continuity and interaction are entirely virtual.

The video game is an extension of Bohm's droplet and glycerin model, but more comprehensive in that it includes patterning from the algorithm hard-wired in the "black box". An actual videogame presents no problem, because 0 is external to the system : the function of the display is to generate a manifest world on which an organism (the player) can operate without performing cumbersome calculations. This is precisely what Kant meant by a-priori-ness - we, the observers, are ourselves hard-wired to see (and require) a hands-on sequential world in which the equivalent of coded information or state functions is resolved into objects.

Consider now what happens if we apply the same model to the whole system, 0 plus $\sim$0, observer and observed. The conventional interpretation of "mind" in science is Helmholzian : it is the activity of a neural computing system. The going interpretation (token functionalism) is that "mind" represents a program : any system which could run the program would "have a mind", whether it consisted of silicon chips, neurons, ectoplasm or pieces of string.

This kind of epiphenomenalism would satisfy both von Helmholz and Turing. It is destroyed by the videogame model in its extended form: here there is no 0 external to the system, for the brain and the computer both consist of quasi-objects (molecules, atoms, subatomic particles) which are not "things" but blips on the screen. Mentation, on which the synthesis of these quasi-objects depends, becomes an activity of the (wholly virtual) gamepieces. In that case who or what is doing the thinking? A simpler way of putting this circularity ("the hyperloop") is that

> 'the human mind, including consciousness and reflective thought, can be explained by activities of the central nervous system ... second, biological phenomena at all levels can be totally understood in terms of atomic physics, through the interaction of component atoms. Third and last, atomic physics, which is understood most fully by means of quantum mechanics, must be formulated with mind as a primitive component of the system' (Morowitz 1980).

Or, as Bohm put it (1979) 'the ultimate perception does not originate in the brain, or in any material structure, although a material structure is necessary to manifest it'.

Thus the "observer paradox", at least as Wheeler formulates it, has philosophical implciations which to say the least require thought. We are not necessarily alarmed that the "hyperloop" is closed, i.e. recursive, because observation ('the Universe seeing itself' in Spencer-Brown's phrase) comes very close to defining reflective "mind", and in dealing with the observer problem an algebra proper to recursion seems appropriate and unavoidable.

A more immediate and less contentious problem for both physics and neuropsychology is how precisely the brain transforms an array into "objects" within the 3d + t Kantian space. The joker in this pack is clearly t, perceived or elapsing time, which is not identical with t as a dimension of space-time, and, as Paul Davies points out, should be firmly replaced where it belongs, in the human brain.

As in Hinduism Mahakali, Great Time, is the goddess who creates the hands-on world (maya) as a display, so for us experiential time is a built-in way of ordering experience, whereas we invincibly treat it as an object of knowledge.

This is not new or alaraming to physicists (de Broglie was pointing it out thirty or more years ago in relation to the en-bloc character of relativistic space-time) but it has been utterly without impact on life sciences, in creating historical-sequential models for evolution, rather than superpositional ones, and even on cosmology, which goes on using the convenient historical-sequential mode, rather as we treat positive as "hot" in circuit diagrams. It does not matter so much in cosmology, but it has a limiting effect on evolutionary biology. Moreover it tends to the persistence of Newtonian-Democritean-Cartesian shorthand which comes to influence scientific thinking: we should not forget that sequentiality drags causality along with it. The purist alternative, of course, has all the drawbacks of trying to play "Asteroids and Spaceships" with the VDT turned off. Moreover since life sciences deal with the characters of the manifest, middle-order, or pheonomenal world the convenience is obvious: it is still however theoretically limiting - the conclusions from seeing evolution as a superposition and seeing it as a rigid sequence are quite different, and the superpositional conclusions are at least as counterintuitive as those of quantum mechanics. As long as the life sciences watch the results drawn from physics rather as a crowd watches an air circus, this is likely to persist and lead to stultification.

I think it interesting that as these philosophical problems were raised again by physics (they have been around more or less explicit form since early Buddhist philosophy) we now look to computer science to solve them. Buddhist philosophy approached them in an entirely different way, based on belief that the Kantian mode of perception is not invincible, and that the subjacent structures can be directly perceived and the object-forming process turned off. This is a possibility which neuropsychologists at least, and possibly also physicists, may require to re-examine. After all, although we cannot draw or visualize a fourth orthogonal, most of us here are quite happy with higher dimensional orders, Hilbert spaces and the like expressed mathematically. There is neurological evidence that the 3-d structures we use in cerebral processing depend on something very similar to matrix algebra performed by "reentrant processing over a multidimensional store" (Edelman and Mountcastle 1979) and we have the potential to assist this kind of facility by creating 'demonic', that is, logical but non-Kantian, computing systems using quantum logics, and multivalued and recursive logics, in the place of the Boolean systems to which present day logic gates are particularly apt. This calls either for greater complexity or different engineering, but it could be done.

What I have done here is to try to summarize a general philosophical agenda, not only for physics, though physics probably feels the necessity most, but for science in general - including life sciences when they catch up with the implications of the quantum theory and its algorithms. The point is that these algorithms are fundamental, whether or not they can be applied at other system levels directly, e.g. in biology.

<u>THE PURSUIT OF AN ALGORITHM</u>

When we turn to specific models, it becomes clear that algorithm-hunting is a double, not a single task. What we ideally require is a definite algebra for T, "alles, was der Fallist". In arriving at this we are obliged to use B, the pocket-computer in our heads, because we have <u>no other</u> equipment (I will not go into the further maze involved in determing what precisely is implicit in phrases like 'we require', 'we are obliged', which use the plural of 'I', the undefined homuncular observer - is 'I' part for the computational system, and, if not, to whom or what is the printout presented)? This being so, to an algebra A_T we have to add a second algebra A_B to unscramble the transformations imposed on T by the B-operation. We know that B imposes a-prioris, treats en-bloc processes by sequential slicing under which 'past' slices are determined and 'future' slices undetermined: it chops probabilistic distributions, reduces field-type arrays to objects, collapses wave functions, and reduces superpositions either to yes-no answers or to iterants. Moreover when we correct some of this elision of off-diagonal terms by mathematics, the results still have to be presented to an 'I' which is a-priori limited, or they will be unintelligible. Rather obviously both models which 'relate events to a basic algebraic structure connected to 3+1 space' (Bastin and Kilminster) and program universe calculations generally fall into the category $A_T . A_B$: they deal with the played, not the unplayed tape. This is wholly reasonable if the limitations is recognized, but is exposed to further reduction if we are able to analyze A_B and reverse the B-transformation.

It would, I think, be generally agreed by everyone, including the authors, that Noyes' model and other competing models deal with the mathematics which expresses our knowledge of particle behaviour. In terms of the videogame model, they describe the behaviour of the virtual "objects" on the screen. The vitally important part of this operation is that it can be and has been computerized, so that the results of computation can be compared with the whole ensemble of data. In Noyes' own words, if the computation works out, it validates itself, but "a failure ... will either point to an area in which to look for a new physics.. or even a way to look beyond physics". In other words, it initiates the process of describing ~ 0 in a way which gives us a prospect of inferring $T = \overline{0}\rceil \sim 0$ therefrom. Noyes' use of quotes around such words at "time", "random" and "universe", and the possibility - in fact the likelihood - that a strict computational approach with linear equations will throw up systems breaks (why, for example, in the VDT model, does an object which goes off to the right of the screen reappear on the left? Is the line scan a closed dimension?) points the way forward. If we get a systems break and reprogram to treat it as true, we have a 'demonic' computer with non-Kantian, higher-dimensional or otherwise counterintuitive 'thinking' and are on the way to make an end run round the brain-imposed transformations. This is not so much going beyond physics as going beyond the videogame to the algorithms in the 'black box', which otherwise looks unopenable. This may well be the major importance of the whole program universe approach, that it opens up 'phenomena' by exposing inconsistencies, and incidentially puts an end to the convention that physics has no truck with metaphysics and can omit the bias imposed by the passage of data through a brain.

PROGRAM UNIVERSE AND THE LIFE SCIENCES

If one looks at Noyes' thinking, in particular, to see how far anything analogous, if not homologous, would be possible in biology, the outlook is quite encouraging. The approach, of course, would have to be different: there are not too many biological areas where mathematical structures are sufficiently well-defined. The two most evident are evolution and morphogenesis. The coincidence immediately links them with physics, and, indeed, with the argument over fundamental versus hidden-parameter models.

The orthodox position is that morphogenesis is programmed but evolution is not - the zygote follows a predetermined path (it contains information), but evolution involves only variation and selection by a kind of market economy. This view works so well over a large area of evolutionary biology that fudges (over the evolution of feathered flight from temperature-controlling scales, for example) are tolerable, if only to preserve a common front against naive kinds of teleology. However, any historical theory, which this is, incurs suspicion - empathic or elapsing time is an element in it. We are accordingly dealing with serial sections of something which, suitably rotated, would appear as a superposition. The question in this case is whether there is any flow of information into the evolutionary process other than selection - is the evolutionary surface in some way configured, as we assume the source of phenomena in physics to be configured, by a structure not describeable exhaustively within space-time?

This type of speculation, which is heretical in biology but not in physics, has nothing to do with vitalism or teleology, simply with the most economical description of phenomena. There are already computer programs which apply selection to random variation, but apart from yielding mathematical data to quantify e.g. selection pressures and to show the results of manipulating them, they do not differ greatly from laboratory selection experiments with actual organisms. These are limited to micro-evolution, over which there are no problems. What might be more instructive would be the construction of macro-evolutionary models, possibly line automata generating partiuclar forms. A line automaton has a basic algorithm - it generates a "configured surface": the aim would be to see whether a configured model for evolution was more conomical than the going stochastic model. There is also, of course, the question of how far up the hierarchy of complexity quantum phenomena such as nonlocality and indeterminacy extend. Most physicists assume that they do not extend beyond the subatomic level, if only because state equations for large aggregations are unmanageable and thermodynamics takes over. This may well be true but is not selfevident. Some physics speculations, such as the "many worlds" interpretation, could not be confined in this way. Biologists, because they are still out of breath from putting down XIX cent. Supernaturalists, tend to see this kind of speculation as dangerous. But if it is proper to tackle the videogame called "middle order reality" it is also proper to tackle the video game called "Darwinsim", and by similar methods.

This is one possible approach. The other, of course, is a neurological and psychological attack on observerhood and the built-in transformational algebra of brain - our B.

We can do this in two ways - by looking for observed system breaks, discontinuities and "sore thumbs" which indicate that all is not as it seems, and by an attack on the actual processing mechanism involved in B: given that it is adapted to simplified, predominantly visual, sensory inputs which pattern 'phenomena' before we encounter them, and which we go on to apply perferentially to inferential as well as visual data, how exactly does it acquire the extra computational capacity to get round its own a-prioris by mathematics? A mathematician, or a yogi, who apprehended a 'thingless' universe would almost certainly be reconstructing it, not seeing it: he would be running A_B backward in simulation. Kant believed that given A_B a non-Euclidean universe was fundamentally unthinkable - it wasn't; why? and how precisely is it thought?

There is some balm in Gilead already, though our knowledge of brain processing is rudimentary. If it does indeed involve either holographic storage (Pribram) or re-entrant matrix processing (Edelman and Mountcastle) that would explain its easy ability to do rotational algebras: a lot of the reductions imposed by B on the data coming in look as if they can be quite plausibly reversed by rotation - unlike the pocket computer, the brain deals easily with not only **i** but **ijk**, and more eleaborate matrix-to vector operations such as spinors and twistors. I am not talking here about the ability to do mathematics involving these algebras, but the fact that the brain may process some data by doing them uninstructed: the paper algebra is an elucidation to us of what is going on in the brain program. This may be germane to Watson's distinction between horizontal and vertical processing of the perceived world: to achieve nirvikalpa perception, multiply experience by **i**.

Premathematical philosophers have for several thousand years tried to represent "mind" intuitively as a quality of mirror-ness in the universe. Now the mirror calls up images both of rotational (noncommuting) algebra and of re-entrant processing. It is fascinating therefore to see re-entrant processing of multivectors turning up in the computational construction of hierarchies - Amson's bi-uroboros. The uroboros of esoteric philosophies did multiple duty - as an image of a cyclical universe, what we would call toroidal space-time with closed t, so that the whole of the system's 'past' coincides with its future, separated from it by a singularity: as an image, in reincarnational philosophies, of serial 'lives' as components of a superposition; and, more relevant for our purposes, as an intuition of 'mind' as a re-entrant process. Wheeler's diagram of cosmology as a loop ending in an eye which inspects its origins is a modern example. The neurological point here is that Amson's construction may not only be a convenient way of programming towards an algorithm, but a reflection of the way that the brain intuitively performs the same operation in extracting (0,1) from a statistical array. It is difficult to extract mathematical detail from the kind of altered-state instrospection which generated e.g. Buddhist philosophy, but with the 'skilful means' provided by mathematics we are in better position than previously to determine the potential relevance of some of these intuitions. These thinkers were as smart as we are, and were only constrained by their time and culture to express intuitions in myth instead of math. Amson's program presents itself to the mathematician who reads the printout. the brain's program presents itself to the homuncular "I", whatever that my be: attempts to mathematize I-ness end in another loop. Buddhist philosophy treats the observer as her~~/~~self virtual, like the phenomena which 'appear' to her. We cannot get into that arugment here, though it remains to be addressed in making sense not only to psychology but of physics and of 'beyond physics'. It is sufficient at this stage to stay with the <u>cogito</u> as given, though the issue will continue to overhang any alternative natural philosophy which does not address it head on. In meditational practice, intense introspection directed at the 'I' causes it to disappear completely, but without cog~~i~~itation or experience ceasing. One wonders whether mathematical physicists should not undertake some of these mental exercises rather as marine biologists now obligatorily learn scuba diving. What was 'ineffable' to mystics might well seem to them expressible and even programmable. The essence of 'mystical' philosophies is not hocuspocus but an intense effort at parasitic analysis of reflective mind by reflective mind - using the B computer to analyse its own working to the limits which Goedel and Feigl imposed on that kind of activity. Like Moliere's citizen who had been speaking prose all his life, many areas of mathematical theory are already into this process without being aware of it, largely because 'pure' number theory and the like are seen as abstract, existing without the inconvenient mediation of a human brain. Enough to say here that the mathematical devices which occur to us in writing programs may be those which are inherently in place as a component of B.

Another resource we may well find productive is Spencer Brown algebra, particularly its development by Kauffman and by Kauffman and Virela. Spencer Brown is the introducer of imaginary numbers and rotational algebras into logic : Kauffman has extended this to superpositions of the type $a \cap \sim a \neq \emptyset$ (the Spencer Brown 'box' becomes a waveform, an alternate iterant: this is how the brain deals with the Necker cube paradox).

I am fully aware that a great deal of what I have said here is not specifically germane to the program universe model. But as a life-scientist I am increasingly unhappy about the radical split which has developed between the very fertile conceptual debate in physics and the almost complete neglect of this debate in psychology and biology. One way of crossing the gap is to educate life scientists, but a more valuable one would be to involve them directly in the Project. Physics is long past the point where it has had to address neurology and psychology - if it has had no help from these disciplines, that is because the neurologists and psychologists haven't been called in, educated, and put to work. Algorithm-hunting is a pan-scientific project - physics has to climb at the front of the rope, but physicists, as brain-users and organisms themselves, need the rest of science to back them up.

anpa/ljc

DIRAC HAMILTONIAN FOR A FREE PARTICLE DERIVED ON STRUCTURAL PRINCIPLES

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ABSTRACT

Our argument is based on the following conjecture: that the models brought into the Quantum Theory (QT) from classical physics, via the Correspondence Principle (CP), might be derivable from the QT itself on certain very general principles. These principles would not make appeal to the CP and would not be expressed in the classical picture language. Such derivations, if they exist, could be interpreted as independent explanations of the classical models.

We begin by referencing a version of the mathematics of QT and we summarise useful notations and results.

To bring in dynamics we assume that the state evolves with a continually increasing scalar time; the remainder of the argument is abstract. Axioms are given which express, in the mathematics of QT, the ideal behaviour of a 'classical system'; that is, completely certain, predictable and bounded evolution. Various alternative expressions are derived but, finally, it is proved that the axioms are inconsistent; thus, classical systems (as defined) cannot exist. This result is discussed in terms of information theory.

Despite the inconsistency, the axioms appear to be a useful guide to theory. By ignoring an inconsistent axiom, that the Evolution Operator (EO) is bounded, we derive a canonical form for the EO of a classical system. We also derive canonical forms which apply when the classical behaviour is only transient and when, in addition, the spectrum of the EO is dense. The latter of these forms is identical to the Hamiltonian of a Dirac free particle in n (spatial) dimensions (certain operators, which happen to have the correct properties, being interpreted as coordinates and momenta etc.). Although, the theory gives no special significance to the case n=3, it is shown that, in the simplest representation, n<5. To derive the forms aproximations are used. The physical interpretation of the approximations is that the results apply only to low energy phenomena.

The implication, that Special Relativity (SR) may derive from QT, is briefly explored. It is shown that the 'particle' must obey QT versions of SR kinetics and kinematics.

1. Formalism

1.1 Elements

The mathematical formalism is that given in [1]; some more advanced results are taken from [2] and the notation of [3] is used on occasion. Specifically: measurable variables are represented by Hermitian (self adjoint) operators on a normed (to unity) vector space; the vectors represent states; the eigenvalues of the Hermitian operators are the values, obtainable by measurement, of the variables; measurement leaves the system in an eigenstate (pure state), at least, momentarilly; an eigenstate is represented by the eigenvector corresponding to the (measured) eigenvalue; the result of a measurement is, in general, subject to chance depending on the state prior to the measurement; vectors represent states in the sense that the inner product of an Hermitian operator with a vector is taken to be the expected value of the measurement given the prior state represented by that vector; since, this statement must also apply to any Hermitian function of an Hermitian operator (see [2] and below), the prior state vector can be used to generate all the moments, and hence the probability distribution, of the measurement [2],[4]; the probability distribution given by an eigenstate is degenerate- all moments are zero save the first and that equals the eigenvalue- so that, when the prior state is an eigenstate, the result of measurement is certain; in order to describe the dynamical evolution of isolated systems the state vector is taken to be a function of an increasing (scalar) time.

1.2 Hilbert Space

In order to discuss operators with dense spectra (eigenvalue sets) we use a particular vector space- the Hilbert space of (normed) modulus squared integrable functions [1]. It is important to realise that operators with dense spectra do not have normed eigenfunctions. So, their eigenfunctions are not in Hilbert space; for practical purposes we can only discuss states for which the variance of an operator is small but not zero.

1.3 Notations, Definitions, Lemmas

1.3.1 Notations

Schrodinger operator A has eigenvalue a and eigenvector $|a\rangle$; Heisenberg operator $A(t)$ has the same eigenvalues and eigenvectors as A; operator function $F(A)$ of A has eigenvalue $F(a)$ and eigenvector $|a\rangle$ [2]. State vector is denoted by $|\rangle$ or $|\phi\rangle$ or $|\phi,t\rangle$ (to indicate dependence on time). In Hilbert space the state function is denoted by $\phi(q_1,q_2,\ldots)\equiv\phi(\underline{q})$ where the scalars $q_1,q_2,\ldots$ are defined in $(-\infty,\infty)$; underline is used to indicate a vector, array or set; $\phi(\underline{q},t)$ indicates dependence on time t. The inner product of operator A with vector $|\phi\rangle$ is denoted, variously, by $(\underline{\phi},A\underline{\phi})\equiv\langle\phi|A|\phi\rangle\equiv\langle A\rangle$; see [1],[3].

1.3.2 Definitions and Lemmas

We assume that the state $|\phi\rangle$ depends on a steadilly increasing scalar time t; this is our only appeal to the CP [1],[3]. It can be shown that this implies that there is always an Hermitian operator H such that

$|\phi,t\rangle=U(t-t_0)|\phi,t_0\rangle$; Stone's Theorem [2] ....(1)

where H commutes with $(t-t_0)$ and

$U(t)=\exp(-itH/h)$; h=(Planck's constant)/(2π) ....(2)

is unitary. The factor 1/h is inserted by convention. We shall call H the Evolution Operator(EO). Result (1) is taken to be valid only for isolated systems (which are not interacting).

We use the Schrodinger representation, in which the operators are constant and the state vector evolves, unless otherwise stated; however, the Heisenberg representation, in which the state vector is constant while the operators evolve, is sometimes useful. The connection between the two representations is defined to be [1]

$A(t)=U^*(t)AU(t)$; * means Hermitian transpose; see (2) ....(3)

The rate operators V_j (j=1,2,...) of A are defined by

$\langle V_j\rangle=d^j/dt^j\langle\phi,t|A|\phi,t\rangle|_{t=t_0}$ ....(4)

from which [1]

$V_1=i(HA-AH)/h$; $V_{j+1}=i(HV_j-V_jH)/h$ ....(5)

V_1 and V_2 being called the 'velocity' and the 'acceleration' of A, respectively.

We require the so called precision operators defined by

$K_j=i(V_jA-AV_j)$ ....(6)

These satisfy the recurrence relation

$K_{j+1}=i(HK_j-K_jH)/h+i(V_1V_j-V_jV_1)$ ...(7)

and since

$A(t)=A+\Sigma(V_jt^j)/j!$; $j=1,2,...\infty$; see(2),(3) ....(8)

it follows that

$i(A(t)A-AA(t))=\Sigma(K_jt^j)/j!$; $j=1,2,...\infty$; see (6) ....(9)

If two Hermitian operators A and B do not commute then they cannot share a complete set of eigenvalues [1]; and, where they do not share eigenvectors, they cannot share eigenstates and so exact simultaneous measurement of A and B is impossible (the probability distribution of an observable is degenerate only if the prior state is an eigenstate). This is illustrated by the inequality [1],[2]

$var(A)var(B)\geq\langle i(AB-BA)\rangle^2/4$; $var(A)=\langle A^2\rangle-\langle A\rangle^2$ ....(10)

known as the Generalised Uncertainty Principle, which can be shown to hold for any two Hermitian operators A and B and any normed vector $|\rangle$. Note that since

$$\mathrm{var}(A)=\langle A^2\rangle-\langle A\rangle^2=\langle (A-\langle A\rangle I)\,|\,(A-\langle A\rangle I)\rangle \qquad \ldots.(11)$$

where I is the unit operator

$$\mathrm{var}(A)=0 \text{ iff } |\rangle \text{ is an eigenvector of } A \qquad \ldots.(12)$$

Note also that $\langle A\rangle$ is stationary, with respect to infinitessimal arbitrary variation of normed $|\rangle$, only if $|\rangle$ is an eigenvector of A; proof by the calculus of variations [1],[5]. It follows that the bounds of the spectrum of A are also the bounds of $\langle A\rangle$ [2].

The Hermitian operator F(A) is said to be a pure function of the Hermitian operator A if it has the same eigenvector set as A [2]; an eigenvalue F(a) of F(A) is then expressible as a scaler function of the corresponding eigenvalue a of A.

This notation generalises. Hermitian operators $A_1,A_2,\ldots$ have a common complete set of eigenvectors iff they commute [1]. A pure function $F(\mathbf{A})$, of the set of commuting Hermitian operators $\mathbf{A}\equiv(A_1,A_2,\ldots)$, may be defined as any Hermitian operator with the same eigenvector set as $\mathbf{A}$ [2]. An eigenvalue of $F(\mathbf{A})$ is then expressible as a scalar function $F(\mathbf{a})$ of the corresponding eigenvalues $\mathbf{a}\equiv(a_1,a_2,\ldots)$ of $\mathbf{A}$. All elements of $\mathbf{A}$ commute with $F(\mathbf{A})$.

We refer, loosely, to $F(\mathbf{a})$ as the spectrum of $F(\mathbf{A})$ when, strictly, the spectrum of $F(\mathbf{A})$ is the image set of $F(\mathbf{a})$. Properties of $F(\mathbf{a})$, such as differentiability, are likewise attributed to the spectrum.

A mixed function of non-commuting Hermitian operators $B_1,B_2,\ldots$, can be defined as a sum of products of pure functions $F_1(B_1),F_2(B_2),\ldots$ of $B_1,B_2,\ldots$, respectively. The extension, to mixed functions of sets of operators $\mathbf{B}_1,\mathbf{B}_2,\ldots$ where the elements of a given set mutually commute but do not commute with those of other sets, is obvious.

1.4 Basic Conjugate Operators

Denote the general state by the normed function $\phi(\mathbf{q},t)$ in a Hilbert space $(L^2(-\infty,\infty))$ of such functions; denote this space by S_q.

1.4.1 Theorem(1)

All bounded Hermitian operators on S_q can be expressed as functions (pure or mixed) of two sets of Hermitian operators

$$\mathbf{P}\equiv(P_1,P_2,\ldots);\ \mathbf{Q}\equiv(Q_1,Q_2\ldots);\ \text{the Basic Conjugate Operators} \qquad \ldots.(13)$$

where:

$$Q_j|\phi\rangle=q_j|\phi\rangle;\ \text{for all } |\phi\rangle \text{ in } S_q;\ j=1,2,\ldots;\ -\infty<q_j<\infty;\ q_j\epsilon\mathbf{q} \qquad \ldots.(14)$$

and

$$Q_jQ_k=Q_kQ_j;\ \ P_jP_k=P_kP_j;\ \ Q_jP_k-P_kQ_j=i\hbar I\delta_{jk} \qquad \ldots.(15)$$

See chapter 4 of [2] for discussion of a proof.

1.4.2 Properties of $\underline{P}$ and $\underline{Q}$

It can be shown that the elements of $\underline{Q}$ and $\underline{P}$ have dense spectra in $(-\infty,\infty)$ [1]. There is an Hilbert space S_p, the dual of S_q, of functions $\psi(\underline{p},t)$ where

$$P_j|\psi\rangle = p_j|\psi\rangle; \text{ for all } |\psi\rangle \text{ in } S_p;\ j=1,2,\ldots;\ -\infty<p_j<\infty \qquad \ldots.(16)$$

ψ being the multivariate Fourier transform of ϕ [1]. Parseval's theorem ensures that if ϕ is normed so is ψ [5]. From the point of view of QT ϕ and ψ are just different representations of the same state vector. In the so called P-diagonal representation

$$P_j \equiv p_j I;\ Q_j \equiv ih\delta/\delta p_j;\ |\rangle \equiv \psi(\underline{p});\ \delta/\delta x \text{ denotes partial derivative} \quad ; \text{ with respect to } x \qquad \ldots.(17)$$

In the Q-diagonal representation

$$Q_j \equiv q_j I;\ P_j \equiv -ih\delta/\delta q_j;\ |\rangle \equiv \phi(\underline{q}) \qquad \ldots.(18)$$

Notice that (15) does not uniquely define $\underline{P}$ since the transformations

$$'P_j = P_j + F_j(\underline{Q});\ 'Q_j = Q_j \qquad \ldots.(19)$$

where the F_j are pure Hermitian functions, leave (15) unchanged.

1.4.3 Theorem(2)

The transformations

$$'\underline{P} = M\underline{P};\ '\underline{Q} = N\underline{Q};\ M,N \text{ are real constant square matrices} \qquad \ldots.(20)$$

where $'\underline{P},\underline{P},'\underline{Q},\underline{Q}$ are treated as column arrays of operators, leave the form (15) invariant if

$$N=(M^T)^{-1};\ T \text{ denotes transpose} \qquad \ldots.(21)$$

Proof- by matrix algebra.

1.4.4 Theorem(3)

$$\exp(iaP/h)F(Q)=F(Q+aI);\ \exp(ibQ/h)G(P)=G(P-bI);\ P\epsilon\underline{P};\ Q\epsilon\underline{Q} \qquad \ldots.(22)$$

for any scalars a,b and pure functions F,G with differentiable spectra. Proof- by Taylor's theorem and the choice of representations (17) and (18) which reduce G and F, respectively, to scalar functions; see Stone's theorem.

1.4.5 Theorem(4)

The operators

$$F_{,P}=i(FQ-QF)/h;\ F_{,Q}=-i(FP-PF)/h;\ P\epsilon\underline{P};\ Q\epsilon\underline{Q} \qquad \ldots.(23)$$

can be derived from the operator F by a process analogous to symbolic partial diferentiation.

Lemma(1)

If $F(\underline{P})$ and $G(\underline{Q})$ are pure functions, with spectra that are differentiable, then $F_{,P}$ and $G_{,Q}$ are also pure functions of $\underline{P}$ and $\underline{Q}$, respectively, with spectra

$$\delta/\delta p F(\underline{p}) \text{ and } \delta/\delta q G(\underline{q});\ p \epsilon \underline{p};\ q \epsilon \underline{q} \qquad \text{....(24)}$$

respectively. Proof- use (17) and (18) and consider the action of each side of the relations (23) on arbitrary functions $\psi(\underline{p})$ (in S_P) and $\phi(\underline{q})$ (in S_Q), respectively.

Lemma(2)

For any two (pure or mixed) operators U and V:

$$(UV)_{,P}=U_{,P}V+UV_{,P};\ (UV)_{,Q}=U_{,Q}V+UV_{,Q};\ P\epsilon\underline{P};\ Q\epsilon\underline{Q}$$

$$(U+V)_{,P}=U_{,P}+V_{,P};\ (U+V)_{,Q}=U_{,Q}+V_{,Q}; \qquad \text{....(25)}$$

are identities. Proof- directly from definitions.

Lemma(3)

For any operator U:

$$U_{,'P,P}=U_{,P,'P};\quad U_{,'Q,Q}=U_{,Q,'Q};\ 'P\&P\epsilon\underline{P};\ 'Q\&Q\epsilon\underline{Q} \qquad \text{....(26)}$$

are identities. Proof- directly from definitions.

1.4.6 Theorem(5)

Products of pure functions of $\underline{P}$ and $\underline{Q}$, respectively, can be expressed as sums of ordered pairs of pure functions. Thus

$$A(\underline{P})B(\underline{Q})C(\underline{P})...=\Sigma F_j(\underline{P})G_j(\underline{Q});\ j=1,2,... \qquad \text{....(27)}$$

for some pairs of pure functions F_j,G_j given (a large class of) pure functions A,B,C.... Proof- the theorem follows directly from (15) when A,B,C,... are polynomials of the elements of $\underline{P}$ or $\underline{Q}$; however, it also follows from (22) when the spectra of A,B,C,... can be expressed as Fourier expansions; (questions about the convergence of products of series are important).

1.4.7 Theorem(6)

The evolution operator satisfies the QT analogues of Hamilton's equations as identities:

$$i(HQ-QH)/h=H_{,P};\ P\epsilon\underline{P};\ Q=\underline{Q} \qquad \text{....(28)}$$

$$i(HP-PH)/h=-H_{,Q} \qquad \text{....(29)}$$

The LHSs of (28) and (29) are the velocity operators of Q and P, respectively; and, as theorem(4) shows, the RHSs of (28) and (29) derive from H by an analogue of symbolic partial differentiation. The physical analogue is complete, however, only if we can interpret H as the Hamiltonian of a system with (operators of) conjugate coordinates and momenta $\underline{P}$, $\underline{Q}$. Theorem(6) then gives a justification for the use of the same scaling factor h in both (2) and (15).

2. Description by Bounded and Unbounded Operators

2.1 An Hypothesis

A satisfactory theory of quantum physics can be expressed, solely, in terms of bounded Hermitian operators (operators with bounded real spectra [2]).

Theorem(1) shows that if, the hypothesis is correct, a satisfactory theory is also expressible, solely, in terms of the sets of unbounded operators $\underline{P}$ and $\underline{Q}$; of course, even if the hypothesis is false, this may still be true. However, if observables are always bounded, we must be careful when ascribing physical significance to elements of $\underline{P}$ and $\underline{Q}$. The notion of 'local approximation' is helpful.

2.2 Local Approximation

Under suitable conditions of differentiability the spectra of bounded pure functions $A(\underline{P})$ and $B(\underline{Q})$ are locally approximated by linear functions of the elements of $\underline{p}$ and of $\underline{q}$, respectively. These linear functions are the spectra of certain pure linear operator functions $L_A(\underline{P})$ and $L_B(\underline{Q})$, respectively. We shall say that L_A and L_B locally approximate A and B, respectively, in suitable subspaces. The subspaces are spanned by the eigenvectors common to $\underline{P}$ and common to $\underline{Q}$ which correspond to eigenvalues $\underline{p}$ and $\underline{q}$ in the neighbourhood of spectral approximation.

3. Classical Variables and Classical Systems

3.1 Axioms

An operator A is defined to be 'classical' or 'classical with respect to H' if:

Axiom(1)- *the system characterised by A and H is isolated.*

Axiom(2)- *A and H are bounded; see Section 2.*

Axiom(3)- *the state at any time $t>t_0$ is an eigenstate of A, given that it was an eigenstate at time $t=t_0$.*

We shall call a system 'classical' if it is characterised by at least one classical variable.

3.2 Alternative Forms of Axiom(3)

Axiom(3) can be expressed in various mathematical forms (labelled by †). For example, where $|a(t)\rangle$ is any eigenvector of A with eigenvalue $a(t)$,

$$A|a(t)\rangle=a(t)|a(t)\rangle \quad † \qquad ....(30)$$

where

$$|a(t)\rangle=U(t-t_o)|a(t_o)\rangle;\ \text{see (2)};\ t\geq t_o \quad † \qquad ....(31)$$

Alternatively

$$A(t-t_o)|a(t_o)\rangle=a(t)|a(t_o)\rangle;\ t\geq t_o;\ \text{see (3)} \quad † \qquad ....(32)$$

It follows from (30) and (32) that

$$(A(t-t_o)A-AA(t-t_o))|a(t_o)\rangle=|0\rangle;\ \text{null vector} \qquad ....(33)$$

If, the eigenvectors of A form a complete set, this implies [1]

$$(A(t-t_o)A-AA(t-t_o))|\phi\rangle=|0\rangle;\ \text{any } |\phi\rangle \qquad ....(34)$$

which gives another alternative form [1]

$$A(t-t_o)A=AA(t-t_o);\ t>t_o \quad † \qquad ....(35)$$

Since, t is arbitrary, we deduce from (35) that

$$K_j=0;\ \text{null operator};\ j=1,2,...;\ \text{see (6), (7) and (9)} \quad † \qquad ....(36)$$

According to (7) these conditions imply that

$$V_1V_j=V_jV_1;\ j=2,3,... \qquad ....(37)$$

which in turn imply that V_1 is a classical variable. Similarly, V_2 and all the other rate operators are classical.

(36) is the form of axiom(3) that we shall use. It appears to be necessary; is it sufficient? Given (9), (36) implies (35) and, providing that the eigenvalues of A are distinct, (35) implies (32) and hence (30) [1],[2].

Summarising: within the limitations imposed on A (complete set of eigenvectors- distinct eigenvalues) (30),(32),(35) and (36) are equivalent.

3.3 Boundedness,Continuity,Differentiability

Axiom(2) is important. If A is bounded so is A^2; and, in consequence, so is the scalar var(A); see (10). It follows, from (10), that if A and H are bounded so is V_1; similarly, all the operators V_j and K_j (j=2,3,...) are bounded as is $A(t)A-AA(t)$. It is then a simple matter to prove that the RHSs of (8) and (9) are bounded.

Now the action of a bounded operator on a limiting sequence of vectors is always continous [2]. So, (8) and hence (32), are differentiable. It follows that the scalar function $a(t)$ is differentiable ($|a(t_o)\rangle$ is constant). Thus, if A is classical, its spectrum is dense.

3.4 The Meaning of the Axioms

The condition (30) ensures that, given $t>t_o$, whenever we measure A we shall find the system in an eigenstate. The result $a(t)$ will, therefore, be exact and certain and, as is shown above, will evolve as a differentiable function of t with bounded variation. We can hardly ask more of a classical variable.

The 'classical virtues' of a bounded, smooth, exact and certain dynamic are important because they are exhibited by many large scale natural phenomena. But, they are also important for a more abstract reason: the observed behaviour of a system must exhibit at least some of the classical virtues if we are to perceive 'natural law'.

3.5 The Axioms are Inconsistent

Differentiate (32) with respect to t and set $t=t_o$

$$V_1|a(t_o)\rangle='a(t_o)|a(t_o)\rangle;\quad 'a=d/dt(a(t));\ \text{see (8)} \qquad \ldots.(38)$$

which shows that the eigenvector $|a(t_o)\rangle$ of A, with eigenvalue $a(t_o)$, is also an eigenvector of V_1 with eigenvalue $'a(t_o)$. Therefore, according to (12), the condition

$$var(A)\rightarrow 0$$

implies

$$|\phi\rangle\rightarrow|a\rangle;\ \text{an eigenvector of A}$$

implies

$$\langle V_1\rangle\rightarrow v_1;\ \text{an eigenvalue of } V_1$$

It follows from (10) that

$$\underset{var(A)\rightarrow 0}{Lt}var(H) \geq h^2\underset{var(A)\rightarrow 0}{Lt}(\langle V_1\rangle^2/var(A))/4 \rightarrow h^2\underset{var(A)\rightarrow 0}{Lt}(v_1^2/var(A))/4 \rightarrow \infty;\ v_1\neq 0 \qquad \ldots.(39)$$

(39) implies that, unless $v_1=0$ in this particular limiting eigenstate, var(H) is infinite; this is possible only if H^2 and (hence) H are unbounded. So, H is unbounded unless all the eigenvalues of V_1 are zero; that is, unless

$$V_1=0;\ \text{null operator} \qquad \ldots.(40)$$

(40) implies that A is a constant of the motion [1].

It follows that axioms (2) and (3) are inconsistent unless A is a constant of the motion. Thus, if H is bounded in nature (see Section(2)), there are no truly classical variables, as here defined, except constants of the motion.

3.6 An Interpretation of the Inconsistency

Suppose, for simplicity, that H has a point spectrum and a complete set of eigenvectors [2]. Write the eigenvalue equation of H as

$$H|\lambda_j\rangle=\lambda_j|\lambda_j\rangle;\ \lambda_j \text{ a real scalar; } j=1,2,\ldots \qquad \ldots.(41)$$

and express the general state vector in terms of the eigenvectors

$$|\phi\rangle=\Sigma a_j|\lambda_j\rangle;\ a_j \text{ a complex scalar; } j=1,2,\ldots \qquad \ldots.(42)$$

Then it is easy to show that

$$\mathrm{var}(H)=\tfrac{1}{2}\sum_{j\neq k} c_j c_k(\lambda_j-\lambda_k)^2;\ c_j=|a_j|^2 \qquad \ldots.(43)$$

where

$$\Sigma c_j=1;\ \langle\lambda_j|\lambda_k\rangle=\delta_{jk};\ j,k=1,2,\ldots \qquad \ldots.(44)$$

Therefore

$$\mathrm{var}(H)<\tfrac{1}{2}\Delta\lambda^2\sum_{j\neq k} c_j c_k<\tfrac{1}{2}\Delta\lambda^2\sum_{j,k} c_j c_k=\tfrac{1}{2}\Delta\lambda^2 \qquad \ldots.(45)$$

where

$$\Delta\lambda=\sup_{j,k}|\lambda_j-\lambda_k| \qquad \ldots.(46)$$

(45) implies that var(H) can be large or infinite only when the spread $\Delta\lambda$ of the spectrum of H is large or infinite.

Consider the Fourier transform (with respect to t) of the general state vector $|\phi,t\rangle$, given by (1) and (2), when H has a point spectrum:

$$|f(\omega)\rangle=\Sigma a_j\delta(\omega-\lambda_j/h)|\rangle;\ \text{Dirac delta; } j=1,2,\ldots \qquad \ldots.(47)$$

The inner product

$$\langle f(\omega)|f(\omega)\rangle \qquad \ldots.(48)$$

can be regarded as a power spectral density [6] of the signal $|\phi,t\rangle$; it has lines at the angular frequencies $\omega=\lambda_j/h$ with relative amplitudes c_j. It follows from (45) that, if var(H) is unbounded, then the range of frequencies spanned by the power spectral density (48) is also unbounded; in the parlance of information theory [7],[8] the bandwidth of the signal $|\phi,t\rangle$ is infinite.

Now it is a fundamental result of information theory that the information carrying capacity of a signal is proportional to its bandwidth; and, the carrying capacity is infinite only if the bandwidth is infinite [7],[8]. This result suggests that, for a classical system, the information carrying capacity of $|\phi,t\rangle$ must be infinite. Why? As we have seen, for a variable A to be classical, we must be able to measure A and the V_j (j=1,2,...) simultaneously (because they commute- see (36)). It follows that it must be possible to determine a(t) exactly, in infinitessimal time, because the derivatives of a(t) (the eigenvalues of the V_j- see (38)) are determined by taking differences. But, this implies that information is transmitted, from system to apparatus, at an infinite rate. If, as we suppose, $|\phi,t\rangle$ carries this information (in a statistical sense) then the bandwidth must be infinite.

4. Quasi-Classical Systems

4.1 Definition

A system is quasi-classical if it is characterised by at least one variable A, which has a dense bounded spectrum and for which

$$K_1=0 \text{ but } K_j\neq 0 \text{ for some } j>2;\ \text{see}(5),(6),(36) \qquad \ldots.(49)$$

4.2 Fundamental Property

Notice that the condition

$$K_1=0 \qquad \ldots.(50)$$

ensures that

$$K_2=0;\ \text{see}(7) \qquad \ldots.(51)$$

Now suppose that

$$t\to t_0 \qquad \ldots.(52)$$

then, according to (9), (50) ensures that condition (35) is approximated. This is true, whatever the nature of $K_3,K_4,\ldots$, providing that these operators are bounded; ($K_3,K_4,\ldots$ are bounded, for example, if both A and H are bounded). So, for a short time after a measurement of A, a quasi-classical system behaves like a clasical system (providing that $K_3,K_4,\ldots$ are bounded).

4.3 Canonical Forms

4.3.1 For Classical Systems

We wish to find solutions, if any, of equation (50). For this purpose we define the following sets of operators:

$$\underline{P}\equiv(P_1,P_2,\ldots);\ \underline{Q}\equiv(Q_1,Q_2,\ldots);\ \underline{C}\equiv\underline{P}\ \cup\ \underline{Q};\ \cup \text{ denotes union of sets} \qquad \ldots.(53)$$

$$\underline{P}_n\equiv(P_1,P_2,\ldots P_n);\ \underline{Q}_n\equiv(Q_1,Q_2,\ldots Q_n);\ \underline{C}_n\equiv\underline{P}_n\ \cup\ \underline{Q}_n \qquad \ldots.(54)$$

with complements defined with respect to the sets (54)

$$^-\underline{P}_n\equiv\underline{P}-\underline{P}_n;\quad \underline{Q}_n\equiv\underline{Q}-\underline{Q}_n;\ ^-\underline{C}_n\equiv\underline{C}-\underline{C}_n \qquad \ldots.(55)$$

First, suppose that (50) is satisfied for every

$A=Q;\ Q\in\mathbf{Q}$ ....(56)

then

$K_1=hH,_{F},_{P}=0;\ P=P_j;\ j=1,2,...$ ....(57)

the solution of (57) being

$H=\Sigma\alpha_j P_j+\beta$ ....(58)

where

$\alpha_j=\alpha_j(\mathbf{Q});\ \beta=\beta(\mathbf{Q});\ j=1,2,...$ ....(59)

are pure functions.

That (58) is a solution of (57), given (56), may be verified directly; see (5),(6),(15) or theorem(4). That the form (58) is unique can be proved at least in the case where the α_j have inverses and are linearly independent; (this is sufficient for our purpose). Suppose that H takes the ordered mixed form

$H=\Sigma\alpha_j F_j+\beta;\ j=1,2,...;$ see theorem(5) ....(60)

where the $\alpha_j(\mathbf{Q}),\beta(\mathbf{Q})$ and $F_j(\mathbf{P})$ are pure. Then, according to (57),

$F_{1},_{F},_{F}+(\alpha_1)^{-1}\sum_{j\neq 1}\alpha_j F_j,_{F},_{F}=0;\ P=P_k;$ for all 1,k ....(61)

Consider the action of the LHS of (61) on functions $\phi(\mathbf{p})$ in S_n. In the $\mathbf{P}$-diagonal representation (see section 1.4) the $\mathbf{P}$ are scalars $\mathbf{p}$, the α_1 are differential operators of some sort (or constants) and the F_j are scalars. If, we hold the $\mathbf{p}$ constant and, we select functions $\phi(\mathbf{p})$ which have the same value but varying derivatives at $\mathbf{p}$ then, the first term on the LHS of (61) generates a constant while the second term generates an arbitrary variable. This is true for all $\mathbf{p}$. If, the α_1 are linearly independent, this is possible only when

$F_j,_{F},_{P}=0;\ P=P_k;$ for all j,k ....(62)

But, in the $\mathbf{P}$-diagonal representation, (62) is the set of partial differential equations

$\delta^2/\delta p^2 F_j(\mathbf{p})=0;\ p=p_k;$ for all j,k QED ....(63)

Second, suppose that, instead of (56), we choose

$A=A(\mathbf{Q})$ ....(64)

as any pure bounded function with a dense spectrum. Then, given (58) and (59), theorem(4) shows that

$K_j=0;\ j=1,2,...;$ see (36) ....(65)

So, (58) and (59) express a canonical form for a *classical* system, rather than a *quasi-classical* system, even though the form is derived from (50).

It may be objected that the choice (64) is not sufficiently general because, ostensibly, we might choose

$$A=A(\underline{P}) \qquad \ldots.(66)$$

or

$$A=A(\underline{C}) \qquad \ldots.(67)$$

The choice (66) merely leads to a conjugate form in which $\underline{P}$ replaces $\underline{Q}$. The choice (67) is inappropriate because mixed functions have discrete spectra.

4.3.2 For Quasi-Classical Systems

From (50), the condition which defines a *quasi-classical* system, we have derived a form for the EO which has all the (consistent) characteristics of a *classical* system (H is unbounded); yet the conditions (36) include condition (50) and, taken together, should be more stringent. Is it possible , therefore, to generalise (58) by taking advantage of the lesser stringency of (50)?

The key to this question appears to lie in restrictions placed on (56) and (64). Suppose that (50) is satisfied for every

$$A=Q;\; Q\in\underline{Q}_n \qquad \ldots.(68)$$

then the same argument leads to

$$H=\Sigma\alpha_j P_j+\beta;\; j=1,2,\ldots n \qquad \ldots.(69)$$

where

$$\alpha_j=\alpha_j(\underline{Q}_n\; U\; {}^{-}\underline{C}_n);\; \beta=\beta(\underline{Q}_n\; U\; {}^{-}\underline{C}_n); \qquad \ldots.(70)$$

Now choose

$$A=A(\underline{Q}_n) \qquad \ldots.(71)$$

as any pure bounded function with a dense spectrum. As before (see (58),(59) and (64)), the α_j and β commute with A and, neither the α_j and β nor A commute with the elements of $\underline{P}_n$. However, the α_j and β do not now mutually commute on account of their dependence on ${}^{-}\underline{C}_n$. The result is that, in general, the conditions (49) are satisfied. (69) and (70), therefore, express the canonical form of the EO for a quasi-classical system.

The Hermitian character of the α_j and β can be deduced as follows:

$$H=H^*;\; P=P^*;\; Q=Q^*;\; * \text{ denotes Hermitian transpose} \qquad \ldots.(72)$$

$$i(HQ_k-Q_kH)/h=-i(Q_kH-HQ_k)/h=(i(HQ_k-Q_kH)/h)^*$$

$$\alpha_k=\alpha_k^* \qquad \ldots.(73)$$

Further, from (72), in the notation of (23)

$$\Sigma\alpha P+\beta=\Sigma P\alpha^{*}+\beta^{*}=\Sigma(\alpha^{*}P-i\hbar\alpha^{*}{}_{,Q})+\beta^{*};\ \alpha=\alpha_j;\ P=P_j;\ Q=Q_j;\ j=1,2,\ldots n \qquad \ldots.(74)$$

Given (73)

$$\beta^{*}=\beta+i\hbar\Sigma\alpha_{,Q};\ \alpha=\alpha_j;\ Q=Q_j;\ j=1,2,\ldots n \qquad \ldots.(75)$$

It follows that β is Hermitian only if

$$\Sigma\alpha_{,Q}=0;\ \alpha=\alpha_j;\ Q=Q_j;\ j=1,2,\ldots n \qquad \ldots.(76)$$

4.3.3 When the EO has a Dense Spectrum

The dependence of the α_j and β on $\underline{Q}_n$ means that (69) is a mixed function of the elements of $\underline{C}_n$. The spectrum of H can therefore be either discrete or dense or both in different parts. To avoid these complications we restrict the rest of the discussion to a most important special case; namely, we assume that the spectrum of H is dense.

The only way to ensure that the spectrum of H is dense is to deprive the α_j and β of their dependence on $\underline{Q}_n$. Specifically, we assume that:

$$\alpha_j=\alpha_j(\neg\underline{C}_n);\ \beta=\beta(\neg\underline{C}_n);\ \alpha_j \text{ and } \beta \text{ bounded};\ j=1,2,\ldots n \qquad \ldots.(77)$$

In consequence:

i) The α_j and β commute with every element of $\underline{C}_n$, but not with each other, and (69) is still the EO of a quasi-classical system.
ii) The spectra of the α_j and β are discrete.
iii) The α_j and β can be represented by matrices with constant elements (because, by (77), they are bounded [2]) in both the $\underline{Q}_n$-diagonal and $\underline{P}_n$-diagonal cases (because they commute with $\underline{C}_n$).
iv) The elements of $\underline{P}_n$ commute with H (see (i)) and are constants of the motion with the same eigenvectors as H.
v) Since the α_j commute with $\underline{P}_n$

$$\alpha_{j,Q}=0;\ j=1,2,\ldots;\ Q\in\underline{Q}_n \qquad \ldots.(78)$$

so, β is Hermitian (see (75)).
vi) In the $\underline{P}_n$-diagonal representation

$$H=\Sigma p_j\alpha_j+\beta;\ p_j\alpha_j=\alpha_j p_j;\ p_j \text{ scalar};\ j=1,2,\ldots n \qquad \ldots.(79)$$

can be represented as a matrix (see (iii)) with characteristic equation

$$|H-\lambda I|=0 \qquad \ldots.(80)$$

whose coefficients (of powers of λ) are polynomials in the elements of $\underline{p}$.
vii) It follows from (vi) that the spectrum of H is dense (as expected) and, at least over restricted domains, differentiable with respect to the elements of $\underline{p}$.

The assumption that the α_j and β are bounded is a tacit assumption that they are observables; see Section 2.1. This is reasonable. If H is observable then so is

$$\beta = H|_{p=\omega};\ p \in \mathbf{p};\ \text{see (79)} \qquad \ldots.(81)$$

The α_j are the velocities of the Q_j and the Q_j are observable if they can be regarded as local approximations to bounded observables like $A(Q_n)$ (see Section 2.2).

4.3.4 The Dense Spectrum Near a Turning Point

We now use only the E_n-diagonal representation. Suppose that

$$\lambda = \lambda(\mathbf{p}) \qquad \ldots.(82)$$

is the spectrum of H. Expand, λ by Taylor' theorem, about a point

$$\mathbf{p} = \mathbf{p}_o \qquad \ldots.(83)$$

in a region in which λ is differentiable (see items (vi) and (vii) of Section 4.3.3)

$$\lambda = \lambda_o + \sum_j c_j(p_j - p_{oj}) + \sum_{j,k} c_{jk}(p_j - p_{oj})(p_k - p_{ok}) + \ldots;\ j,k = 1,2,\ldots n \qquad \ldots.(84)$$

where, without loss of generality, the $n \times n$ matrix (c_{jk}) can be taken to be symmetric. We now make three assumptions:

a) $\mathbf{p}_o$ is a turning point of λ so that

$$c_j = 0;\ j = 1,2,\ldots n \qquad \ldots.(85)$$

b) the expansion begins with second (as opposed to third or fourth etc.) order terms. That is

$$c_{jk} \neq 0 \text{ for some } j,k \qquad \ldots.(86)$$

c) the expansion is confined to a small neighbourhood

$$p_j \to p_{oj};\ j = 1,2,\ldots n \qquad \ldots.(87)$$

With these assumptions (discussed below)

$$\lambda \simeq \lambda_o + \sum c_{jk}(p_j - p_{oj})(p_k - p_{ok});\ p_j \to p_{oj};\ j,k = 1,2,\ldots n \qquad \ldots.(88)$$

(88) may be further simplified by permitted constant linear transformations. Firstly,

$$p_j \text{ replaces } p_j - p_{oj} \to 0\ ;\ \text{see (19)} \qquad \ldots.(89)$$

Secondly, an orthogonal transformation which diagonalises the matrix (c_{jk}). Thus

$$\lambda=\lambda_0+\Sigma\mu_j p_j{}^2;\quad p_j\to 0;\quad j=1,2,\ldots n \qquad \ldots.(90)$$

where the μ_j are eigenvalues of the matrix (c_{jk}); see (20) and (21) in the case where

$$\mathbf{M}=\mathbf{N} \qquad \ldots.(91)$$

Notice that it is not assumed that the eigenvalues are all of the same sign, otherwise, a transformation of the type (20) and (21) could reduce the matrix to a scalar.

(90) is taken to be a canonical form for a dense spectrum near a turning point; in particular, it applies to quasi-classical systems.

Consider the assumptions (85),(86) and (87). We suppose that, in nature, H is bounded; see Section 2.1. Therefore, when the spectrum is dense (and differentiable), λ is a bounded function with at least two points of zero gradient; we assume that these occur for finite values of $\mathbf{p}$. We have chosen to expand λ about one of them. Since (79) is unbounded it is, at best, a local approximation to nature; hence the restriction (87). The assumption (86) is discussed in Section 5.1.3.

5. The Dirac Form

5.1 Matching Canonical Forms

5.1.1 The Conditions for Second Order Matrices

Suppose that we try to match the forms (79) and (90). That is, the matrix H, given by (79), is to have an eigenvalue, given by (90), for arbitrary $\mathbf{p}$. If this condition is possible it must impose constraints on the α_j and β. First, we assume that the matrices are of order

$$r=2 \qquad \ldots.(92)$$

Substitute (79) and (90) into (80), expand the determinant, neglect terms in products of the elements of $\mathbf{p}$ of order greater than two and then equate the coefficients to zero:

$$(\beta_{11}-\lambda_0)(\beta_{22}-\lambda_0)-\beta_{12}\beta_{21}=0;\ \text{constant term} \qquad \ldots.(93)$$

$$(\beta_{11}-\lambda_0)\alpha_{j22}+(\beta_{22}-\lambda_0)\alpha_{j11}-\alpha_{j21}\beta_{12}-\alpha_{j12}\beta_{21}=0;\ p_j\ \text{term} \qquad \ldots.(94)$$

$$\mu_j(2\lambda_0-\beta_{11}-\beta_{22})+\alpha_{j11}\alpha_{j22}-\alpha_{j21}\alpha_{j12}=0;\ p_j{}^2\ \text{term} \qquad \ldots.(95)$$

$$\alpha_{j11}\alpha_{k22}-\alpha_{j21}\alpha_{k12}+\alpha_{j22}\alpha_{k11}-\alpha_{j12}\alpha_{k21}=0;\ p_jp_k\ \text{term};\ j\neq k \qquad \ldots.(96)$$

For simplicity, we can adopt the β-diagonal representation

$$\beta_{12}=\beta_{21}=0 \qquad \ldots.(97)$$

Also, we may add an arbitrary scalar to H without changing either, the eigenvectors of H or, the state represented by the LHS of (1); (normed eigenvectors are undetermined to within a complex number of modulus unity). So, we can always ensure that

$$\beta_{11}+\beta_{22}=0 \qquad \ldots(98)$$

From (93) to (98) we deduce that:

$$\beta_{11}^2=\beta_{22}^2=\lambda_o^2;\ \text{see (93), (97) and (98)} \qquad \ldots(99)$$

$$\alpha_{j11}=\alpha_{j22}=0;\ \text{see (94), (97) and (99)};\ \lambda_o\neq 0 \qquad \ldots(100)$$

$$2\mu_j\lambda_o=\alpha_{j12}\alpha_{j21}=|\alpha_{j12}|^2;\ \text{see (95), (98) and (100)} \qquad \ldots(101)$$

$$\alpha_{j12}\alpha_{k21}+\alpha_{j21}\alpha_{k12}=0;\ \text{see (96) and (100)};\ j\neq k \qquad \ldots(102)$$

According to (99) λ_o is one of two eigenvalues of β that have equal magnitude and opposite sign. (101) shows that the μ_j all have the same sign as λ_o. It follows that a transformation is possible (see (20) and (21)) which reduces the matrix (c_{jk}) (see Section 4.3.4) to scalar form. In consequence, with a suitable choice of the basic conjugate operators,

$$\mu_j=1/(2m);\ m \text{ is a real scalar};\ j=1,2,\ldots n \qquad \ldots(103)$$

Further, inspection of (100) and (101) shows that the eigenvalues of α_j are

$$\pm|\alpha_{j12}|=\pm(2\mu_j\lambda_o)^{1/2}=\pm(\lambda_o/m)^{1/2};\ j=1,2,\ldots n;\ \text{see (103)} \qquad \ldots(104)$$

So, the eigenvalues of the α_j are *all of the same magnitude* and occur in equal and opposite pairs. Call this magnitude $c>0$ and

$$\lambda_o=mc^2 \qquad \ldots(105)$$

These results can be summarised in matrix form:

$$\beta^2=\lambda_o^2 I;\ \text{see (99) and (105)} \qquad \ldots(106)$$

$$\alpha_j^2=c^2 I;\ j=1,2,\ldots n;\ \text{see (100), (101) and (104)} \qquad \ldots(107)$$

$$\alpha_j\alpha_k+\alpha_k\alpha_j=0;\ j\neq k;\ \text{see (100) and (102)} \qquad \ldots(108)$$

$$\alpha_j\beta+\beta\alpha_j=0;\ \text{see (97), (98) and (100)} \qquad \ldots(109)$$

5.1.2 Conditions Independent of Matrix Order

Suppose that we square (79) at the same time making use of (106) to (109):

$$H^2=\sum_j (p_j^2\alpha_j^2+p_j(\alpha_j\beta+\beta\alpha_j))+\sum_{j\neq k} p_jp_k\alpha_j\alpha_k+\beta^2=(\sum_j c^2p_j^2+\lambda_o^2)I;\ j,k=1,2,\ldots n \qquad (110)$$

Since, the RHS of (110) is scalar, (110) must be the characteristic equation of H. The eigenvalues of H are therefore

$$\lambda=\pm(\Sigma c^2 p_j{}^2+\lambda_0{}^2)^{1/2}$$

$$\simeq \Sigma p_j{}^2/(2m)+mc^2;\ p_j\to 0;\ j=1,2,\ldots n \qquad \ldots(111)$$

where the indeterminate sign is absorbed in m. The first expression on the RHS of (111) gives the *exact* eigenvalues of H; the second (approximate) expression is of the same form as (90) (with the μ_j all identical). So, the conditions (106) to (109) do indeed ensure that (79) has eigenvalues of the form (90).

An interesting feature of this calculation is that it depends only on the matrix relations ((106) to (109)) and not on either, the order of the matrices or, the chosen basis. Yet, if we repeat the calculation of Section 5.1.1 for order r>2, the resulting four conditions (see (93) to (96)) no longer determine the matrix elements, in detail, as they do when r=2. The conditions are not expressed naturally as matrix equations but, rather, as relations between the invariants of the matrices. The form of these relations depends on the order; see [9] p. 87,88 for expansion of determinants by their diagonal elements. This situation is quite unsuitable for QT. In QT we require the rules to be expressed as relations between operators (in this case matrices) with as much freedom as possible in the choice of the representation.

It appears, therefore, that the conditions (106) to (109), expressed for arbitrary order, are necessary as well as sufficient to define a *canonical form for the EO of a quasi-classical system which has a dense spectrum.*

Another view that may be taken of this derivation, is that we require the operator

$$H'=(\Sigma\mu_j p_j{}^2+\lambda_0)I;\ j=1,2,\ldots n;\ \text{see (90)} \qquad \ldots(112)$$

to have the same eigenvectors and eigenvalues as (79) (given small values of the arguments $\underline{p}$) on condition that none of the matrices α, and β shall be scalar or zero; (since H' is scalar any non-null vector is an eigenvector). We need, therefore, to compare the matrices (79) and (112) in a way that invokes the approximation. Direct comparison of H' with H does not allow us to do this (and, otherwise, yields α_j zero and β scalar), but comparison of higher powers of the operators does. For example, neglect products of order greater than two and equate coefficients on each side of

$$H'^2=H^2 \qquad \ldots(113)$$

to obtain (106),(108) and (109) together with

$$\alpha_j{}^2=2\mu_j\lambda_0 I;\ j=1,2,\ldots n \qquad \ldots(114)$$

in place of (107). In fact (107) may be derived from (114). The eigenvalues of the square of an Hermitian matrix are non-negative and, therefore, the μ_j have the same sign as λ_0; the rest of the argument is like that which leads from (101) to (107).

Given (106) to (109), higher *even* powers of H' and H are both scalar (see (110)) and approximate each other (to second order). Higher *odd* powers, however, do not compare. We can cancel through by the next lower even power to be left with the direct comparison of H' and H.

5.1.3 The Starting Terms in the Series for the Spectrum

We need to justify the assumption that (90) begins with second (as opposed to third or fourth etc.) order terms; see Section 4.3.4 item (b).
Suppose that

$$H'=(\sum_{jkl}\mu_{jkl}p_jp_kp_l+\lambda_o)I;\ \mu_{jkl}\text{ a real scalar; } j,k,l=1,2,\ldots n \qquad \ldots(115)$$

because the Taylor's expansion of the spectrum begins with third order terms. Then, comparing H'^3 with H^3, we find (among other relations)

$$\beta^3=\lambda_o{}^3I \qquad \ldots(116)$$

which, because β is Hermitian, implies that β is scalar. This is contrary to hypothesis. Therfore, the Taylor's expansion of the spectrum of (79) cannot begin with third order terms.

The case in which the series begins with fourth order terms is more complicated. We have

$$H'=(\sum_{jklm}\mu_{jklm}p_jp_kp_lp_m+\lambda_o)I;\ \mu_{jklm}\text{ a real scalar; } j,k,l,m=1,2,\ldots n \qquad \ldots(117)$$

Comparing H'^4 with H^4, we find (among other relations)

$$\beta^4=\lambda_o{}^4I \qquad \ldots(118)$$

$$\beta^2(\alpha_j\beta+\beta\alpha_j)+(\alpha_j\beta+\beta\alpha_j)\beta^2=0;\ j=1,2,\ldots n \qquad \ldots(119)$$

$$\beta^2(\alpha_j\alpha_k+\alpha_k\alpha_j)+(\alpha_j\alpha_k+\alpha_k\alpha_j)\beta^2$$

$$+(\alpha_j\beta+\beta\alpha_j)(\alpha_k\beta+\beta\alpha_k)+(\alpha_k\beta+\beta\alpha_k)(\alpha_j\beta+\beta\alpha_j)=0;\ j,k=1,2,\ldots n \qquad \ldots(120)$$

Since, β is Hermitian, (118) implies (106). So, (119) implies (109) and (120) therefore implies (108); but, (120) also implies

$$\alpha_j{}^2=0 \qquad \ldots(121)$$

Taken together (106),(108),(109) and (121) give

$$H^2=\beta^2=\lambda_o{}^2I \qquad \ldots(122)$$

which means that the eigenvalues of H are $\pm|\lambda_o|$ and that the μ_{jklm} vanish.

Similar arguments apply, and similar inconsistencies are found, when the series begins with higher order terms. We conclude that, to be consistent with (79), the series for the spectrum *must* begin with second order terms (the linear terms having been eliminated).

5.2 The Matrices $\alpha_1, \alpha_2, \ldots \alpha_n$ and β

5.2.1 The Eigenvectors of H and the Minimum of r

When $r>2$ the spectrum of β is associated with an $(r-2)$-fold degeneracy because the matrix has only two distinct eigenvalues. This means that while the eigenvectors are linearly independent they are not, necessarilly, mutually orthogonal [1]. However, given r such eigenvectors, the Gram-Schmidt process may be used to generate a complete orthonormal eigenvector set for β. The same remarks apply to H and the α_j.

Suppose that we express an eigenvector of H in terms of the eigenbasis of β:

$$|\lambda\rangle=\Sigma a_j|\rangle_j; \; a_j \text{ scalar}; \; j=1,2,\ldots r \qquad \ldots(123)$$

where

$$H|\lambda\rangle=\lambda|\lambda\rangle; \; \lambda \text{ real scalar}; \; \langle\lambda|\lambda\rangle=1 \qquad \ldots(124)$$

and

$$\beta|\rangle_j=b_j|\rangle_j; \; b_j \text{ real scalar}; \; {}_j\langle|\rangle_k=\delta_{jk}; \text{ Kronecker} \qquad \ldots(125)$$

We note that the b_j have only two distinct values, $\pm|\lambda_o|$. Define

$$\alpha=\Sigma p_j\alpha_j; \; j=1,2,\ldots n \qquad \ldots(126)$$

Then

$$\alpha\beta-\beta\alpha=0; \text{ see } (109) \qquad \ldots(127)$$

and therefore

$$\beta(\alpha|\rangle_j)=-b_j(\alpha|\rangle_j); \text{ see } (125) \qquad \ldots(128)$$

from which we deduce that: the order r is even; $(\alpha|\rangle_j)$ is an eigenvector of β; the positive value of the eigenvalue may be assigned (say) to $b_1, b_2, \ldots b_{r/2}$ and the negative value to $b_{r/2+1}, b_{r/2+2}, \ldots b_r$. Thus

$$b_j=-b_{j+r/2}>0; \; j<r/2 \qquad \ldots(129)$$

and

$$\alpha|\rangle_j=e_j|\rangle_{j+r/2}; \; e_j \text{ scalar} \qquad \ldots(130)$$

because an eigenvector is not determined to within a scalar. Since

$$H=\alpha+\beta \qquad \ldots(131)$$

we may form the inner product of ${}_k\langle|$ with (124) to give

$$\lambda a_k=|\lambda_o|a_k+e_k a_{k+r/2}; \; k<r/2 \qquad \ldots(132)$$

$$\lambda a_k=e_{k-r/2}a_{k-r/2}-|\lambda_o|a_k; \; k>r/2 \qquad \ldots(133)$$

from which

$$a_{k+r/2}=e_k a_k/(\lambda+|\lambda_o|)=(\lambda-|\lambda_o|)a_k/e_k;\ k\leq r/2 \qquad \ldots(134)$$

and

$$e_k=\pm(\lambda^2-|\lambda_o|^2)^{1/2};\ \text{see (111)} \qquad \ldots(135)$$

It is clear, from (134), that r/2-1 of the a_k are at our disposal; this is a symptom of the degeneracy [9]. Therefore set

$$a_k=0;\ 1<k\leq r/2 \qquad \ldots(136)$$

giving

$$|\lambda\rangle=a_1|\rangle_1+a_{1+r/2}|\rangle_{1+r/2};\ |a_1|^2+|a_{1+r/2}|^2=1 \qquad \ldots(137)$$

Substitute (134) into the second of equations (137) and

$$a_1=e^{i\theta}((1+|\lambda_o|/\lambda)/2)^{1/2};\ \theta \text{ an arbitrary real scalar} \qquad \ldots(138)$$

giving

$$|\lambda\rangle=a_1(|\rangle_1+((\lambda-|\lambda_o|)/e_1)|\rangle_{1+r/2});\ \text{see (135)} \qquad \ldots(139)$$

Now λ and e_1 have arbitrary and independent signs. So, the formula (139) gives four distinct and linearly independent eigenvectors:

$$|\lambda\rangle_1=\exp(i\theta_1)(u|\rangle_1+v|\rangle_{1+r/2}) \qquad \ldots(140)$$

$$|\lambda\rangle_2=\exp(i\theta_2)(u|\rangle_1-v|\rangle_{1+r/2}) \qquad \ldots(141)$$

$$|\lambda\rangle_3=\exp(i\theta_3)(v|\rangle_1+u|\rangle_{1+r/2}) \qquad \ldots(142)$$

$$|\lambda\rangle_4=\exp(i\theta_4)(v|\rangle_1-u|\rangle_{1+r/2}) \qquad \ldots(143)$$

where

$$u=(1+|\lambda_o|/|\lambda|)^{1/2} \text{ and } v=(1-|\lambda_o|/|\lambda|)^{1/2}\ ;\ \text{real} \qquad \ldots(144)$$

and θ_1, θ_2, θ_3 and θ_4 are real and arbitrary.

As is to be expected these normed eigenvectors are not all orthogonal to each other. $|\lambda\rangle_1$ is orthogonal to $|\lambda\rangle_4$ and $|\lambda\rangle_2$ is orthogonal to $|\lambda\rangle_3$. Nevertheless, the set can be rendered orthonormal by the Gram-Schmidt process. We conclude that to represent the full behaviour of (79) the matrix order r must be at least 4.

5.2.2 The Matrices of Order 2 and 4

It appears, from the argument of the previous section, that, to represent the *simplest* form of quasi-classical system for which H has a dense spectrum, the matrices α_j and β must be of order 4. The properties of matrices of orders 2 and 4, which satisfy rules like (106) to (109), are discussed and tabulated in [10]. The three matrices of order 2 are called the Pauli matrices [1],[10]. The matrices of order 4 include those which Dirac used in his theory of the electron [3]. Concerning the latter we note, in particular, that:

i) there are exactly 15 non-scalar matrices of order 4;
ii) the matrices can be generated as direct products of the Pauli matrices and the unit matrix;
iii) they are linearly independent and non-singular (see Section 4.3.1.);
iv) products of the matrices are proportional to others in the set or to the unit matrix;
v) the largest subsets of anti-commuting matrices, which square to a scalar, contain five elements and are six in number;
vi) there two sets of three matrices with multiplication properties isomorphic to the Pauli matrices;
vii) the matrices vi) either generate or are homomorphic to the rotation group and other symmetry groups which have been used in the study of geometry and in theories of the elementary particles [10].

An obvious conclusion to be drawn from (v) is that the 'dimensionality' n of the 'simplest quasi-classical system' cannot exceed 4.

5.3 Interpretation

5.3.1 Cardinal Points

We make the following physical interpretation of the formalism:

a) In the case where r=4 and n=3 we take (79) to be the Dirac Hamiltonian [3]. This operator is supposed to represent a 'free' spin ½ particle embeded in the 4 dimensional space-time.
b) Accordingly, the elements of $\underline{P}$ and $\underline{Q}$ are local approximations to the conjugate (cartesian components of) momenta and coordinates.
c) m is the rest mass and c the upper bound on coordinate velocity.
d) In the case where n=4 the particle is embedded in a 5 dimensional space-time. This is taken to be the 5-space of the (classical) Klein-Kaluza theory of charged particles [11]. The modern theory of charged particles 'quantises' the Klein-Kaluza theory by a process similar to the Dirac factorisation [3],[11].
e) The particle is said to be 'free' (see item (a) above) because the spectrum of (79) is dense and differentiable; in the case of a 'bound' particle the spectrum of the Hamiltonian is discrete or discontinuous.
f) The spectrum of (69) is, in general, discrete. So, (69) is taken to be the canonical form of the Hamiltonian of a particle moving in a vector field; the dependence of the α_i and β on Q_n characterises the field.
g) The method of their derivation shows that (69) and (79) are unbounded *local approximations* to what, we suppose, are bounded operators; see Section 2. The eigenvalues of the Hamiltonian are usually taken to be the possible values of the system energy. We conclude that (69) and (79) are suitable to describe only *low energy phenomena*

5.3.2 Remarks

We have proved (see (103) and (104)) that the rest mass m and the upper bound on coordinate velocity c do not depend on the choice of coordinate; the inertia and the limiting speed are the same in all directions. But, there is nothing in the theory to show that c is a universal constant; it might just as well be a parameter, like m, peculiar to the particle. We can, however, scale the coordinate set $\mathcal{Q}$ for any particle so that c is common to all particles. It appears that we cannot do the same for m, by scaling $\mathcal{P}$, because the elements of $\mathcal{P}$ have already been scaled to ensure that the same constant h appears in (15) for every suffix $j=1,2,\ldots$.

The case n=3 has no special status in the theory. But it is only in a 3-space that the operators for the coordinates, the momenta, the angular momenta, the spin and the isospin behave like the components of vectors. All these operators either appear in the theory or derive from operators that do so.

There is, apparently, no upper limit to n. This does not mean that we could use (69) as the basis for a GUT! The GUT must deal with the strong short range forces and strong gravity; that is, with very high energy phenomena.

Condition (50) leads to unbounded H (see (60),(69) and (79)). Therefore, if in nature H is bounded, in nature K_1 is not zero. So, (50) and (36) are highly artificial conditions. Although, it is reasonable to assume that *some* axiom of stability, such as axiom(1), is approximated in nature, there is no guarantee. Thus there is no guarantee that either (50) or (36) approximate anything in nature; they must be judged by the results they give. As it turns out (50) produces recognisable structures, whereas, (36) does not.

6. Special Relativity

6.1 Relation of the Dirac theory to SR

Dirac's theory of the electron is based on Special Relativity (SR) and the theory of electromagnetism [3]. In particular he derives his Hamiltonian for a free particle from the SR relation between energy and momentum

$$H^2=\lambda^2 I=\Sigma c^2 p_j{}^2+m^2c^4 I;\quad j=1,2,\ldots n;\quad n=3 \qquad \ldots(145)$$

Since, this Hamiltonian is here derived on quite other principles, it is reasonable to expect that, given the interpretation of Section 5.3, we may also derive some SR relations.

6.2 Single Particle Kinetics

We have already derived (145) for general n (see (105),(110) and (111). Paraphrasing an argument given by Dirac [3]:

$$U(t)=\cos(-itH/h)+i\sin(-itH/h)=I\cos(t\lambda/h)-i\lambda H^{-1}\sin(t\lambda/h);\quad \text{see (2), (145)}$$

The Heisenberg operator of the coordinate velocity is, therefore,

$$\alpha(t)=\alpha\cos^2(\lambda t/h)+i\lambda\sin(\lambda t/h)\cos(\lambda t/h)(H^{-1}\alpha-\alpha H^{-1})$$

$$+\lambda^2H^{-1}\alpha H^{-1}\sin^2(\lambda t/h);\quad \text{see (3)};\ \alpha=\alpha_j;\ j=1,2,\ldots n \qquad \ldots(146)$$

The long term time average of (146) is

$$v_j=\tfrac{1}{2}(\alpha+\lambda^2H^{-1}\alpha H^{-1})=\tfrac{1}{2}(\alpha H+H\alpha)H^{-1}=cp_jH^{-1};\ \text{see (79),(107),(108),(109)} \quad \ldots(147)$$

This is the SR relation, between coordinate velocity v_j and momentum p_j, expressed in operator form. The derivation presupposes that the experiment to measure velocity takes a time much greater than h/λ and that, therefore, only a time average is recorded.

The operators that appear in (147) mutually commute and are all scalar in the $\underline{P}_n$-diagonal representation. (145) and (147) may, therefore, be inverted to give

$$\lambda=mc^2(1-\Sigma v_j^2/c^2)^{-1/2};\ j=1,2,\ldots n \quad \ldots(148)$$

$$p_j=mv_j(1-\Sigma v_j^2/c^2)^{-1/2} \quad \ldots(149)$$

the familiar SR formulae which express energy and momentum in terms of velocity.

Inspection of (147) shows that the upper bound of Σv_j^2 is c^2. Therefore, (148) is unbounded, as is required by (145). But this means that if, in nature, H is bounded then the classical formula (148) is an approximation! Now (148) is known to hold to high precision- a fact which seems to contradict the boundedness hypothesis. However, it is difficult to say whether or not an experiment to investigate (148) would be valid, once the energy reached a level where *any* interaction resulted in inelastic scattering (the creation and transmutation of particles). Further, it may be that the approximation only breaks down at cosmological energy levels.

6.3 Single Particle Kinematics

6.3.1 Permitted Transformations

Postulate a Q_0 which satisfies

$$P_0=H/c;\ Q_0P_0-P_0Q_0=ihI;$$

$$Q_0 \text{ commutes with } \underline{P}_n \text{ and } \underline{Q}_n;\ \text{spectrum of } Q_0 \text{ dense in } (-\infty,\infty) \quad \ldots(150)$$

The definitions are consistent because the eigenvalues of Q_0, like those of (79), lie anywhere in $(-\infty,\infty)$ [1]. Notice that the elements of the set $(P_0,Q_0,\underline{Q}_n)$ obey the rules (15); Q_0 is conjugate to P_0.

Now (145) gives

$$m^2c^2I=P_0^2-\Sigma P_j^2;\ j=1,2,\ldots n \quad \ldots(151)$$

Consider permitted transformations of the sets

$$(P_0,\underline{P}_n) \text{ and } (Q_0,\underline{Q}_n) \quad \ldots(152)$$

Transformations of $(P_0,\underline{P}_n)$ must leave both the form and the value of (151) unchanged. The form depends on the properties assigned to the elements of the set $(P_0,Q_0,\underline{Q})$; and, providing that these properties are preserved, the form is invariant. The value of the constant on the LHS helps to define the state; and, we cannot allow transformations which alter the state. At the same time, transformations of (P_0, P_n) must be accompanied by transformations of (Q_0, Q_n) which leave the form of (15) unchanged. Consider, for example, constant linear transformations of (P_0, Q_0, C_n) of the form (20); these must obey conditions of the form (21) in order to preserve (15) and (150).

Explicitly the invariance of (151), under constant linear transformations, requires that

$$M^T(s)M=(s);\ M \text{ is the real } (n+1)\times(n+1) \text{ matrix of transformation} \quad \ldots(153)$$

where (s) is the $(n+1)\times(n+1)$ diagonal matrix derived from the signature of (151) $(1,-1,-1,\ldots,-1)$. Whereas, (21) requires that

$$N=(M^T)^{-1}=(s)M(s)^{-1} \quad \ldots(154)$$

so that

$$N^T(s)N=(s);\ (s)=(s)^{-1} \quad \ldots(155)$$

Thus

$$Q_0^2-\Sigma Q_j^2;\ j=1,2,\ldots n \quad \ldots(156)$$

is also invariant under permitted constant linear transformations.

6.3.2 Lorentz Transformations of the Spatial Coordinates

Although the above argument shows that (156) is invariant under the transformation

$$'Q_j=\Sigma N_{jk}Q_k;\ j,k=0,1,2,\ldots n \quad \ldots(157)$$

the elements of the matrix N are not determined. To give values to these we need to consider the operator

$$T=-Q_0/c \quad \ldots(158)$$

which satisfies the familiar condition (1)

$$HT-TH=i\hbar I \quad \ldots(159)$$

T is a time operator; but, it is not the background evolutionary time t of (1). It represents measurement of epochs of events in the system characterised by H; and, to make such measurements, we must interfere with the system. So, (1) does not apply and T, unlike t, does not commute with H.

Now form the commutators with respect to H on each side of (157) and take expectations:

$$'u_j=\Sigma N_{jk}u_k;\ u_j=\langle i(HQ_j-Q_jH)/\hbar\rangle;\ u_0=c;\ 'u_j=\langle i(H'Q_j-'Q_jH)/\hbar\rangle \quad \ldots(160)$$

$u_1, u_2, \ldots u_n$ and $'u_1, 'u_2, \ldots 'u_n$ are the expected velocities of the spatial coordinates referred to cartesian axis sets O and 'O (say), respectively. (155) and (160) determine a form for N; this may be interpreted as the general

Lorentz transform with 4-velocities $\underline{u}$ and $'\underline{u}$ in the two coordinate systems. We sketch a method by which this form can be deduced.

Suppose that the transform differs from identity by only an infinitesimal amount. Then

$$N=I+E \qquad \ldots(161)$$

where the elements of E are small compared to unity. (155) requires that

$$E(s)=-(s)E \qquad \ldots(162)$$

Suppose also that, for simplicity,

$$'u_j=w_j;\ u_j=0;\ E_{jk}=0;\ j\neq k;\ j,k=1,2,\ldots n \qquad \ldots(163)$$

That is, the particle is (expected to be) at rest in O with expected components of velocity $w_1,w_2,\ldots w_n$ in 'O. Further, the 'O axes are similarly oriented to the O axes. Then

$$E_{00}=0;\ E_{jj}=0;\ E_{0j}=E_{j0}=w_j/c;\ j=1,2,\ldots n;\ \text{see (160) and (162)} \qquad \ldots(164)$$

N is now completely determined in terms of $\underline{w}$; but, it represents only a Galilean (infinitesimal Lorentz) transformation. Following the method of [10] we may now compute the full Lorentz transformation (for the case (163)) as the compound of an infinity of identical infinitesimal transformations

$$N=\underset{j\to\infty}{\mathrm{Lt}}\ (I+E/j)^j=\exp(E) \qquad \ldots(165)$$

where E is still given by (163) and (164), but its elements are no longer small. The means, by which the exponential (165) can be reduced to a simple matrix, are illustrated in [10]; however, that reference uses different conventions of signature and notation.

7. Claims

The axioms investigated are:

I) the conventional QT hypothesis, that the state of an isolated system is a function of a continuous scalar time;
II) that it is sufficient to describe physical systems in terms of bounded measures;
III) absolute determinism, expressed in QT terms;
IV) transient determinism, expressed in QT terms;
V) that the evolution operator (EO) has a dense spectrum.

The claims are that:

a) Axioms (I),(II),(IV) and (V) lead to a canonical form for the EO which is identical, in structure, to the Hamiltonian of the Dirac free particle.

b) Although, the dimensionality n, of the space (as opposed to space-time) which contains the particle, is not determined, the values for which $n<5$ have special significance.
c) The particle obeys SR kinetics (relations between energy, momentum and mean velocity) expressed in terms of operators.
d) The particle obeys SR kinematics in the sense that:

i) the Minkowski metric, expressed in terms of space-time coordinate operators, is invariant to permitted constant linear transformations;

ii) the permitted constant linear transformations are identical to Lorentz transformations in which the classical velocities are replaced by expectations.

It is worthy of note that the Born-Jordan commutation rules are essential to the argument that leads to these results.
e) Removal of axiom (V), from the scheme, leads to a canonical form for the EO which might be interpreted as the Hamiltonian of a Dirac particle moving in an unspecified vector field; this needs more investigation.
f) Axiom (III) yields an unrecognisable EO; its conditions appear to be too stringent for the description of nature.
g) The methods of analysis involve approximations; and, if the physical interpretations are correct, the results apply only to relatively low energy phenomena. In particular, the SR relation between kinetic energy and speed should break down at very high energy.

The implication of these results is that Relativity Theory may derive from QT. If this so, to 'quantise' Relativity Theory is to work backwards!

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Cybernetic Consciousness

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ABSTRACT

Cybernetics from the study of abstract systems can make a holistic contribution to the age-old quest to understand consciousness. Being conscious of consciousness is inherent in defining consciousness using usual models from mathematics, physics and myth. A formal description of consciousness is presented in terms of a Kuratowski closure for standard topological spaces.

INTRODUCTION

BACKGROUND

Without some form of consciousness there is no meaning to anything. It is not surprising therefore that the subject of consciousness is to be found on the epistemological shopping list of many philosophers. Indeed whole schools of thought --such as the cartesianism of Descartes, Malebranche and Spinoza devoted to the study of the integration of the intelligible world with the human mind-- may be viewed as a contemplation of consciousness. Nevertheless no conclusive results seem to have emerged from the many scores of works that over the centuries have poured forth on the subject of consciousness. It is not then that we can expect easily to succeed today where great minds in the past have faltered, but rather that there is a duty for cybernetics to provide a contribution to the debate on the nature of consciousness from its own view point. For while the contribution may be but a humble snapshot the observation post of cybernetics is a lofty one.

In particular cybernetics can bring to bear an up-to-date integrating view which incorporates a more modern science unshackled from the classical physics framework of the Universe that constrained most early philosophers.

A cybernetic notion of consciousness has to be distinguished from knowledge that might be sought in a single discipline such as in psychology, where consciousness is of prime interest but at a different level of investigation. There the prime concern is to investigate human intelligence and considerations of other consciousnesses -for example in animal intelligence- are directed to that end. Consequently the various qualitative models to be found in psychology and even the more quantitative descriptions to be found in neural systems form just one ingredient of the cybernetic cake. For while cybernetics depends to some extent on disciplines such as philosophy and psychology, it also subsumes them and to some extent gives them context and indepedent validation. But why the urge to define consciousness?

NEED TO DEFINE CONSCIONESS

The urge to understand, to reduce the unknown in terms of the known is itself perhaps a corollary of the nature of consciousness. To turn Descartes through 180 degrees we might say sum ergo cogito. That consciousness is a consequence of existence probably might not be accepted by many. Yet it is implicit in basic scientifically accepted principles such as the law of gravity. Gravity is the recognition by matter of the existence of every other piece of matter in the Universe. No existence without consciousness is a very basic point in fundamental terms. However it is somewhat controversial because it does not conform to our cultural experience and understanding. However the paradigm shift brought about by Einstein and even more so by quantum mechanics is gradually changing the climate of thought. The philosophy of Bishop Berkeley has found new expression in the anthropic principle[1]. The anthropic principle which considers the place of man in the physical universe comes in two strengths. In its weak form the principle merely states that post eventum the existence of man is no more remarkable than any other circumstance, but in its strong form it has been interpreted[2] as meaning that all existence depends on human consciousness. The meaning of existence and its connection with self-consciousness are more advanced topics and we need to look first at the foundations.

A more contemporary need to delineate the meaning of consciousness arises from its connection with intelligence. In that context this paper is a sequel to the contribution on defining intelligent systems[3]. There were a number of points put forward in that discussion which are relevant here but which will not be examined further here. These are:

a. A diversity of disciplines from genetics to cosmology, psychology to law, education to robotics, etc, etc are dealing with lifestyle activities which share common features that can to their mutual benefit be compared and contrasted within cybernetics.
b. that the basic unit of the intelligent system is more tractable than the abstract quality of intelligence itself.
c. that definitions of intelligent systems presuppose some definition of consciousness.

KINDS OF DEFINITION

Before embarking on a discussion of definitions of consciousness we should perhaps first consider the kind of explanation we are seeking. For words like 'meaning' and 'definitions' are relevant to the concept of consciousness itself. It is always said to be good practice to define one's terms. When dealing with a definition of consciousness we need to define what we mean by a definition. The current way is to think in terms of models. It is possible to divide models into three broad categories in so far as they utilize:

a. Mathematics;
b. Physics; or
c. Myth.

and each will be briefly considered in turn.

MATHEMATICAL MODELS

Mathematical models may involve geometry with graphical methods , or algebra with symbols, or arithmetic with the 'natural' and 'real' numbers, so-called although they are to be found neither in nature nor in reality. Mathematical models may be built one on another in the way that probability is built on arithmetic and statistics is built on probability. These models are conceived in the human mind and all constructed as shown by workers such as Russell and Whitehead in the Principia Mathematica, from classical logical primatives. The astounding success of mathematical models has bred the feeling that mathematics is real, whereas the only existence that mathematics possesses is that it lies in the mind of a human mathematician who happens to inhabit the Universe. Why mathematics works is not because the laws of physics are based on mathematics but rather that both the Universe and mathematics are systems --one physical the other metaphysical. For under the basic tenet of cybernetics any system behaves much like any other system. The disadvantage of the mathematical model is that it is never precise when it models reality.

PHYSICAL MODELS

Physical models involve the use of the physical universe including all experiments whether in the field or in the laboratory as well as in the use of well-loved archetypal iconic models such as the model railway. Physical processes are always exact although there will be inaccuracies involved in any measurement. The main limitation with physical modelling is one of locality. Results can only be obtained on parts of the Universe not on the Universe as a whole. This partiality introduces familiar problems like those of scale and sampling.

MYTH MODELS

The myth is the modelling of physical and metaphysical concepts in human terms. It is the basis of literature and of all art forms. A simple form of myth is the fable but the highest form of pure myth is probably to be found in the theological parable. There are a few conspicuous examples of the direct use of myth in science such as 'Maxwell's demon', Schroedinger's cat' and the like. (Some question whether these human and animal models might not obscure rather than illuminate the scientific issues they are trying to illustrate.) What is much more prevalent is the indirect use of myth. For it exists whereever natural language is employed. All science including mathematics and the exact sciences rely on natural language. Mathematics and computer science as well as much physical theory makes use of words of natural language as primatives in equations etc. Natural language is often thought to be informal and language models considered to be imprecise and to lack rigour. This is much too harsh a judgement on natural language, for subject areas such as the Law show that natural language possesses a real-world exactness that cannot be matched by mathematics. In fact natural language is the primary medium of natural logic, where the mode of inference is analogical reasoning.

REALITY OF MODELS

These three types of model can be epitomized as 'maths, matter or myth'. A definition may employ any one or any combination of the three. In practice it is difficult to find a definition that does not involve the myth. This is how we can be conscious of consciousness. Reality is only real to us through the models in our consciousness. We can go on finding better models as in the models of the atomic nucleus which began with some very simple representations such as the liquid drop model to describe fission and fusion, followed by slightly improved models such as the cloudy crystal ball model, the nuclear shell model and so on until the more recent and much more complicated quantum chromodynamic and super string models. The only perfect model of anything is the thing itself. Thus reality may be considered as the limit (in a mathematical sense) of the sequence of models that model it.

Of course in a way all models are real in that they are to be found in the Universe, if only in the human mind. From that point of view the physical models are ontological while the mathematical and the myth are epistemological. The distinction is to be found in consciousness. To pursue this fully would take us into self-consciousness and existence which is further along the intelligence-consciousness-existence road than we can perhaps go at present. We need first to explore the beginnings of that road.

INTELLIGENCE AND CONSCIOUSNESS

DEFINITIONS OF INTELLIGENCE

Let us take a look at the role consciousness plays in the notion of intelligence. The cybernetic definition as given in the Fatmi-Young version is as follows:

> intelligence is that faculty, of mind, by which order is perceived in a situation previously considered disordered.[4]

The three difficulties seen in this definition were:

a. its negentropic character:
b. the time dependency; and
c. the oblique anthropomorphism.

These three difficulties can now be identified respectively within the three models of mathematics, physics and myth. The definition of intelligence draws on all three models (although the mathematical aspect is not explicit) and is only satisfactory in so far as those models are acceptable. The interesting point is that these three difficulties arise because the definition follows paths into the three models which stop at a common point and that common point is consciousness.

META-INTELLIGENCE

If a definition of intelligence rests on the concept of consciousness then consciousness may be presumed to lie beyond intelligence and be considered a 'meta-intelligence', that is the dimension in which intelligence is embedded. That the cybernetic definition of intelligence has an inbuilt concept of consciousness is apparent from the use of the words:

...by which order is perceived...

This is an example of the operational nature of consciousness. Consciousness acts as a closure in this definition. From a cybernetic point of view no system is closed. This is essentially the third law of thermodynamics. It is not possible to partition one part of the Universe from any other. In passing it may be noted that if there are metaphysical activities going on within the Universe these must also be included. In most scientific system investigations it is necessary either to confine one's interest to phenomena well away from the horizon or to employ some form of closure which performs the function of wrapping up in a simple form the significant effect of the residue of the Universe. A good example of this is the concept of randomness. A Kolmogorov understanding of randomness is as a (maximal) ordering of the Universe. Investigations that make use of different classes of random distributions, normal, poisson, binomial, etc are using these as a closure operation providing for the effect of the rest of the Universe on the system in question. This closure may be looked on as an infinite source/sink. Where it is particularly significant is in providing a channel between different levels. It is this characteristic as a 'hole' (black or white) that has been given names such as 'mind','spirit','soul', etc.

CONSCIOUSNESS IN SCIENCE AND HUMANITIES

It is perhaps useful to examine examples of the phenomenon of consciousness in various fields of human endevour where the way it operates can give us some insight into its nature.

CONSCIOUSNESS IN MATHEMATICS

An important example is in mathematics. Because of the very formal nature of the subject it might be thought that there would be little difficulty in deciding whether consciousness played any part at all. But this, surprisingly enough, does not seem to be the case. Mathematics teams with the exercise of the consciousness of the observer. A statement such as 2+2 = 4 requires a consciousness to recognize that '2+2' is the same entity as '4'. From early on in Euclid, proofs on congruent triangles require an observer to make identifications. Taking different axiomatic approaches only results in shifting the 'hole ' to other parts of the argument, because mathematics is a tautology. Any proof by induction has a similar requirement for consciousness to be exercised. Equating coefficients is another common example. An awareness of the necessary assumptions is needed to apply a reductio ad absurdum argument. Even Goedel's proof itself demonstrates the phenomenon in identifying logical statements with applications of the fundamental theorem of arithmetic.

The gaps can be found in any mathematical textbook. The better textbook writers make them explicit. For example consider the following from a standard graduate text on vector and tensor analysis:

> ...you can get a firm intuitive feeling for a point, line, and plane in the pictorial representation that you are no doubt familiar with from your high school geometry. Depending on previous training, the use of undefined terms may or may not seem strange. Indeed in his original collection of mathematical works Euclid "defined" point, line, and plane. However with the evolution of logical thought over the centuries, it became evident that these definitions were meaningless. In fact, a significant development in man's mode of thinking is the realization that certain concepts must be taken as undefined; otherwise a circular reasoning necessarily results[5].

The student of mathematics is therefore expected to fill the fundamental gaps in the argument from his 'intuitive feeling' to provide the necessary closure.

One 'hole' that keeps on appearing which has to be filled has a formal basis in mathematics. It is sometimes referred to as the axiom of choice but there are a number of diverse equivalent statements stretching back to Zermelo[6] such as the Hausdorff Maximal Principle, the Zermelo postulate, the Zorn dilemma, etc. These all tell us that there is in any system an ordering that cannot be reached by mathematics, as we know it. Yet the whole of mathematics rests on that undefined point. In practice this anomaly does not cause many problems. The reason for this seems to be that it can be dealt with by a mathematical reduction processor and one exists. Namely the understanding of the human mind that can close the gaps wherever they appear.

The significance for consciousness of the axiom of choice is that in one of its forms it means that a set cannot be numbered. So the cardinality of a set may exceed its maximal ordinality however many ways it is counted. So a count of the parts of a human system by disection might sum to less than the whole unless we can find parts like the mind to include. The same is true for larger human systems such as society which seems to exhibit a separate will and wisdom of its own beyond that of the sum of that possessed by the individuals that compose it. Take a two-human system as in joint authorship of some work. It is common for both authors to be aware of some entity beyond the two of them put together which is often the source of their creativity. Again the whole is greater than the sum of its constituent parts.

It seems that mathematics may only work because it is performed by mathematicians. This may be an example of the anthropic principle in its strong form. An objection may be raised that mathematics exists in the workings of nature. The fact that mathematics can be used in so many scientific and engineering applications may be thought of as proof that the mathematics exists independent of the mind of the mathematician for the same mechanisms can be found to be working in a myriad of situations independently of whether man decides to carry out calculations on them. This point has already been touched on above. The weakness in the applicability argument is that mathematics never quite works in nature but is only an approximation. So it is the human mind that makes it apply. The reason why we are able to do this has already been given. It is because of the fundamental principle of cybernetics that any system behaves like any other system so nature behaves in a logical fashion rather like the metaphysical constructed model in the mind of the mathematician. It is simply that we happen to live in a least action Universe.

Mathematics works because of the consciousness of the mathematician. The relationship with the physical universe can perhaps be seen better in the consciousness which is present in the development of physics.

CONSCIOUSNESS IN CLASSICAL PHYSICS

The main difference between physical reasoning and mathematical reasoning appears to be a different kind of consciousness. A mathematician rests his reasoning on axioms found in his inner consciousness while a physicist relies more on empirically based principles ascertained through some consciousness of the external physical world. However where the understanding comes from may be somewhere different. For it seems that more than is usually admitted comes from within. As with mathematical texts there is a related effect apparent in physics education. The learning of very established physical principles turns out to be subjective. For instance the question of whether mass is conserved separately to the conservation of energy is surprisingly still debatable by eminent physicists. Thus Sir Herman Bondi and others claim that there are two conservation laws[7] and that many text book writers are quite misleading on the way that they treat the equivalence of mass and energy. Professor Rudolf Peieris of Oxford on the other hand maintains that there is only one conservation law[8] but others have gone further than Bondi and have claimed:

> The ideas taught about energy are quite literally incoherent nonsense. Interference of waves, entropy, distribution laws, radioactivity, electrical conduction and many other topics are widely misrepresented.[9]

Despite such severe inadequacies in their education, physicists brought up on these methods with these textbooks are still able to aspire to a very high level of scientific attainment! It is as though physics teaching is not didactic but more a maieutic ritual drawing out from the students own consciousness a true understanding of the subject quite independent of what is taught. This may explain the phenomenon where physics experiments tend to be carried out to confirm the results expected. It may be that the scientific method which relies on hypothesis and prediction is often a mere justification for an understanding derived from an internal consciousness. Where does the hypothesis come from?

There is a succession of well known examples of intuitive derivations such as Kekule's structure of the benzene ring, James Clerk Maxwell's expression for the kinetic gas distribution of energy correct but based on faulty reasoning, Einstein's assumption of e=mc2 in the steps of one of his published proofs of that same formula, etc. Today some famous justifications for both the scientific method and for the principles that underpin it are taking a hard knock. One 'prediction' to fall recently relates to the discovery of what at the time was thought to be the planet Pluto. Its triumphant discovery in 1930 from confident predictions made from observations of the irregular orbit of the planet Neptune now seems to be a fluke. Recent determinations of the mass and size of Pluto suggest[10] that it is probably a satellite of Neptune that would need a density greater than that of gold for its existence to be detectable from the motion of Neptune.

Black holes themselves, if they at all exist, are examples par excellence of points of consciousness in the Universe. That is points in the Universe but not of the Universe, which act as channels through to some other existence outside of our own space-time. They are points that give the Universe a cardinality greater than its maximal ordinality. That their existence is predicted as singularities by Einstein's general theory of relativity is significant in the context of gravity as information but there are some doubts about their nature and whether Einstein's general theory can be applied in areas of such strong gravity. Observations[11] such as the spectra of stars orbiting within apparently steep gradients near the centres of galaxies are cited as evidence for black holes but we may need to know more before we can apply them to add to our knowledge of consciousness. Indeed the beauty of cybernetics is that mappings are invertible. Our knowledge about human consciousness can be used to suggest that black holes may be considerably more complicated than a Schwarzschild singularity.

CONSCIOUSNESS IN QUANTUM PHYSICS

The role that consciousness plays in quantum physics is perhaps of even more significance than its place in the development of classical physics[12]. It is also topical because of the proposal for a quantum computer[13] where essentially the computation is based on a physical model in terms of definitions discussed above rather than a mathematical model of computation used by present computers. Classical physics is so different from quantum physics that it needs to be treated separately, although cybernetics can provide an amalgam between the two types of physics with fuller insight into the physicist's elementary correspondence principle. It was Wiener who was able to show that there were mathematical foundations to physical concepts such as Dirac's function. Wiener's theorem, a converse of the Fourier theorem that non-periodic discontinuous functions may be represented by a continuous set of periodic functions, generalizes the Dirac function. Wiener's recognition of this equivalence of discontinuity and continuity representations led him to develop cybernetics by applying the equivalence in mathematical statistics and information theory. Nalimov[14] took these methods into the realm of consciousness by applying stochastic theories to the subject. The problem there is the meaning to be attributed to the concept of probability. It now seems that statistical mechanics does not lead far enough but quantum mechanics seems more promising.

The duality of particles and waves in quantum physics is a similar complementarity equivalence. The Heisenberg Uncertainty Principle may be the most oft quoted form of this duality in philosophical discussions but of much more importance for the notion of consciousness is the Einstein-Podolsky-Rosen[15] effect, which takes various forms as in the 'two slit paradox', the experiments of Aspect or Bohm, and the theory of Bell. These all relate to the the characteristics in a system of supraluminal correlation when there is no supraluminal connection between the parts of the system. This is the phenomenon of self-consciousness. A significant proof by Bell that the EPR effect is not due to hidden variables suggests that the origin of consciousness lies within the system itself. It should be noted however that the most popular versions of quantum mechanical explanations are a one level description. Alternative theories do exist however such as the discrete hierarchical combinatorial model[16] which provide more structure related to phenomenological levels.

CONSCIOUSNESS IN AI

In dealing with the subject of cybernetic consciousness, it might seem strange to omit the subject of Artificial Intelligence. So perhaps a few words are required on consciousness in AI.

AI however is probably a true exception that proves the rule. For in AI it is hardly possible to find an example of any natural consciousness in the sense so far described. What can be found are strong doses of human consciousness injected into formal systems. These are in the main mathematical models. The human consciousness is introduced through the interpretation of the symbols which in languages like LISP and PROLOG are expressed as atoms of natural language, i.e. by the use of myth. Robotics that employ sensors to control their behaviour are being guided by real world events and exhibit a form of consciousness amounting to a physics model. There is some sense of consciousness in pattern matching employing usually statistical methods (i.e. arithmetical models) and the beginnings of a cognitive consciousness using database technology[17]. The latter seems to integrate models of myth and mathematics rather than have them operating separately as in declarative programming languages.

AI has a different emphasis from cybernetics. AI is directed towards currently available realizable machines. This means in general the electronic digital computer based on a von Neumann architecture. Cybernetics on the other hand perhaps somewhat fortunately went through a development phase before computers were readily available and was not side tracked into one type of device but rather through the influence of early eminent cyberneticians such as Ashby concentrated on the machine in general form. So while consciousness is a property of the general system or machine it is not very characteristic of the limited machinery employed in AI. Despite reports of lambda calculus implementations,prototype theorem provers and imminent systems, we do not yet see computers performing algebra. The transition from arithmetic to algebra requires a conceptualization which amounts to consciousness. It may be that AI is waiting for the development of other types of machine such as the quantum computer -before we see the arrival of artificial consciousness.

CONSCIOUSNESS IN THE LAW

The humanities, particularly any form of art, abound in the phenomenon of consciousness. Quotations have been given already which illustrate examples in education. The instances in the humanities are so varied that it is not possible even to survey them here. Instead attention will be restricted to just one area, the Law.

There is the story of the judge whose judgments were respected by all as being of great distinction and correctness but he never gave any reasons for his decisions and when prevailed upon to do so his legal reasoning was so bad that he had to be dismissed from his post. The judge was apparently in touch with some order of justice that he could not explain. The Law is an opportunity to investigate the same phenomenon of consciousness in a very different context and may be seen as an example from the human sciences. For Law is a kind of exact science of the humanities[18]. Indeed at times the Law is called on to play the role of a meta-science when it has to adjudicate between scientists. This is very common when a court has to decide between conflicting evidence from forensic science, not uncommon with disputed medical evidence in accident cases and occasionally with other sciences when there are scientific issues as in actions for defamation.

How are legal decisions arrived at? This depends at first sight on the legal jurisdiction. If the world is roughly divided into the three main families of Law, the common law, the civil law, and socialist law the methods appear very different. The common law operates by stare decisis through a hierarchical system of binding precedent, the civil law by judicial interpretation of a statutory code and the socialist law by dialectic argument. However if we examine them a little closer we find that they reduce to a human consciousness of a system of legal norms with striking similarities of interpretation. They are all based on some collective consciousness of what is just. Judges by their natural skills, training and experience are entrusted by society with being the mouthpiece for public knowledge. While there is a difference of emphasis across the three families they all recognize human capabilities at gaining access to some other order-determining level. Even young children seem from an early age to know when 'it's not fair!' and to have right of access to that level of consciousness.

Work on Celtic legal systems suggests that before writing, Law in western Europe was more of an art form. Judgments were sung and founded on precedents handed down in an alliterative form of verse[19]. It appears that the present day practice of the oral extempore decisions of a judge in court has a continuous history back to the days when a druid would divine judgments. The druidic order is important in archaeological cybernetics for the druids practised a holistic view of life not distinguishing between law, medicine, religion, etc. The Law holds special interest as a discipline because it has a continuity throughout history which cannot be matched by other ancient disciplines such as mathematics or medicine. The divination is of course not specially Celtic. The origin of the Mosaic Law had a similar provenace. Divination was very prevalent in Ancient Greece, one of the most rational societies we know to have inhabited this planet. Plato even has a taxonomy of divination, distinguishing between sane divination and insane divination[20]. It may be that what the ancients called divination we call creativity. The cybernetic conclusion might be that both creativity and divination are a form of consciousness.

CHARACTERISTICS OF CONSCIOUSNESS

It is time now to pull together the different aspects that we have seen about consciousness to try to draw some conclusions and to try to find some kind of definition. Grammar of consciousness needs to be sorted out. It describes the following:

a. an operation; and
b. a state.

This is not particularly surprising. The dichotomy between the prescriptive and the descriptive is very common. The manner in which a descriptive matrix can perform as a prescriptive operator is at the essence of modern group theory. Associated with both an operation and a state are further dichotomous distinctions between openness and closure, between a source and a sink and between external and internal. Yet they are not disjunctive dichotomies but rather conjunctive in the way that they fuse one with the other. Throughout these there is a basic function of ordering. It is a natural ordering. As we have seen it has to be introduced in mathematics through the axiom of choice; in physics it arises through the observation of particular events in nature or by use of a general natural ordering to be found in 'random' distributions.

CONSCIOUS SYSTEMS

With a definition of intelligence it was found more appropriate to investigate intelligent systems so it may be easier to focus on systems which exhibit these various characteristics of consciousness. We need some definition of a system and then to investigate it for the properties and characteristics so far described. Although cybernetics is built on the theory of systems and the concept of the abstract machine, there is no commonly accepted basic unit. No 'systeme' or 'cyberneme' corresponding to the lexeme, phoneme, etc, in the nomenclature of linguistics. If the 'systeme' is a general system and a 'cyberneme' a component of a systeme, this conveys the current level of attention but retains the point that each is a system in its own right. Thus in considering a theory with the brain as the systeme and the neurons as the cybernemes it should not be forgotten that each neuron is not an elementary particle but is itself an entity with a complexity that has been described as more like a minicomputer. The brain itself is a cyberneme when considered as a part of the human body systeme. A human is a cyberneme in a social society systeme, and so on. It may be necessary for the systeme-cyberneme relationship to be recursive. In set-theoretic mathematical terms there is the set and its elements, but recursion can only be achieved by some contrivance for a set cannot be a member of itself.

A very fundamental question in considering any systeme composed of cybernemes is what kind of matter are we concerned with. As soon as we start distinguishing between cybernemes and systemes we are departing from a holistic approach. We are introducing discreteness. This is a weakness because it is such a strong assumption with so many consequences. There are theories with other kinds of matter such as Parker-Rhodes' theory of indistinguishables [21] which leads to interesting quantitative results which correlate well with some fundamental constants in physics. The Casimir[22] effect of substratum radiation in a vacuum also provides us with information about energy of matter at the level of nothing. These seem rather promising but unfortunately they are part of a stream of thought which virtually dried up with the pre-socratic philosophers and until these are further developed we are prisoners of the all-pervasive aristotelean way of thinking. However set theory is not so restricted to the point version which is its usual form. A set does not have to consist of identical items. We can perhaps make some progress by applying it without eliminating the element which amounts to the conscious cyberneme. Therefore this paper will be concluded with a mathematical model of conscious systems. This of course from what has already been said will still rest to some extent on the use of the myth.

A MATHEMATICAL MODEL OF CONSCIOUSNESS

The 'systeme' we will use is a topological space. This has the advantage of allowing us to use a considerable amount of well tried and tested standard mathematics. For fuller details of the background, reference should be made to any convenient textbook in topology[23].

Given a topological space $\{X, T\}$ to represent a system where X is a set and T is the family of open sets of X. The term 'set' is used in its widest sense to be found in mathematics. X does not need to be finite. The members of the set are the cybernemes. These are not restricted to points or numbers, but may consist of any form of physical or metaphysical matter or abstraction. If cybernemes are denoted g,

$$\text{then, } g \in X$$

these cybernemes as previously mentioned are not necessarily homogeneous, the most elementary cyberneme g_o is given by

$$g_o \in \emptyset$$

where $\emptyset$ is the null set. This is Casimir's vacuous particle/wave, Parker-Rhodes' 'indescribable', the nothing that is the ultimate material of everything.

Interesting cybernemes are those that belong to the topology T, that is the class of subsets of X that are open relative to T. T determines the structure of the space and therefore represents the characteristics of a system so defined. The systeme is a collection of cybernemes with structure. The simplest structure is given where the set X and the null set are the only open sets, for these two sets are always open (and also closed). This is the indiscrete topology. This may represent the lowest form of consciousness, such as the gravitational force 'felt' by every particle in the Universe in Newton's theory, or better expressed by the gravitational self-attraction in Einstein's General Theory. That is the consciousness of inanimate objects. The highest form of consciousness will then be in the discrete topology where every subset of the system is open.

A set $A \subset X$ is open if its complement $X \sim A$ is closed. A set $A \subset X$ is closed if it contains A', the set of cybernemes that limit it. A~ The closure of $A \subset X$ is the union of A and this set of its limits, that is A~ = A U A'; also if A is closed, A = A~, and if A is open it contains only interior cybernemes. A neighbourhood of a cyberneme is any set (which need not be open), $N_o \subset X$ that contains an open set of which the cyberneme is a member. In the lowest consciousness case of the indiscrete topology there is only one neighbourhood, $N_{\smile} = X$, the space itself. At the other extreme in the discrete topology, every set the cyberneme belongs to is a neighbourhood. The family of neighbourhoods defines the extent of consciousness, its event horizon.

Kuratowski[24] has shown that a given topology on X may be determined by a closure operator subject to four axioms which therefore in this context define a system. The four axioms are:

a. 0 = 0~;
b. $A \subseteq$ A~
c. (A U B)~ = A~ U B~; and
d. (A~)~ = A~

The first, the null-closure axiom, provides the lower limit to consciousness, this closure relates to the perception of nothing and deals with the g_o cybernemes which are less than nothing. It relates to the origins of a system, creation of matter, etc and provides the connection between existence and consciousness which is more than we can be concerned with here. The second axiom tells us that a set always has a consciousness greater than or equal to itself. This then applies to any cyberneme by considering a singleton A = { g }.

The third axiom is important because it provides us with information about the sum of consciousnesses. The combination of two consciousnesses leads to a third combined consiousness, as in the example of joint-authorship quoted earlier. The principle applies in general to the union of any finite number of sets $\bigcup A_i \subset X$ where i is a positive integer. The final axiom, that a closure of a closure is itself, is a proposition of transfinite logic. It describes self-consciousness; that is how we can be conscious of our own consciousness. Without this proposition this paper on consciousness would itself be totally without meaning.

This provides us with a mathematical model of consciousness to be found in the T-openness of a topology. Consciousness may be defined as the closure operator which defines the topology.

CONCLUSIONS

It is time briefly to take stock of where we are and where we need to go in the future. In attempting to model the notion of cybernetic consciousness we have had to look carefully at the limitations in modelling and acknowledge that any model reduces to a process of consciousness and the model has to recognize this recursiveness. This process is implicit in the models of physics and myth but we have seen that it is possible to make it explicit in a mathematical model by defining a conscious system as a topological space with a topology relative to a closure operator which can be identified with consciousness. It has not yet been possible to make the connection with intelligence. This is because we have looked only at simple topologies on sets which have no order. Consciousness is a meta-intelligence when it operates as a closure mapping on to ordered sets in uniform topological spaces. These more sophisticated systems await further elucidation.

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Special Relativity and a Calculus of Distinctions

Louis H. Kauffman

I. Introduction.

The purpose of this essay is to place the *mathematical pattern* of the theory of specal relativity in a new context. The pattern of the Lorentz Transformation, the Poincaré Group and related invariants arises historically from the ground of electromagnetic theory and Einstein's extraordinary formulation of these ideas in his special theory [6]. The next reformulation, due to Minkowski [14], led to the concept of *spacetime* as we know it today. Bondi [2] has pointed out the extraordinary simplicity of the theory, once it is formulated in light-cone or radar coordinates.

I will take this simplicity a step further and show that the mathematical pattern, and the pattern of ideas related to it can be regarded as arising from *consideration of the properties of a distinction.* These ideas, necessarily informal in the beginning, become more definite as we fit them out with mathematical clothing. In the course of this development, the mathematics of special relativity appears quite naturally, but with a non-physical interpretation. This provides a new ground and a new language for cradling the old physical ideas.

It is commonplace in mathematics to find one formalism holding a multitude of interpretations. Each such interpretation is potentially useful as a lens through which all the others may be seen.

For the reader unfamiliar with special relativity, I have included in the appendix to this paper a quick and self-contained introduction along the lines of [2] and [9]. In the body of the paper I have restricted my constructions to the context of distinction, and to the progression toward special relativity.

The basic idea of this paper is very simple. Let a distinction be given. Assume that the sides of the distinction are evaluated by A and by B. (A and B can be real numbers.) Denote this by $[A, B]$. Another observer may change the emphasis given by A and B. Thus we must consider transformations of the form $\mathcal{O}[A, B] = [RA, SB]$ where R and S denote the changes of emphasis created by the reference frame of the second observer. Then we see that choosing $RS = \rho^2$ and $R/S = \lambda^2$ we can re-write the transformation as $\mathcal{O}[A, B] = \rho[\lambda A, \lambda^{-1} B]$. If we are concerned only (projectively) with relative values, then it is sufficient to take $\rho = 1$. This leaves us to consider transformations of the form $O[A, B] = [\lambda A, \lambda^{-1} B]$, *giving the structure of the Lorentz group to a calculus of distinctions!*

The paper is organized as follows. Section 2 discusses distinctions, indistinguishables and concepts of evaluating and weighting a distinction. This is considered with respect to the ideas of text and context, figure and ground. Section 3 re-examines these ideas in terms of description/perception of a ground-form that is susceptible to multiple interpretation. Along with this we construct an algebra of the iterants $[A, B]$.

An *iterant* $[A, B]$ is a distinction viewed as an alternation or superposition of its two sides. (The iterant itself is neither one side nor the other.) Thus [Figure, Ground] as iterant becomes the process of viewing the figure and the ground and the figure and ... Similarly [Self, World] is the superposition of self-reference and world-seeing. Section 4 introduces the idea of considering an iterant as a superposition of *temperance* exemplified by $[1, 1]$, and *polarity*, exemplified by $[1, -1]$. In *temperance* the sides of the distinction are equally emphasized. In polarity, they are oppositely emphasized. Any iterant, any ordered distinction, can be regarded as a superposition of temperance and polarity. On changing to the reference frame of another observer, these quantities transform in the *pattern* of time and space in special relativity. Time corresponds to the degree of temperance, and space to the degree of polarity. Thus the *the formal structure of special relativity is in fact present continually in the affairs of decision and evaluation that occur at all levels of human discourse and action.* That this is so, has surely been felt intuitively again and again. This paper provides a mathematical language that explicates this domain, giving conscious access to its implications.

The first appendix outlines the structure of physical relativity. The second appendix shows how spinors and Clifford algebras arise in this calculus of distinctions. Since it is quite significant that spinors (in the form or Pauli spin matrices and their generalizations) should appear so closely with the concept of distinction, I will give a quick sketch of how this comes about: After considering pairs $[A, B]$, it is natural to consider *pairs of pairs.* This arises at once if the pair or iterant is coupled with its conjugate $[B, A]$. This is the level of four-dimensional space. A single polarity such as $\sigma = [1, -1]$ becomes generalized into three basic double pairs

$$\sigma_1 = \begin{array}{ccccccccc} \ddots & & & & & & & & \\ & 1 & & & & & & & \\ & & -1 & & & 0 & & & \\ & & & & 1 & & & & \\ & & & 0 & & & -1 & & \\ & & & & & & & 1 & \\ & & & & & & & & -1 \\ & & & & & & & & & \ddots \end{array} \qquad \longleftrightarrow \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$
\sigma_2 = \begin{array}{ccccccccc}
\ddots &&&&&&&& \\
& 0 &&&&&&& \\
&& 0 &&&&&& \\
&&& 0 &&& 1 && \\
&&&& 0 &&&& \\
&& 1 &&& 0 &&& \\
&&&&&& 0 && \\
&&&&&&& 0 & \\
&&&&&&&& \ddots
\end{array}
\qquad \longleftrightarrow \qquad
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
$$

$$
\sigma_3 = \begin{array}{cccccccc}
\ddots &&&&&&& \\
& 0 &&&&&& \\
&& 0 && \sqrt{-1} &&& \\
&&& 0 &&&& \\
&&&& 0 &&& \\
& -\sqrt{-1} &&&& 0 && \\
&&&&&& 0 & \\
&&&&&&& \ddots
\end{array}
\qquad \longleftrightarrow \qquad
\begin{bmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}
$$

each a special case of the conjugate labelling of the two sides of the distinction created through the waveform pattern associated to an iterant $[a, b]$:

$$
\begin{array}{ccccccccc}
\ddots &&&&&&&& \\
& a &&&&&&& \\
&& b &&& z &&& \\
&&& a &&&&& \\
&&&& b &&&& \\
& \overline{z} &&&& a &&& \\
&&&&&& b && \\
&&&&&&& a & \\
&&&&&&&& \ddots
\end{array}
\qquad \longleftrightarrow \qquad
\begin{bmatrix} a & z \\ \overline{z} & b \end{bmatrix}
$$

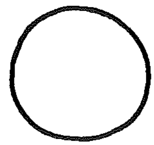
The pattern of the spinor occurs at the level of distinguishing distinguishing. That is, it comes about when a given distinction is seen as the instruction to create a wave-form boundary that can bring forth new distinctions - new forms in a never-ending process of generation.

There are two short appendices on relations with language, conversation and visual perception, and a third discussing observation in quantum mechanics and beyond.

It gives me great pleasure to thank Alex Comfort, David Solzman, James Flagg, Eddie Oshins, Anne Dale, Gary Berkowitz and Haj Ross for helpful conversations leading to this paper.

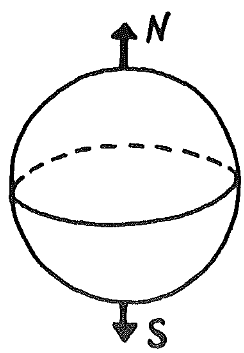
II. Considering a Distinction.

Consider a distinction. For example, consider the distinction between inside and outside that is made (indicated) by a circle drawn in the plane.



Here inside and outside are distinguished by difference in geometric form (bounded inside, unbounded (potentially unbounded) outside).

If the circle is drawn as an equatorial circle upon the surface of a sphere, then the two sides (topological disks) into which the sphere is divided by the equator are in all respects identical, except that one has the pole labelled N (north) while the other has the pole labelled S (south).



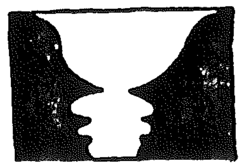
In this case, appropriate indication is required to let the division become a distinction. Without labelling, the upper and lower hemispheres are *indistinguishable*, and this is underlined by the existence of a rotational symmetry (180° turn about an axis through antipodal points on the equator) that interchanges them.

In practice it is the observer who makes the difference, calling out a distinction between the symmetrical and indistinguishable hemi-spheres. This distinction may arise from context, such as the orientation and spin of that sphere.

In other circumstances a choice involving foreground and background is made. One side of the distinction is given prominence, and this may be agreed upon by a group of observers. Thus it is common to foreground the human figure against its three-dimensional background.

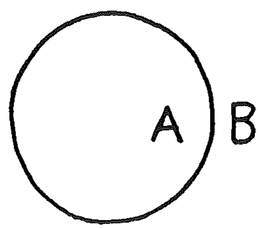
Only rarely, in ordinary speech, does one regard a person as identical to his/her surrounding space rather than the space of the body. Even in sketching, the formal outline of a profile is enough to bring forth the distinction *head in space*.

It is by playing on this tendency that one creates situations where there is an alternation of figure and ground, as in the faces/vase illusion:

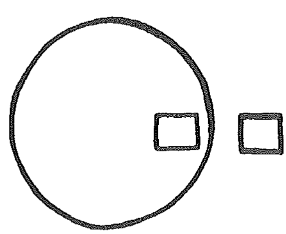


Rather than delineating the fine structure of distinctions (as is done in set theory) I wish to concentrate on a model for the pattern of valuing the sides of a distinction, and the possible alternation of sides.

Let a distinction be given, with sides labelled A and B.



These labels may be distinct, or they may themselves be indistinguishable. Thus the illustration below shows a distinction (the large circle) whose sides are labelled by indistinguishable squares.



Let the distinction with its labels A and B be indicated by the ordered pair $[A, B]$.

Let it be noted that an unlabelled, typographical ordered pair

$$[\ ,\ ]$$

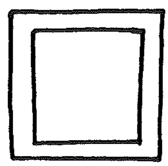
consists in a distinction between interior right-half $,\ ]$ and interior left-half $[\ ,$ plus a distinction between inside and outside that is carried by common conventions about the use of brackets, parentheses and linear notation. Thus the given distinction can be taken to be the unlabelled ordered pair itself.

In fact (compare [3]) this entire discussion could be carried out using only the empty ordered pair. For example,

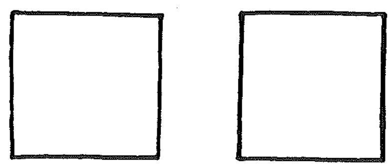
$$\Big[\ ,\ [\ ,\ ]\ \Big]$$

is a labelling that distinguishes the right-half from the left-half via the presence of an empty ordered pair in the right-hand compartment.

Note that while any two empty ordered pairs are indistinguishable, they may nevertheless become distinguished from one another through their mutual relationship in the indicational space. Thus any two boxes □ are typographically indistinguishable. Nevertheless, nested boxes



are clearly distinct from adjacent boxes



with regard to the context of the plane. In this way distinction and indication are mutually intertwined. (Compare [18].)

To return to the ordered pair $[A, B]$, we imagine that A and B are *evaluations* of the distinction. An evaluation may be denoted by word(s) in context (fine, precise, strange), by conventional symbols (!, *, A^+, ?), by numbers ($-3, 100\%, 3.14159$) or by another distinction (He wears his own picture on the security badge.).

Once the sides and evaluations of a distinction are symbolically or perceptually fixed, there is still lee-way for an observer to emphasize one side and (correspondingly) de-emphasize the other. Thus I shall model *the action of an observer* by the transformation

$$\mathcal{O}[A, B] = [\lambda A, \lambda^{-1} B]$$

where λ is a real or complex number. Implicit in this model is the "stationary observer" who sees $[A, B]$ and the "moving observer" who sees $[\lambda A, \lambda^{-1} B]$. This transformation is natural. Let me tender persuasions.

Note that $\lambda A / \lambda^{-1} B = \lambda^2 (A/B)$. The ratio λ^2 gives the change in the ratio A/B. If we are concerned always and only with relative evaluation then it is λ^2 that is the important quantity. A transformation of the form $[RA, SB]$ can be re-written as a constant multiple of one of the form $[\lambda A, \lambda^{-1} B]$. Since we are only concerned with relative changes of evaluation, the form $\mathcal{O}[A, B] = [\lambda A, \lambda^{-1} B]$ is sufficient at a numerical or algebraic level.

III. Iterants and Invariants.

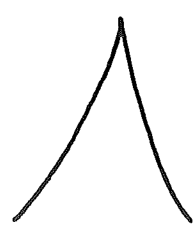
Let me summarize the discussion so far. We take as given a distinction and a context or "stationary observer" that assigns value to the sides of this distinction in the form $[A, B]$. We further assume that other observers will numerically or algebraically change the emphasis on these sides from $[A, B]$ to $[\lambda A, \lambda^{-1} B]$. Since we are only concerned with the ratio $\lambda A / \lambda^{-1} B = \lambda^2 (A/B)$ it suffices to allow the transformation to take the form $\mathcal{O}[A, B] = [\lambda A, \lambda^{-1} B]$.

If A and B are themselves numerical evaluations, then *the product*

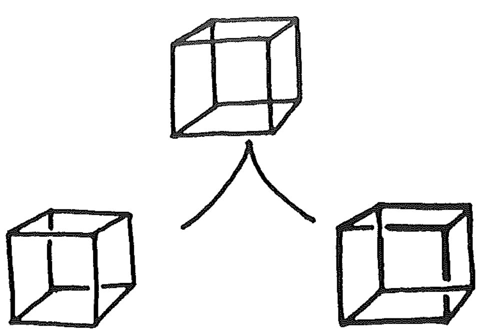
$$(\lambda A)(\lambda^{-1} B) = AB$$

is an invariant of the transformation. This remark re-opens the original source of the discussion in another way, for notice that $AB = BA$ implies that $[A, B]$ and $[B, A]$ receive the same invariant product. The forms $[A, B]$ and $[B, A]$ correspond to reversal of figure and ground as in

$$\cdots ABABABABA \cdots$$



$[A, B]$ $\qquad\qquad\qquad$ $[B, A]$



Here $[A, B]$ denotes the distinction that emphasizes the order $A \to B$ (a freeze of the oscillation $\cdots ABABABA \cdots$).

The product $I[A, B] = AB$ is invariant under conjugation

$$\overline{[A, B]} = [B, A].$$

Furthermore,

$$I[A, B] = I[B, A],$$

and the transformation

$$\mathcal{O}[A, B] = [\lambda A, \lambda^{-1} B]$$

extends this invariance. That is, $I(\mathcal{O}[A, B]) = (\lambda A)(\lambda^{-1} B) = AB = I([A, B])$.

In this context, I call $[A, B]$ an *iterant*. An iterant is neither A nor B but indicates a directionality from A to B, or a distinction between A and B. Both $[A, B]$ and $[B, A]$ are indicators or views of the underlying vibration

$$\cdots ABABABA \cdots$$

By choosing an ordering BA or AB, they become distinct. By obliterating this ordering as in the product $I[A, B] = AB = BA = I[B, A]$, this distinction is lost. But invariance is gained!

And from this first invariance arises the group of transformations

$$G = \{\mathcal{O}_\lambda | \mathcal{O}_\lambda[A, B] = [\lambda A, \lambda^{-1} B]\}$$

with $I\mathcal{O}_\lambda[A, B] = I[A, B]$ and

$$\mathcal{O}_\lambda \mathcal{O}_\mu = \mathcal{O}_{\lambda\mu} \quad (\text{i.e. } \ \mathcal{O}_{\lambda\mu}[A, B] = [\lambda\mu A, \lambda^{-1}\mu^{-1} B])$$

for all λ, μ.

Thus from the primitive condition of observation of a ground that may be distinguished in conjugate ways, arises an invariant of conjugation and from this an entire group of transformations preserving this invariant. This group G is isomorphic to the Lorentz group for the Minkowski plane (See Appendix 1.).

IV. Temperance and Polarity.

The analogs of time (t) and space (x) in this discussion are *temperance* and *polarity*. The exemplar of temperance is the pair $1 = [1, 1]$. The evaluations on the sides of $[1, 1]$ are indistinguishable. The sides are distinguished only through the framework of the brackets. The exemplar of polarity is $\sigma = [1, -1]$. The sides are directly opposite.

Let pairs be added coordinate-wise, and multiplied coordinate-wise. Thus

$$[A, B] * [C, D] = [AC, BD]$$
$$[A, B] + [C, D] = [A + C, B + D].$$

Note that $\sigma * \sigma = [1, -1] * [1, -1] = [1, 1] = 1$.

Proposition 4.1. Any pair $[A, B]$ can be written uniquely as a superposition (sum) of a *tempered iterant* $[t, t] = t1 = t$ and a *polar iterant* $[x, -x] = x[1, -1] = x\sigma$. That is, $[A, B] = t + x\sigma$ where

$$t = (A + B)/2 \text{ and } 1 = [1, 1]$$
$$x = (A - B)/2 \qquad \sigma = [1, -1].$$

Hence

$$t + x = A$$
$$t - x = B.$$

PROOF: Certainly,

$$[t + x, t - x] = [t, t] + [x, -x] = t + x\sigma.$$

Conversely, if $t + x = A$ and $t - x = B$, then $t = \frac{1}{2}(A + B)$ and $x = \frac{1}{2}(A - B)$. This completes the proof.

The value t will be called the *temperance* of $[A, B]$ while $x = \frac{1}{2}(A - B)$ will be called the *polarity* of $[A, B]$. We wish to see how the temperance and polarity transform under an element $\mathcal{O}_\lambda$ of the *Iterant Group*

$$G = \{\mathcal{O}_\lambda | \mathcal{O}_\lambda[A, B] = [\lambda A, \lambda^{-1} B]\}.$$

To see this, note that $\mathcal{O}_\lambda$ may be regarded as multiplication by the element $[\lambda, \lambda^{-1}]$. That is $\mathcal{O}_\lambda[A, B] = [\lambda, \lambda^{-1}] * [A, B]$. $([C, D] * [A, B] = [CA, DB])$.

Definition 4.2. The *velocity* v of a pair $[A, B]$ *is the ratio* $v = x/t$ of its *polarity* to its *temperance*: $v = \left(\frac{A-B}{A+B}\right)$. In this terminology,

$$v[1, 1] = (1 - 1)/(1 + 1) = 0$$
$$v[1, -1] = 2/0 = \infty$$
$$v[-1, 1] = -2/0 = -\infty$$
$$v[1, 0] = 1$$
$$v[0, 1] = -1.$$

Proposition 4.3. The quantity $t^2 - x^2$ is invariant under transformations of the iterant group $G = \{[\lambda, \lambda^{-1}]\}$.

PROOF: $[A, B] = [t + x, t - x]$.

$$AB = (t + x)(t - x) = t^2 - x^2$$

is invariant. Q.E.D.

Proposition 4.4. Let $v = v(\lambda)$ denote the velocity of the pair $[\lambda, \lambda^{-1}]$. Then

$$[\lambda, \lambda^{-1}] = \frac{1 + \sigma v}{\sqrt{1 - v^2}}$$

where $1 = [1, 1]$, $\sigma = [1, -1]$ are the basic tempered and polar iterants.

PROOF: Let $[\lambda, \lambda^{-1}] = [t + x, t - x]$. Then $1 = \lambda\lambda^{-1} = (t + x)(t - x) = t^2 - x^2$. Hence $\frac{1}{t^2} = 1 - (x^2/t^2) = 1 - v^2$. Hence $t = 1/\sqrt{1 - v^2}$. Thus

$$\begin{aligned} [\lambda, \lambda^{-1}] &= t + \sigma x \\ &= t(1 + \sigma v) \quad (v = x/t) \\ &= \frac{1 + \sigma v}{\sqrt{1 - v^2}}. \end{aligned}$$

Proposition 4.5. Let $[A, B] = t + \sigma x$ be any iterant with temperance t and polarity x. Let $\mathcal{O}_\lambda = [\lambda, \lambda^{-1}] \in G$ be an element of the iterant group with velocity v. Then

$$\mathcal{O}_\lambda[A, B] = t' + \sigma x'$$

where

$$\begin{aligned} t' &= (t + vx)/\sqrt{1 - v^2} \\ x' &= (x + vt)/\sqrt{1 - v^2}. \end{aligned}$$

PROOF: From 4.4 we have

$$\mathcal{O}_\lambda = [\lambda, \lambda^{-1}] = \frac{1 + v\sigma}{\sqrt{1 - v^2}}.$$

Thus

$$\begin{aligned} \mathcal{O}_\lambda[A, B] &= \left(\frac{1 + v\sigma}{\sqrt{1 - v^2}}\right) * (t + x\sigma) \\ &= \frac{t + vx\sigma * \sigma + (x + vt)\sigma}{\sqrt{1 - v^2}} \\ &= \left(\frac{t + vx}{\sqrt{1 - v^2}}\right) + \left(\frac{x + vt}{\sqrt{1 - v^2}}\right)\sigma \quad (\sigma * \sigma = 1). \end{aligned}$$

Hence

$$t' = \frac{t + vx}{\sqrt{1 - v^2}}, \; x' = \frac{x + vt}{\sqrt{1 - v^2}}.$$

As the reader will undoubtedly recognize, the transformations in 4.5 are exactly the classical Lorentz transformation when the speed of light is normalized to unity. In conjunction with the remarks in Appendix 1 to this paper, this constitutes a derivation of these transformations based on the iterant calculus.

The mathematical ground of our discussion has been the concept of distinction, and how the evaluation of distinctions leads naturally to the concepts of iterant, iterant group, temperance and polarity.

Remark. It is worth noting the following formula for an element $[\lambda, \lambda^{-1}] \in G$. If v is the velocity of $[\lambda, \lambda^{-1}]$, then $v = \frac{\lambda - \lambda^{-1}}{\lambda + \lambda^{-1}} = \frac{\lambda^2 - 1}{\lambda^2 + 1}$. Hence $\lambda^2 = \frac{1+v}{1-v}$. Since λ is real we must have $|v| \leq 1$. Only by extending G to complex values can we obtain velocities greater than 1.

It may be quite interesting epistemologically to compare the concept of tachyon ($v >$ light speed) in physics with the concept of imaginary values ($\sqrt{-1}$) in the evaluation of distinctions. Note also that an iterant with $v > 1$ has more polarity than temperance. Real transformations live in the realm where temperance dominates polarity.

Remark. It is easy to see that if $v_1 = \text{velocity}[\lambda_1, \lambda_1^{-1}]$ and $v_2 = \text{velocity}[\lambda_2, \lambda_2^{-1}]$ then $v = \text{velocity}[\lambda_1, \lambda_1^{-1}] * [\lambda_2, \lambda_2^{-1}]$ is given by the formula $v = \frac{v_1 + v_2}{1 + v_1 v_2}$. It is worth thinking about this formula for the relativsitic addition of velocities in the context of velocity as ratio of polarity to temperance. In fact, the velocity of an arbittary pair $[A, B]$ is given by the formula

$$v[A,B] = \frac{\text{polarity}[A,B]}{\text{temperance}[A,B]} = \frac{(A-B)/2}{(A+B)/2} = \frac{A-B}{A+B}$$

and we have the

Proposition 4.6. Let $v_1 = v[A,B]$, $v_2 = v[C,D]$ and $v_3 = v([A,B] * [C,D])$ Then

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2}.$$

PROOF:

$$\frac{v_1 + v_2}{1 + v_1 v_2} = \frac{\left(\frac{A-B}{A+B}\right) + \left(\frac{C-D}{C+D}\right)}{1 + \left(\frac{A-B}{A+B}\right)\left(\frac{C-D}{C+D}\right)}$$

$$= \frac{(A-B)(C+D) + (A+B)(C-D)}{(A+B)(C+D) + (A-B)(C-D)}$$

$$= \frac{AC + AD - BC - BD + AC - AD + BC - BD}{AC + AD + BC + BD + AC - AD - BC + BD}$$

$$= \frac{2AC - 2BD}{2AC + 2BD} = \frac{AC - BD}{AC + BD}$$

$$= v[AC, BD]$$

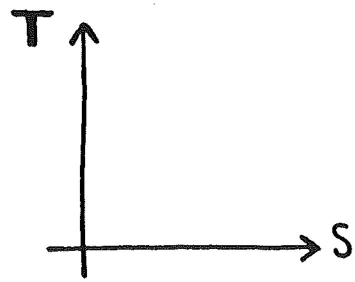
$$= v([A,B] * [C,D])$$

$$= v_3.$$

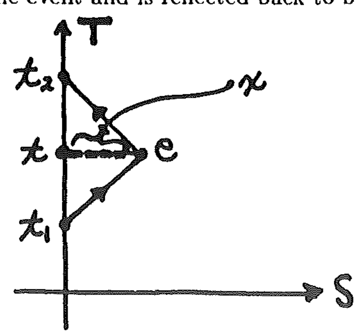
Q.E.D.

Appendix 1. Physical Relativity.

Let



denote a two-dimensional time-space frame. Let the speed of light be taken conventionally to be unity. An event e is determined, for a given observer, by two times $[t_2, t_1]$. The time t_1 is the time at which a signal is sent. The signal moves outward at light-speed, encounters the event and is reflected back to be received by the observer at time t_2.

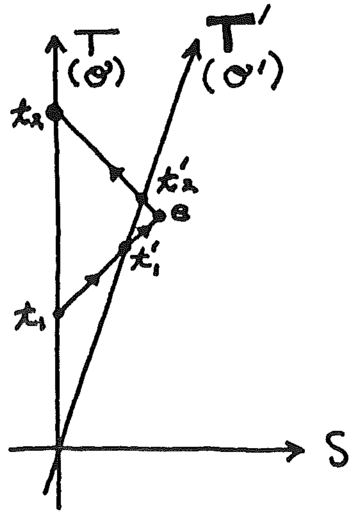


$$e = [t_2, t_1] = [t + x, t - x]$$
$$e = t + x\sigma$$

It is manifest from the diagram above that $t_2 = t + x$ and $t_1 = t - x$. Thus the event e may be identified as the iterant $[t_2, t_1]$ and the *time t of e* is the *temperance of e* while the *distance x of e* is the *polarity of e*. This gives a physcial interpretation of iterants in terms of space time coordinates.

That the Lorentz transformation has the form $T[t_2, t_1] = [\lambda t_2, \lambda^{-1} t_1]$ follows from the principle of relativity: (See [2], [6].) If observers $\mathcal{O}$ and $\mathcal{O}'$ have a constant velocity v of separation then light signals separated by time Δt in $\mathcal{O}$'s frame will be received in separation $\Delta t'$ in $\mathcal{O}'$'s frame. The ratio $K = \Delta t'/\Delta t$ is dependent only on the relative velocity. $K = \frac{\Delta(\text{signals received})}{\Delta(\text{signals sent})}$. It then follows as shown in the diagram below that

$$[t_2', t_1'] = [K^{-1} t_2, K t_1].$$



$$t_1' = Kt_1$$
$$t_2 = Kt_2'$$
$$(\Rightarrow t_2' = K^{-1}t_2)$$

In this diagram an event e receives a signal sent by the observer $\mathcal{O}$ at time t_1. This same signal can be regarded as sent by $\mathcal{O}'$ at time t_1'. Similarly, the reflected signal is received by $\mathcal{O}'$ at t_2' and by $\mathcal{O}$ at t_2. Here t_1, t_2 refer to times in $\mathcal{O}$'s frame and t_1', t_2' refer to times in $\mathcal{O}'$'s frame. Then we see that $t_1' = Kt_1$ and $t_2' = K^{-1}t_2$. Thus the group of Lorentz transformations is identical to the group of iterant transformations.

Appendix 2. Spinors and Clifford Algebra.

In this appendix I wish to discuss how the mathematical structures of spinors and Clifford algebras arise naturally from our discussion of distinctions and iterants. To see this, recall that an iterant $[A, B]$ is resolved into its temporal (t) and polar (x) components via

$$\begin{aligned}[A, B] &= [t + x, t - x] \\ &= t[1, 1] + x[1, -1] = t + x\sigma.\end{aligned}$$

We let 1 denote the iterant $[1, 1]$ since this iterant acts as an identity element in the algebra of iterants. We let $\sigma = [1, -1]$ and note that $\sigma * \sigma = [1, -1] * [1, -1] = [1, 1] = 1$.

In special relativity σ *corresponds to a direction in space.* This provides the clue for a generalization. Let V^n be an n-dimensional vector space over the real numbers. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a basis for V^n and let us suppose that elements $\sigma = x_1\sigma_1 + \ldots + x_n\sigma_n \in V^n$ are part of an algebra structure $\mathcal{A}^n \supset V^n$ satisfying the following properties:

1. $\mathcal{A}^n$ is closed under multiplication and addition.

2. If $\sigma = x_1\sigma_1 + \ldots + x_n\sigma_n$ and $||\sigma||^2 = x_1^2 + x_2^2 + \ldots + x_n^2 = 1$ then $\sigma * \sigma = 1$.
3. $\mathcal{A}^n$ is associative under multiplication and addition. Multiplication distributes over addition. For real numbers $x, y : (xv)(yw) = (xy)(vw)$ where $v, w \in V^n$. Scalars such as 1 form an extra dimension so that $\{1, \sigma_1, \ldots, \sigma_n\}$ is linearly independent.

Call $\mathcal{A}^n$ a *direction algebra for* V^n. Any direction σ in V^n will have square equal to 1. Thus 1 and σ can be used as an iterant basis for any spatial direction σ.

Proposition. $\mathcal{A}^n$ will be a direction algebra exactly when

(i) $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2 = 1$

and

(ii) $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for $i \neq j$.

Hence $\mathcal{A}^n$ is a Clifford algebra.

PROOF: (i) follows immediately from condition number 2. To see (ii) note that if $\sigma = x_1\sigma_1 + \ldots + x_n\sigma_n$ with $x_1^2 + \ldots + x_n^2 = 1$ then

$$1 = \sigma * \sigma = x_1^2\sigma_1^2 + \ldots + x_n^2\sigma_n^2 + \sum_{i \neq j} x_i x_j \sigma_i \sigma_j$$

$$1 = 1 + \sum_{i \neq j} x_i x_j \sigma_i \sigma_j.$$

Hence $0 = \sum_{i<j} x_i x_j (\sigma_i\sigma_j + \sigma_j\sigma_i)$. Since this equality is true for all choices of x_i and x_j, it follows that $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$ whenever $i < j$. This completes the proof.

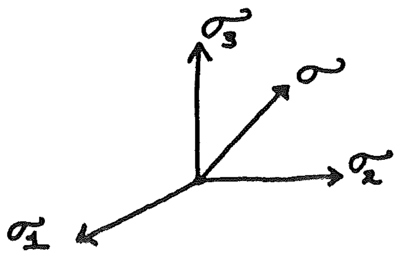
This argument is formally identical to Dirac's production of spinor formalisms in his theory of the electron [5]. Here however, we have made the argument on the ground of distinctions, iterant algebra and motivations from special relativity. In this generalization we have seen how to associate vectorial elements $t + x\sigma$ where σ ranges over an n-dimensional vector space.

Curiously, the basic iterant structure has now become parametrized by the space V^n. We have weighted distinctions or iterants of the form

$$[A, B]_\sigma = \left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right)\sigma.$$

Each distinction has become equipped with a vector direction in a higher dimensional space.

All of this is perfectly understandable in the context of special relativity. Here $n = 3$ and we can even obtain closure in $V = \mathbf{R}^3$ via $\sigma_1\sigma_2 = -\sqrt{-1}\sigma_3$, forming the Pauli algebra.



$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$$
$$\sigma_1\sigma_2 = -\sqrt{-1}\sigma_3$$
$$\sigma_2\sigma_3 = -\sqrt{-1}\sigma_1$$
$$\sigma_3\sigma_1 = -\sqrt{-1}\sigma_2$$

In fact the Pauli algebra is represented by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$$

so that

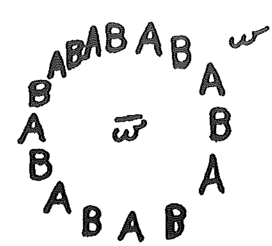
$$e = t + x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} t+x & y+\sqrt{-1}z \\ y-\sqrt{-1}z & t-x \end{pmatrix}.$$

In this way an event is 4-dimensional spacetime and is expressed as a Hermitian matrix

$$H = \begin{pmatrix} t+x & y+\sqrt{-1}z \\ y-\sqrt{-1}z & t-x \end{pmatrix}.$$

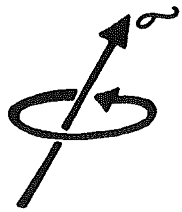
This is an extraordinary form. We see, within it, the original iterant structure $[t+x, t-x]$ displayed as the main diagonal. This diagonal is flanked by a complex number $w = y + \sqrt{-1}z$ and its conjugate $\overline{w} = y - \sqrt{-1}z$. Thus the original iterant becomes the boundary dividing the sides of a distinction labelled by $\overline{w}$ and w:

$$\begin{array}{ccccc} & \vdots & & & \\ & A & & & \\ & B & & & \\ & A & & & \\ \overline{w} & B & w & \text{or} & \\ & A & & & \\ & B & & & \\ & A & & & \\ & \vdots & & & \end{array}$$



Here we move to the realm on pattern. The boundary has resolved into vibration, and the sides are conjugate imaginaries.

This is but one take, one possible viewpoint, on the event as complex pattern. By associating a direction or *axis* σ to a distinction D,



$$D = [A, B]_\sigma$$

that distinction partakes of a larger domain of interaction, and of the possible combination and recombination into new patterns in a hierarchy of form. (It would be well to compare this abstraction with the image of a gyroscope creating its own distinct axis in space.)

It is here that this discussion touches the structure of the combinatorial hierarchy [1], but I have only indicated the general principles of this correspondence. More work is needed in this domain.

Appendix 3. Language.

This appendix begins a discussion of certain issues in linguistics that are related to our relativistic calculus of distinctions, and to the algebra of iterants. Iterant pairs occur continually in language. For example: "scotch and soda." One does not say "soda and scotch." One can make a list:

scotch/soda

observer/observed

here/now

heaven/earth

name/that which is named

...

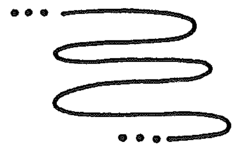
It is fascinating to speculate on the nature of these orderings. They do not constitute a preference for one member of the pair - only the fact of ordering. This provides the first non-numerical level of our structure. It remains to be seen whether the patterns of algebraic/numerical relativity apply in the domain of speech.

In terms of iterants, one may speculate that a sentence is a freeze on the process exemplified by its own unfolding - through repetition, sounding, and the associations of meaning. The simplest unfolding is pure repetition. Thus "I am that." unfolds to become the process

...I am that I am that I am that I am...

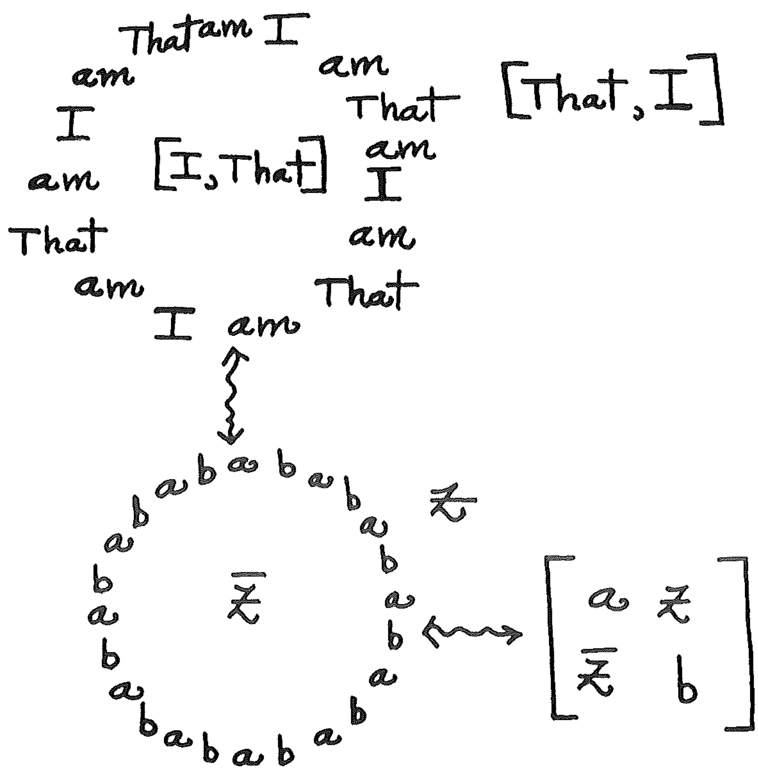
In this sense our speech is a necessary freeze or projection of the vibratory biological process of being in the world.

What process does the mind echo in response to each fragment of speech? The simple sentence "I am that." tempts this author to read it forward, then backward, then forward, in a weaving process



that forms

...I am that am I that am I that...



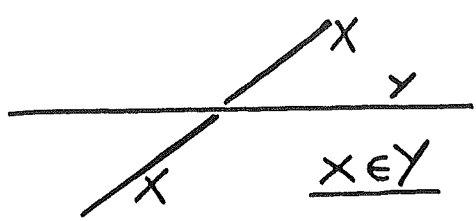
In this explication of the sentence, we already see the advent of the higher dimensional event-structure exemplified by the Hermitian matrix of the last section. The specifics of *that* and I can be chosen, ordered and frozen out. But being (am) is verbal, vibratory. To tell the story well it had best remain symbolized as boundary, glue, process and divider.

In this sense even a single sentence becomes the possibility for an event that is a conversation, a conversation among those embodiments of mind that would pick up its multifold components and discuss them in the chorus of its parts.

It is in the arena of ***conversation*** that the ideas and formalisms discussed in this paper come forward most strongly. A conversation is a taking of turns, a passing back and forth between the viewpoints of two observers. Yet, in the form of the conversation, these two viewpoints become unified into a context that is the conversation. The multitude of transformations of emphasis become one pattern of agreement that unfolds into the particpants. The participants themselves become sides of the distinction that the conversation is. And "they" move back and forth in the change of emphasis that is the rhythm of exchange, the taking of turns.

In the conversation there is ample room for a multidimensional patterned space of relativity, less restricted than the more numerically-based space for physics. These are only hints in the direction of linguistic context, yet even in the domain of hints the ideas beckon to be pursued and articulated. I believe that the key to this articulation will arise through the dvelopment of appropriate non-numerical (diagrammatic, geometric, topological) mathematics, and through the interest generated by these patterns of interconnection. The world of our speaking is the world of our being.

A few words about the non-numerical approach: Diagrammatic formalisms can be of great use. Thus the *linking* of participants A and B in conversation can be diagrammed as A  ^{B}. This is actually the diagram of a link in 3-space, but it can also be interpreted as the mutual relationship that each participant has to the conversation as a whole. In fact, we may formally let $X \in Y$ denote the relation that X *undercrosses* Y:



This gives rise to a non-standard set theory where a set may belong to itself:

$X \in X$

$X = \{X\}$

and two sets may each be members of the other:

$A = \{B\}$

$B = \{A\}$

Here is a non-numerical context for the structures of conversation. And within this context the transformations and patterns of relativity will appear again. (Note that the two viewpoints $A = \{B\}$, $B = \{A\}$ correspond to A *listening to* B and to B *listening to* A - two possible interpretations of $[A, B]$ and $[B, A]$. Compare [9], [11], [13].)

Appendix 4. Binocular Vision.

I believe that it is worthwhile to compare the calculus of distinctions given here with a model for binocular visual perception outlined in *Foundations of Cyclopean Perception* [7] by Bela Julez. Julez proposes a model for individual eye's perceptions as a grid or lattice of polarity choices. Thus at each lattice point is chosen $[1, -1]$ or $[-1, 1]$. The two grids (left eye and right eye) are compared by a superposition that (in analogy to magnetic fields of the magnets $[1, -1]$, $[-1, 1]$) tends to *rotate* each of the individual polarities in the lattice - trying to bring the two views in alignment. In our terms, the result is to replace each individual polarity $[1, -1]_\sigma$ by a *transform* $[\lambda_\sigma, -\lambda_\sigma^{-1}]$ with the value of λ_σ corresponding to the apparent *depth* of this point in the perceptual field. Here is seen a deeply contextual version of our model, and much possibility for further questions and creations.

Appendix 5. Observations and Quantum Mechanics.

We have used an abstraction of an observer as an operator that changes the emphasis on two sides of a distinction as in $\mathcal{O}[A, B] = [\lambda A, \lambda^{-1} B]$. In this sense the (abstract) observer is also a distinction, and we have made the identification $\mathcal{O} = [\lambda, \lambda^{-1}]$ in the iterant algebra so that $\mathcal{O}[A, B] = \mathcal{O} * [A, B] = [\lambda, \lambda^{-1}] * [A, B]$. In this epistemology the observer is an operator *at the same level* as that which is observed.

It is also possible to consider observation of the sides of a distinction. Thus, in the case of ordered pairs, we can define $\pi_1[A, B] = A$ and $\pi_2[A, B] = B$. When the elements of the ordered pairs are themselves distinctions, this is an appropriate move. Note that this sort of operation can also be identified with the extreme case of an observer who gives *zero-emphasis* to one side or the other as in

$$P_1[A, B] = [A, 0]$$

$$P_2[A, B] = [0, B].$$

We have tacitly excluded operations of this type from the discussion because they are not invertible - information about the contents of the side that is assigned zero is lost. Let us call operations of type P_1 and P_2 *projections*. Note that

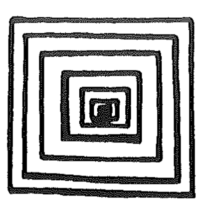
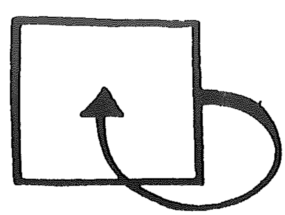
$$\begin{aligned} P_1^2 &= P_1 \\ P_2^2 &= P_2 \qquad \text{and} \\ P_1 + P_2 &= 1 \end{aligned}$$

in the sense that $P_1[A,B] = [1,0] * [A,B]$ and $P_2[A,B] = [0,1] * [A,B]$ so that $P_1 \equiv [1,0]$, $P_2 \equiv [0,1]$ and $[1,0] * [1,0] = [1,0]$ etc. Projections of this type form the basis of the model for observation in quantum theory [5]. (That is, in quantum theory an observable is an operator H on a Hilbert space and the process of observation corresponds to the projections to the eigenspaces of this operator.)

In the realm of value and distinction this spearation into "relativistic" and "quantum mechanical" operators is clearly a matter of choice. Both are special cases of multiplication in the iterant algebra. Nevertheless, even here the meaning takes a subtle shift. Since projection loses information it must be treated differently than a balanced change of emphasis. It is this basic mathematical difference that informs the greater differences in the corresponding physical theories.

At the level of patterns of distinction the consideration of an operator P such that $P(P(X)) = P(X)$ for all X ($P^2 = P$) leads to the more general idea of solutions to $P(V) = V$. If V is non-numerical, then P may be a rotation or other self-similarity transformation. Thus $\text{Rot}[\rightarrow] = \leftarrow$ (rotate by 180°) implies that $\text{Rot}[-] = -$ since an undirected segment is *indistinguishable from itself* after a rotation by 180°.

Similarly, if $P =$ "put a box around it" as in P  = , then $PJ = J$ when J is an infinite nest of boxes:

$J =$  $=$ 

J is an *eigenform* for the operator P. The notion of eigenform goes back to Heinz von Foerster in the cybernetic domain [16]. See also [7] and [9], [11], [12], [13].

The point I want to make is that as soon as we begin to consider projection operators, other domains begin to open up before us. In the realm of distinction and pattern the development is natural and rapid - producing many-valued logics, fractal geometry and recursions of all kinds. In physics the pattern of the projection operation is at the heart of quantum theory, and these formal developments have been restricted by the requirement of numerical eigenvalues (versus eigenforms) to ***hints in the formalism*** such as Dirac's formal projection operators $|\phi\rangle\langle\psi|$. (Here $\langle\psi||\phi\rangle = \langle\psi|\phi\rangle = 1$ so that if $P = |\phi\rangle\langle\psi|$ then $P^2 = |\phi\rangle\langle\psi||\phi\rangle\langle\psi| = |\phi\rangle 1 \langle\psi| = |\phi\rangle\langle\psi| = P$ whence $P^2 = P$.) In the patterned realm we can go ahead and investigate the entire panoply of distinctions: creating, evaluating and projecting process and possibility. The resonance with physics will resound throughout in a great fugue.

One last comment about eigenvalues: In Appendix 2 I pointed out how the event in 4-dimensional spacetime naturally takes the form

$$H = \begin{bmatrix} a & z \\ \bar{z} & b \end{bmatrix} = \begin{bmatrix} T+X & Y+\sqrt{-1}\,Z \\ Y-\sqrt{-1}\,Z & T-X \end{bmatrix},$$

and that H could be viewed as a second-level iterant (pairing conjugates across the vibratory pattern boundary $\ldots ababa \ldots$).

However, H is a Hermitian matrix and, as such, one can enquire of it its numerical eigenvalues. They are: $T \pm \sqrt{X^2+Y^2+Z^2}$. In other words, the event *when operating on* (observing) *itself* can determine its time T and geometrical radius $R = \sqrt{X^2+Y^2+Z^2}$. In so-doing the event becomes a local observer. In this sense, the local observer is linked with quantum mechanics. (Compare [9]).

On extending consideration to include the eigenforms, the relations of our discussion with patterns of language and communication become more significant. A word, a sentence, a paragraph or a book, each is a structure whose implicit reference is unfolded (projected) by context and observer. Thus distinction and pattern work throughout. Each form may be seen as a pattern of patterns just as the sentence unfolds into words and these words into ideas that inform the sentence that inform the idea of the sentence, that inform the words. This implicate structure unfolds in deep resonance with creation of eigenform from its own operator. For more about this context see [9], [11], [12], [13], [17].

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What Do the Bits in the Combinatorial Hierarchy Mean?

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While I am delighted at the accelerating progress in extracting the physics out of the combinatorial hierarchy, the mechanist in me is unhappy. It wants to know what physical interpretation to give to the bits which figure so prominently in the combinatorial hierarchy's view of the world. Undoubtedly I am indulging a certain prejudice here toward the kind of model represented by the computational metaphor, where bits are bits, as it were. My aim with this contribution is to get at the relationship between the two models. I will use, in the following, the string world created by Program Universe, which I require to be homomorphic to the algebraic construction of the combinatorial hierarchy.

I will first demonstrate that the bits of this string universe are not state bits in the usual sense, i.e., *direct* encodings of some physical reality. The bits in the bit strings have two sources, Tick and the events resulting from Pick. Tick appends an *arbitrary* bit to each string. The events resulting from Pick create new bit strings, each bit of which is the exclusive-or of the bits in the two Picked strings. If we go back to the very beginnings of the generation of the string universe, it is possible to argue that the strings of level 1 (01, 10, 11) have a certain inevitability. But as we can cross from level 1 to level 2 only via Tick, from then on every bit, each of which is created via exclusive-or, can be traced back to two bits, from two different strings, which themselves came from Tick. Thus each and every bit in each and every string is fundamentally arbitrary, it being moot whether this arbitrariness is direct (coming from Tick) or indirect (coming from Pick). Since the very basis of scientific endeavor is that what we see in the world is *not* arbitrary, the bits in the bit strings cannot be state bits in the usual sense of the term.

This is not of course to imply that the bit strings are not faithfully tracking the development of our reality. I take it as more or less an article of faith that the combinatorial hierarchy [hereafter, "CH"] does so track reality. For example, certain strings are correlated with certain others precisely by the 'indirection' created by Pick's events, and this correlation is reflected in the resulting discriminately closed subsets and labelled ensembles. Rather, given that these bits are not state bits in the way I use the term, the question is what is the relationship between these bits and "real" state bits.

I claim that the Reader/Writer interaction shown above is exactly the mechanism by which arbitrary outcomes can be generated. This claim rests on the following reasoning: the initial states of the Reader, the Writer, and the Memory are all known, and yet the outcome of the decision at the right (which tests the leading bit in the arriving message) is unpredictable. In the case at hand, the Reader's state might be >01>> (i.e., the Writer wrote before the Reader read), or it might be >0>> (i.e., the Reader read [nothing] before the Writer wrote). It can be shown that by assuming only the indivisibility of Reading/Writing over the memory (i.e., that these are *discrete* operations) and equiprobable arrival (from the point of view of the memory) of the Reader and the Writer, the probability of either of the decision outleg's being taken by the arriving Reader message is 0.5 apiece.

Pierre Noyes has objected to this reasoning because the term "before" is not well defined, in that temporal relationships in the CM appear first at the process, and not at this basic actor, level. I accept the criticism but not its rationalization, or put another way, I think that the intuitive notion "randomness" has its roots in unpredictability, and there is hence very definitely a notion of sequence, and hence of time, implied. We therefore could simply accept the scenario at face value and say that process interaction without an accompanying time scale is *by definition* unpredictable.

Howsoever, I think it is possible to relate this mechanism to the CH. If we go back to the states involved, we could say that there are two distinguishable outcome states, which can be expressed equivalently, either in terms of which outleg of the decision actually comes to carry the Reader message, or in terms of the two Reader process states >0>> and >01>>. These two distinguishable outcomes are the result of two *indistinguishable* input states: (Reader = >0>>, Writer = >1>>) and (Reader = >0>>, Writer = >1>>), which are identical. So, in CH terms, we go from (e.g.) "1" to "10" and "11" as a result of the interaction. In fact, one can imagine an extended network of such memories, and (say) two input messages, and use the CH to classify the various outcomes at the end.

If this last is right, then we have here the basic connection between a state oriented model, exemplified by the computational metaphor, and the combinatorial hierarchy. Two things should be mentioned about the mechanism just proposed: (1) that the result of the non-deterministic interaction is 'observed' by an observer process (positioned at the decider outputs) which is part of the same system, and hence this observer is not a 'separated observer' in the standard quantum mechanical sense [3]; (2) while this mechanism and interpretation is compatible with McGoveran's views [4], the latter are I think preferable for theoretical work at the current stage of development.

Perhaps I can make the intended meaning of "real" state bits clearer by the following example, which is the computational metaphor's [hereafter, "CM"] way of expressing the interaction between two processes:

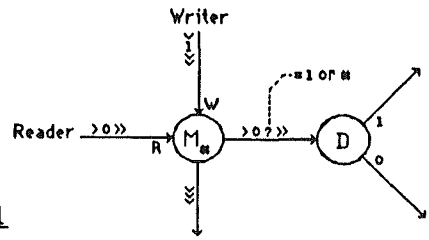

Figure 1

Here we see a Reader process with initial state >0>> heading (as the >'s indicate) from left to right toward a memory which is initially empty (#). From above, a Writer process with initial state >1>> is also heading for the memory. Let us for the moment assume that the Writer writes its '1' in the memory before the Reader attempts to read the memory. After the Writer writes, the memory's state is '1', and the Writer's "state" is null, which is interpreted to mean that the Writer process as such has ceased to exist. After the Reader reads the memory, the Reader's state is >01>> and the memory's is '#'.

In my view the CH is a remarkable scheme for classifying the various ways that things can differ from each other. Exclusive-or is the operator which distinguishes whether two things are the same or different, and the bits in the CH bit strings reflect the distinctions which exist, or might exist, as a system evolves. On the other hand, as is hopefully made clear in the Reader/Writer interaction above, the bits in the CM do *not* have this character. They are what they are, and do not reflect any kind of *difference* between things. It is in this sense that I feel that the CH is an abstraction, and the CM is concrete. It is in this sense that the bits of the CM are "real" and those of the CH not.

Let me come at this another way. The CH, like all other combinatorial entities, appeals to the proverbial balls and urn as a way of concretizing the arbitrary selection processes of which it avails itself. The actual *mechanism* by which balls are chosen from said urn is however never specified. An appeal is made to the intuitive notion of "random selection" or "arbitrariness", and the resulting combinatorics thereafter presented as a faithful portrayal of the *results* of such a selection process. I do not doubt the faithfulness of the portrayal. Rather, it is the lacking detail of the selection mechanism which interests me.

I have generally presented the CM in terms of a *fixed* digraph of actors. Such a graph, I have shown, can exhibit structure in the form of conservation laws. These laws ground, in the end, in the conservation of synchronization 'sticks', which are virtual (*system* state) bits not appearing in any message (and hence *process* state). Sticks correspond to access permissions for resource use. Since these sticks can only be conserved on directed cycles in the digraph, the structure which emerges is that of a hierarchy of nested cycles or cycle complexes[1]. This hierarchy is called the cycle hierarchy [1,2]. I would now like to discuss how this structure can be connected to the CH.

Consider first the following diagram of the CH as generated by Program Universe:

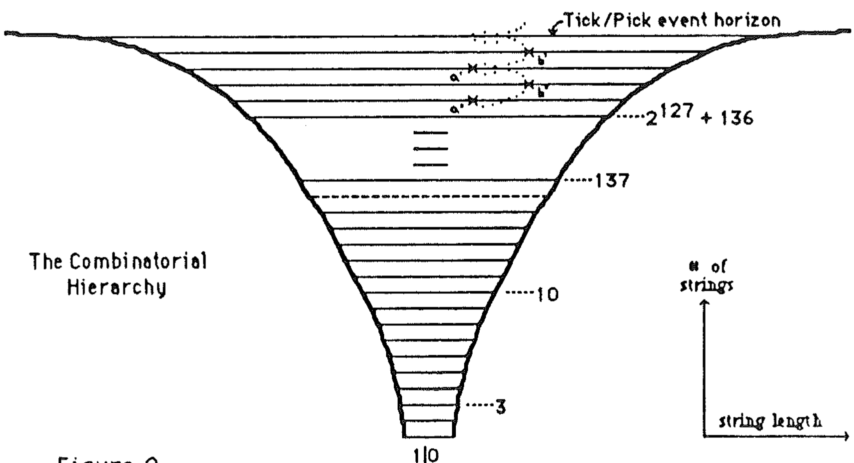

Figure 2.

Let a^0, a^1, b^0, and b^1 be events in the CH/Program Universe sense. Let q be some conserved quantity, e.g. a quantum number, however defined, and assume that q is conserved in the event sequence $a^0 \to b^0 \to a^1$. Suppose further that q is conserved in the event sequence $b^0 \to a^1 \to b^1$. In fact, suppose that q is conserved in all sequences $a^n \to b^n \to a^{n+1}$ and $b^n \to a^{n+1} \to b^{n+1}$. Then in the language of the CM, the spiral indicated in the above CH diagram, which tracks the conserved quantum number, is equivalent to the closed cycle complex in Figure 3 below, in which a (e.g.) single stick is repeatedly exchanged between the processes represented by the messages >x>> and >y>>.

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[1] The word 'emerge', by the way, was chosen to emphasize the emergent character (whole > sum of parts) of conservation relationships.

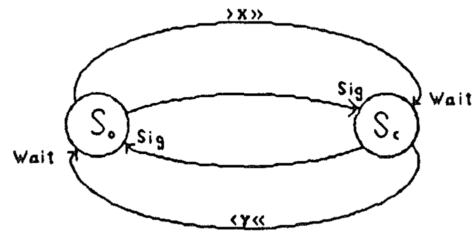

Figure 3

In the figure, the stick is currently "in" the synchronizer on the left ("o" = "open"). Once >y>> goes through that synchronizer ("Waits"), both synchronizers will be closed ("c" = "closed") and the stick will be "in" the message >y>>. When >y>> Signals the right-hand synchronizer, this action places the stick "in" that synchronizer, which results in its being set to open. At this point, the situation is the reverse of the original system state.

Besides the intellectual satisfaction of showing a connection between the CH and the CM models, each model has something to say about the other. The embedding of the cycle hierarchy in the CH gives a hint about how to talk about dynamic actor nets, since the string matching mechanism of the CH is what ultimately specifies which entities interact. It is of course these interactions which form the basis of the CM. In terms of the CH diagram above, the various a's and b's could be 'anywhere'. The 'breaking' of a spiral indicates the breaking of a conservation relationship and hence the formation, "bottom up", of a new cycle hierarchy. Given that the cycle hierarchy is a representation of the (apparent) constancy of our macrocosmic world, this reflects the fact that it is ultimately the activity at the sub-atomic level (cycle-hierarchy-wise) which forms our macrocosmic world.

Since the cycle hierarchy is ultimately based on synchronization relationships, the connecting of the CH to the CM and its cycle hierarchy tells us something about conservation laws in the CH (in whatever form they eventually take) which is not readily apparent, at least to me. In the CM, the act of synchronization between two processes does not affect message *content* (this is what was meant when I said that a synchronization stick is a virtual bit, and as well motivated the quotes around "in" in the above example). A synchronization can therefore*only* be detected in terms of correlation with other processes, i.e., in terms of the system as a *whole*. It follows that conservation itself is intrinsically a non-local phenomenon. In the light of the extreme non-local way in which Program Universe operates, the *necessary* non-locality of conservation, versus the actual locality of (CH) events when they do actually occur, is unapparent. Furthermore, the fundamental relationship between conservation and synchronization is invisible. Given the central position occupied by conservation laws, I herewith appeal to the CH mathematicians to tell us the form(s) invariances can take.

Finally, since synchronization does not exchange information between the Waiting and Signalling processes, neither Wait nor Signal messages are light speed limited. Said in terms of David McGoveran's operator ordering calculus, the attributes describing conservation are not as rich as those describing material (i.e., memory oriented) interactions. This is reflected in the computational world by the fact that pure synchronization nets (e.g., actor nets lacking memory actors of any kind) do not have the computational power of a Turing machine. It must be emphasized, however, that it is precisely this 'lack' of richness which allows interaction exceeding light speed and leads inevitably to the view that our universe is multiply connected. We can therefore hope that, via the multiple similarity and distance relationships now rigorously definable, we have arrived on the doorstep of understanding the basis of so-called psychic phenomena and the magical laws of contagion and similarity.

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THE THERMODYNAMICS OF COMPUTATION -

AN UNDERLYING SCENARIO FOR A THEORY AND PRACTICE OF STOCHASTIC MACHINES

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ABSTRACT

The brain is the most sophisticated and powerful computerlike machine on Earth, but its components have proved far more perplexing than those of present day computers, and the elementary mechanisms they use to process their information is currently unknown. This paper postulates such an elementary mechanism, holochory from the Greek, "holos" = whole, and "choros" = field, and lays the foundation for a control theory of machines utilising stochastic processes. This theory is based on Lie algebras, rather than Boolean algebras that are the basis for present day computers and deterministic automata, but it embraces, as Dirac has shown, both classical and quantum physics, and so both the digital processes of computation i.e. universal Turing machines, and the quantum processes of computation i.e universal quantum machines. Such a theory describes therefore stochastic machines (or indeed deterministic machines) capable of reconstructing or restoring any physical or natural field, and has already been experimentally validated by novel application in acoustics, aerodynamics, macroscopic quantum electrodynamics and chemistry.

INTRODUCTION

Much has been said and will be heard about the transfer of the knowledge of the human expert to the machine and its strategic value to companies, industry and mankind i.e. the technology of expert systems, however little is yet known about how the human expert captured that knowledge in the first place. This is the question I want to address today. The question of experential knowledge and of finding what I describe as intelligent solutions.

FINDING INTELLIGENT SOLUTIONS

In relation to any predetermined goal, the question is not simply is there a solution, but is there an intelligent solution in relation to the environment in which the goal must be achieved; that is in relation to the resources that are available or potentially available.

For many classes of problem such solutions are known, and so providing that the environment in which the goal is set, can be accurately identified with a specific class of problem then an intelligent solution can be computed, and it is for such tasks that digital computers are ideally suited once the analyst has done his/her work.

However in many practical cases, the environment in which the goal is set, is changing so rapidly or is so ill determined, that the problem class cannot be readily identified and in any case, even if it could, it might well correspond to a problem class in which there are no known solutions. And yet it is in these cases where the computational power of the human mind can and does excell. Creativity, innovation (and perception) are characteristics of the human brain that so far no computational machine even comes close to matching. This human ability to perceive or master order in situations previously considered disordered, is what I shall define as intelligence. This definition, first proposed by Fatmi and Young in 1970, has been accepted in the Journal Nature[1] and is strongly supported by experimental evidence in relation to human reasoning, guesswork and intelligence as has been demonstrated by Professor H.B. Barlow at Cambridge.[2] And it is about an understanding and possible automation of these human abilities that I want to talk. Our final objective in the European Cybernetic Machine Group on whose behalf I prepared this paper is nothing less than a technology of Cybernetic Machines with some measure of just such abililities.

The case that I shall be attempting to make today therefore is that artificial intelligence as carried in any form on digital computers can only exhibit intelligent behaviour as already understood, and for which there already exist intelligent solutions. That is, I am saying, that digital systems cannot be truly intelligent in the way the human brain can ie be creative and innovative. This is again confirmed by Fatmi and Young's intuition that a digital computer being mindless ie quite separate from its programmer, can have no claim to such real intelligence but can only exhibit intelligent like behaviour ie that which the programmer has chosen to formally include within the program. What we need therefore is a fundamentally new technology of Cybernetic and Intelligent Machines, if we are seeking to automate the intelligent solution to new classes of problem. The ultimate question is then whether it is intelligent ie wise to seek such a form of automation.

THE PROCESSES OF NEURAL COMPUTATION

The central problem of computation, expert systems and artificial intelligence as it is practiced today is to understand, I believe, how the processes of neural computation take place, for to paraphrase J.J. Hopfield and D.W. Tank,[3] the computational power routinely used by nervous systems to solve perceptual and other problems must be truly immense given the massive amounts of sensory data continuously being processed, the inherent difficulty of the recognition tasks to be solved, and the short time (milleseconds to seconds) in which answers must be found. Biological systems such as the human brain therefore have some operations that are astounding from the engineering technical viewpoint.[4] These serve as existence proofs that carbon-based and apparently analogue chemical computation does work, and we know from the evidence of 25 years reserach into the above fields underlined, that most general purpose digital computers will fail to provide the combination of speed and power that such neural systems possess, even with their substantially slower circuit elements, the neurons.

Such evidence is causing the Cybernetic Machine Group to look beyond digital to new extended theories of computation such as the Quantum Theory of Computation towards a theory and practice of Cybernetic Computers as stochastic machines or well formed heat engines working in a steady energy flux as autonomous systems. Such machines could have operations beyond those which are digitally possible. Machines able to process data as waves or fields continuously in totally parallel modes. The problem is then to understand how the new processes of computation work and how such novel machines can be built. A breakthrough in this area is impending, and I believe such a novel technology could indeed overtake that of digital computers by the year 2000.

Such a view is apparently also taken by the US IEEE which in June sponsored a huge conference of 300 delegates, and 300 or so papers on the subject of Neural Networks under the banner "The Dawn of a New Age", and by US private venture capital concerns which are financing new companies such a Synaptics, which has attracted Frederico Fagin designer of the first microprocessor and Professor Carver Mead the semiconductor silicon luminary from CALTECH.

However what this subject currently lacks is a comprehensive control theory for stochastic processes that corresponds to that based on Boolean algebras which already exists for deterministic machines and automata. I intend to conceptualise and illustrate for you - all the time allows - what I and my European colleagues, believe is the basis for such a control theory for such stochastic machines, and shall briefly state here our belief demonstrated in a paper entitled, "A Theory of Cybernetic and Intelligent Machines based on Lie Algebras"[5] submitted for publication, that this theory can be considered as extending the scope of what can be considered to be computable beyond the domain of digital computability to encompass for example processes in the domain of the quantum theory of computation as annunciated in 1985 by David Deutsch in the Proceedings of the Royal Society.[6] Deutsch's theory concerns non-Turing reproducible operations and is a theory of generalised measurement in which the observer no longer stands outside the system under measurement but is an active participator in the measurement process, the point made by Fatmi and Young.

Such a control theory will therefore ultimately allow us to exploit the control of a system not merely by sending control signals but by using the actual experimentally verified non-locality of quantum phenomena by which supraluminal synchronisicity is possible even though supraluminal signal communication is not. That is, it will be possible to use not only the amplitude of signals ie their intensity, but their quantum phase as well for control purposes. Such a possibility instantly resolves such problems as visual reconstruction which is the central problem of visual perception, because quantum amplitude and phase provide the complete direct experiential knowledge we can have or measure with respect to the visual properties of any object within the range of signals available to the measuring instrument - in our own case - the eye, from the object.

This is not hype or a long term dream, but something that will take place in the immediate future as I hope I can make clear today - all the elements for such a new technology are already in place.

Let me therefore summarise, the fact that computation is fundamentally a physical process will be used to conceptualise a new attack on the central problem of artificial intelligence, the combinatorial explosion. This leads to a new control theory for intelligent machines based on Lie algebras rather than Boolean algebras, and can be interpreted as concerning generalised holochory, from the Greek words "holos" = complete and "choros" or "chorion" = field, since the theory concerns the reconstruction or restoration of a field of any physical nature ie acoustic, electro-magnetic, gravitational, quantum electral dynamic, aerodynamic or even magneto-hydrodynamic or electroweak. And the possibilities that can be conceptualized in the final case, electroweak are astonishing, right out of StarTrek - direct matter to electrical energy convertion for either power or for signal communication ie teleportation.

And as is well known, classical mechanics can be considered with Goldstein,[8] as the "geometrical-optical approximation of wave-mechanics, in the sense that the Hamilton-Jacobi equations reveal classical mechanics as the geometical-optical limiting case of wave movement. And so if our theory is right it spans the experimentally well validated theories of both classical and quantum systems.

However a well known example of holochory concerns the optical processes of generalised holography in which the holograms are not merely passive material artifacts which when illuminated reproduce or replicate the visual properties of the original object, but dynamic fields of electromagnetic energy or matter. Such processes represent a continuation of the work of Dennis Gabor, that was begun in Imperial College in the late fifties and early sixties with the construction of a prototype universal filter, simulator and predictor mechanism.[9] But all the technological elements to implement such generalised holographic processing by optical means now exist as can be seen from such Journals as Scientific American, in articles such as "Optical neural computation" March 1987,[10] and "Optical Phase Conjugation" December 1985.[11] Even experiments concerning and demonstrating the non-locality properties of the quantum phenomenon are in progress. These are taking place via so called "squeezed light" see Schleich and Wheeler, Nature 326, 1987, pages 574-7,[12] which show that it will soon be possible to create such generalised dynamic holograms by optical means in the vacuum itself, because "squeezed light" enables the signal in the vacuum to be actually reduced below the noise of the vacuum in particular localities if it is raised at others. An awesome prospect - that soon we will be able to control for ourselves the very vacuum of the Universe itself by optical means for our own purposes; where such control involves quantum processes and therefore in principle quantum computation, which as I have said Deutsch[6] has shown is more general than digital computation.

Hence my profound certainty that such machines can be developed now, given the political and scientific commitment and funding to such a technological development.

The Dawn of a new Age indeed.

THE BASIS FOR THE NEW THEORY

What is the basis for the quantum theory of computation and the theory of generalised holochory, I am proposing?

It is that computers are physical machines, and that computation is above all a physical process as well as being an abstract formal and logical one such as Turing described using the Universal Turing machine.[13] That is, I am saying that no logical operation can be executed unless a corresponding set of physical conditions in relation to the thermodynamics of the logical operation is satisfied first so that the physical requirements of the computational process are paramount.[14]

How can these physical conditions be conceptualised?[15]

Firstly the signal to be processed must exceed the thermodynamic noise within the system in which the operation is to take place if it is to take place at all, implying that a condition with respect to the thermodynamic entropy must be met. Secondly the communication requirements of the system must be fulfilled so that the signal or signals must be in the right place in the system, implying a condition with respect to the metrical aspects of the entropy, and finally even if the signal is in the right place, it must also be in the right form - that is, there must exist the appropriate signal structure implying an appropriate condition with respect to the topological aspect of the entropy of the system. Only then will it be possible to transform the signal from a probalistic configuration of energy into what we could appropriately term "digital information" with a gating operation giving the result "true" or "false" or alternatively "zero" or "one". Such a process can be thought of as representing the conversion of experiential knowledge into formal logical form. It has in respect of getting "data" in the right place and the right form ie corresponding to the communication and data structure requirements of a system, its appropriate analogues in the formal model of digital computation. However note the absence of the first condition from the formal model and that fundamentally, formal knowledge is being created from experiential knowledge, without the need for any other formal knowledge to exist. Thus we see that any formalised system of knowledge, say mathematics must necessarily be accompanied by and proceeded by, in general, an 'informal' body of concepts, and language ie its experiential knowledge through which most often "an intuitive or informal proof" proceeds before the more formal proof is achieved.

And it was in fact that computation is fundamentally a physical process, that lead Deutsch[6] to formulate the Quantum theory of computation, based on his understanding of the physical world as working entirely through quantum processes, by which as we have said, quoting Goldstein,[8] the classical mechanics and therefore the classical, macroscopic world is also an approximation, or subcategory.

In outline the process Deutsch employed was simple. He took the mathematical formalism of the quantum theory as it is normally applied to valid experimentally validated models of physical systems, and applied it to Turing's model of computation, the Turing machine, generalising it, and then deduced the principal properties of such new quantum machines. This showed that such machines are indeed universal machines in the computational sense, and contain the universal Turing machines as a strict subset. This is as it should be for universal quantum computers if they exist and/or can be constructed, must describe actual physical processes and systems. And so the new theory in abstract mathematical form would indeed explain why the mathematical model of the Turing machine can be translated into an actual physical machine, the digital computer.* However Deutsch shows that this new extended model of the computational process is now based on non-Turing reproducible operations and that it is only under special conditions that these operations perfectly simulate those essential to digital computation. These special conditions are of course those which I have already conceptualised for you in terms of the thermodynamics of the computational process. In more general conditions ie thermodynamic conditions, continuous processes much akin to analogue computation apply.

HOW DO SUCH CONSIDERATIONS APPLY TO SUCH MATTERS AS ARTIFICIAL INTELLIGENCE AND PERCEPTION?

Well clearly since both the models of the quantum theory of computation and the thermodynamics of computation, are more general than the digital model of computation and concern continuous physical processes and not just discrete ones, it immediately suggests that we can extend our models of the brain from that of the digital computer to that of a quantum or thermodynamically possible one. And it suggests that there exist forms of analogue processing or computation that until now were not recognised and are not understood. This conclusion is confirmed by the recent work of J.J. Hopfield and D.W. Tank[3] for example, and by our knowledge of the brain itself.

It suggested to me, that we should adjoin the laws of physics to the mathematical principles already understood as the basis for digital

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* in a sense therefore the digital computer provides itself, an experiment validation and physical existence proof, of Deutsch's theory; and digital computers are, of course quantum machines, but a restricted class of such machines.

computation as the general principles that are required to make the combinatorial explosion more manageable.

We may ask therefore is such an adjoining likely to result in the properties we require?

Well we certainly know that although the mathematical models of digital computation contain no such parameter, computation is limited by the speed of light and that this fact alone is a fundamental practical limitation to the solution of many problems ie to ignore it, is to ignore the obvious.

Secondly in a particular solution evironment which as far as I am concerned always concerns a physical environment, we know that each combinatorial possibility will correspond to a logically valid information flow, but that not every logically valid information flow can be a physically valid energy flow when we take the resources of the environment into proper account. And hence the combinatorial explosion of possible solutions is indeed in principle made more manageable by introducing appropriate physical considerations ie initial conditions.

But thirdly for any general physical environment that has been experimentally validated, we know that the laws are regulated by the Principle of least Action, and so we can regard this Principle as making for us the specific choice of the combinatorial optimization procedure we desire to find. That is, nature is apparently "lazy" and always does things in a way so as to utilize as little energy or more appropriately as little action as possible. But the most general version of the Principle of least Action, that discovered by Richard Feynman[16] applies not only to classical mechanics, but to quantum mechanics and to relativistic quantum mechanics at that; it should therefore provide just the kind of modelling relations we are seeking, and the fact that Feynman's proof of the generalised principle relies conceptually on Huygen's theory of secondary sources, or wavefronts turns out to be a key feature of the new theory we are seeking.* However it is an area which we must enter on a note of caution recognising that the quantum theory as it is generally understood and experimentally validated concerns only closed systems, and that if we were to use the existing quantum theory as the basis of our model, then we require an extension of the theory such as Prigogine[18] has proposed in order to deal with the open and non-linear systems such as we now propose to enter. Another viewpoint of this limitation of existing experimentally validated quantum models is that the wavefunction in such models apparently describes matter at the absolute zero of temperature,[19] and so such models, while correct, neglect to take into full account, the Third Law of Thermodynamics that says that "any finitely realizable physical system cannot by any finite process be reduced to a condition of zero temperature or entropy" And it is noteworthy that Deutsch had to invoke this Law to substantiate and prove his theory. As we are looking for a thermodynamic model therefore, it is essential that any new proposal should take this Law

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* In fact G. Grossing[17] shows that his theory, which can again be mapped onto the theory I shall propose, is, a further generalisation of the Principle of Least Action.

fully into account, as it is this Law which imposes the inseparability of physical systems from one another ie the supraluminal synchronicity that characterises quantum systems. It imposes holism, it demands context dependent models, it says, as I have said earlier that an intelligent solution must concern the resources available or potentially available in the environment in which the problem must be solved. And it is the feature which is usually absent from deterministic, and discrete models.

WHAT THEREFORE ARE THE FEATURES OF THE NEW THEORY FOR WHICH WE ARE SEARCHING?

They are as follows:

(i) a thermodynamic representation, and therefore a stochastic representation in terms of stochastic operators,

(ii) a stochastic representation appropriate not only to classical descriptions of thermodynamic systems, but to quantum systems as well,

(iii) a theory of stochastic operators, therefore in which every logical operation or inferential mapping is governed by three generic conditions suitable for the modelling of quantum and thermodynamically possible processes,

these three requirements immediately suggest that Lie algebras may well be suitable for such a representation because of the fundamental role which the Lie product

$$[a,b] = ab - ba$$

plays in the quantum theory, and in this case the three generic conditions corresponding to our conceptualization will be

$$[x,x] = 0$$

$$[x,y] + [y,x] = 0$$

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 \quad \ldots \quad \text{the Jacobi identity.}$$

And as we shall see, it is possible to treat both closed and open systems by this formalism using the proposed theory, and so it may be appropriate to display the three generic conditions, which Deutsch[6] considers regulate his model of quantum processes and the mappings that exist beteen quantum and Turing computability. These concern logical reversibility and are

$$F(r,s) = F(s,r) \quad \text{commutativity}$$

$$F(r,F(s,t)) = F(F(r,s),t) \quad \text{associativity}$$

and what I shall call discrimination

$F(r,r) = 0$ as required by the Third Law always to be falsifiable.

They of course apply only to closed systems, where we note that the Universe is logically a closed system, that contains open systems within it. A property that is also a property of binary Boolean logic - see "Conservative Logic" Fredkin and Toffoli[20] who show that the system of "Fredkin gates and unit wires" of their conservative reversible logic contains all the operational functionality of the standard binary logic gates, in general leading to irreversible, or one to many transformations.

What we are looking for therefore is a general theory of inference or control based on Lie Algebras, and this suggests that we need a theory based on category theory to represent the modes of inference or control required which is in some way combined or overlaid with an appropriate alegebraic structure or structures based on Lie algebras so as to constitute the complete set of relationships between natural systems and formal systems and the three generic conditions.

But by a stroke of serendipity such a theory already exists, and can be interpreted as concerning the generalization of Huygen's Principle. It was discovered by G. Resconi and M. Jessel and is referred to as General System Logical (meaning inferential) Theory[21] and it is a generalisation of General Systems Theory. Note that an essential aspect of the movement to any open system is symetry splitting.

THE GENERAL SYSTEM LOGICAL OR INFERENTIAL THEORY DIAGRAM I

In this theory, the basic atomic inferential element is called, an elementary logical system or ELS

ELS

$$\begin{tikzcd} X \arrow[r, "R"] \arrow[d, "T_1"'] & Y \arrow[d, "T_2"] \\ X' \arrow[r] & Y' \end{tikzcd}$$

so that $Y = RX$
and $Y' = QX'$

and $X' = T_1 X$ $Y' = T_2 Y$ where R, Q, T_1, T_2 are operators and hence

if $(X, T_1 X)$ is true, then $(Y, T_2 Y)$ is true, where Y, X are immersed in some

appropriate vector field so as to allow the building of the appropriate algebraic structure we need to support the Lie algebras.

And where an inferential or logical system or LS consists of a network of ELS for example

$$
\begin{tikzcd}
X_1 \arrow[r, "I_n"] \arrow[d, "T"'] & X_2 \arrow[r, "I_{n-1}"] \arrow[d, "T"] & X_3 \arrow[d] \arrow[r, dash] & \cdots \arrow[r] & X_n \arrow[r, "I_1"] \arrow[d, "T"] & X_{n+1} \arrow[d, "T_{n+1}\ LS"] \\
X'_1 \arrow[r, "I_n"] & X'_2 \arrow[r, "I_{n-1}"] & X'_3 \arrow[r, dash] & \cdots \arrow[r] & X'_n \arrow[r, "I_1"] & X_{n+1}
\end{tikzcd}
$$

where $T_{n+1} X_{n+1} = TX_{n+1} + [I_1 I_2 I_3 \ldots\ldots I_n, T] X_1$ ie there is recursion.

This theory is a generalization of General Systems Theory.[12]

X is the input to a system R and Y is the output

$$X \longrightarrow \boxed{R} \longrightarrow Y$$

Now how can such a representation be applied to a specific problem and in particular be used to represent Huygen's Principle of secondary sources or wavefronts?

The Huygen's Principle can be stated as follows:

> "The perturbation that goes out through a surface Σ that contains the wave source S_{or}, is identical to the perturbation that can be obtained by cutting off the source and substituting it by appropriate secondary sources distributed on the surface Σ"

DIAGRAM II

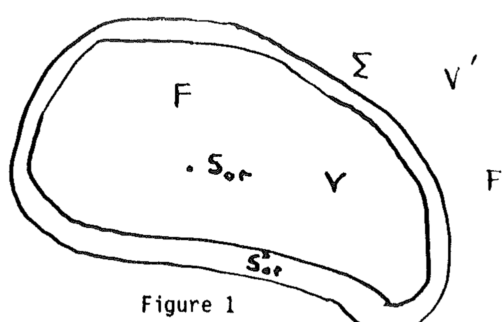

Figure 1

Thus if for example we consider an operator s for which s = 1 for $x > \Sigma$ ie outside the surface, V' and s = 0 for $x < \Sigma$ ie in side, V, then we may put

$$OPF = S_{or}$$

where F is the field and S_{or} is the source of the field and OP is some operator, and hence the ELS is

$$\begin{tikzcd} F \arrow[r, "OP"] \arrow[d, "s"'] & S_{or} \arrow[d] \\ sF \arrow[r, "OP"] & sS_{or} + S_{or}^{secondary} \end{tikzcd}$$

but since $OP\,(sF) \;=\; sOPF + (OPs - sOP)F$

$\qquad\qquad\qquad\qquad = \; sS_{or} + [OP,s]\,F$

then $S_{or}^{secondary} \;=\; [OP,s]\,F$ that is the Lie product represents the system of secondary sources that is required.

And hence if the source S_{or} is inside the surface Σ as above in Figure 1,

then $sS_{or} = 0$ inside the surface and

$S_{or}^{secondary} \;=\; [OP,s]\,F$ are different from zero on the surface since

$s = 1$ in V' and $(OPs - s\,OP)F$ is zero outside Σ .

Thus $OP(sF) \;=\; S_{or}^{secondary}$ and therefore $F = OP^{-1}\,(S_{or}^{secondary})$

outside the surface.

This tells something quite remarkable that it is possible using secondary sources on a surface to completely cancel the effect of a source of waves inside the surface throughout the whole of the volume V continuously (except for a small transient time needed for the waves to mix ie reach one another). This is not then simply a local interference effect, but a complete cancellation throughout say a particular three dimensional domain. Such an interference represents the generalised holochoric phenomenon by which the new theory is characterised.

The Lie product representing a surface Σ of secondary sources or holochor or generalized hologram, is therefore the universal physical element with the Lie product symbolizing its logical manifestation, that in this theory of Lie algebras, corresponds with the universal logical element, the NAND gate of computational circuitry described by Boolean algebras. It tells us how abstractly we must construct now the circuitry for the class of stochastic machines that can be designed via General System Inferential Theory.

In this theory therefore the model of the neuron in no way directly corresponds to a two state device, although as we said from our model of the thermodynamics of the computational process, such devices will emit or receive signals corresponding if we like to the truth values, true and false. They will function as a surface of secondary sources and this can in various circuit configurations performs all the functions (and more) that we would expect to find in digital machines ie

(i) it can act as a processor of signals,

(ii) it can act as a receiver or transmittor of signals,

(iii) it can act as a memory device and

(iv) and with its own source of energy, it can act as a creator of signals

This means since the 'information' reaching the stochastic machine is in the form of a wave or field, that experiential knowledge of an environment can be stored directly without any need to transform such knowledge into formal knowledge in the form of digital information as we understand it. In this theory letters, words, and sentences are sequences of nested holochoric structures (see Diagram II figure for LS) or generalised holograms, which ultimately refer to a virtual structure ie the context in which the communication is being made.

Moreover in principle as much knowledge may be accumulated as one likes on a simple holographic surface. This would explain how it could be that when a monkey's hand and no other hand enters the monkey's visual field, one single cell in the monkey's cerebral cortex is known to fire on every such occasion.[23] And indeed the whole brain could function in exactly the same way, with the whole surface of the cortices of the brain representing the totality of our conscious experiences so that the 'cancellation' of this field, in an uncontrolled way, by a blow to the head, or in a controlled way as in sleep, would result in what we term as unconsciousness until the field was re-established by the inner mechanisms of the brain.

Further the ability to process holochorically is effectively the ability to process in a totally parallel mode an unlimited number of digital representations. Think of a visual hologram, it clearly represents the visual properties of an object from an unlimited number of perspectives, where we can consider each perspective as constituting an individual digital representation.

In terms of experimental knowledge therefore a single neuron may be able to out perform the largest Cray man can produce. * T. Poggio & C. Koch, Scientific American May 1987.

WHY CAN SUCH MACHINES EMBRACE THE CLASS THAT DEUTSCH HAS CALLED QUANTUM COMPUTERS?

The Meissner effect,[24] that underlies the physical behaviour of superconducting quantum objects or SQUID, may be simply described in terms of this theory of stochastic machines, as can many other well validated quantum models. There is therefore no reason to believe that such machines should not embrace all the functionality that Deutsch has shown, would make quantum computers universal machines more general than digital computers.

THE MEISSNER EFFECT

We know that if a metal is cooled down in the presence of a magnetic field past the temperature at which it become superconducting, then provided the field is not too strong, all the magnetic field lines are expelled from the metal when it becomes superconducting. Thus if we denote the magnetic field by H, we will have the ELS

$$\begin{tikzcd} H \arrow[r, "OP"] \arrow[d, "s"'] & S \arrow[d] \\ sH \arrow[r, "OP"] & sSor + [OP,s]\, H \end{tikzcd}$$

where sH is operator for which sH = H outside the metal and sH = o inside the metal. The model therefore says that there will be a set of secondary sources $S_{or}^{secondary}$ = $[OP,s]$ H that will exist in the surface of the metal so that the magnetic field within is neutralised. Any source of magnetic field is usually an electric current, and in fact a current of so-called Cooper electron pairs is indeed found in the surface of the metal and only very close to the surface in a small finite thickness of the surface of the superconductor. Hence

$$[OP,s]\, H = J_{cooper} \qquad \text{a current}$$

And the presence of the Lie product in the equation is of course indicative of stochastic behaviour, and of quantum levels, because in general stochastic operators have eigenvalues. But now we see an example of a quantum phenomenon on the macroscopic scale. Such superconducting elements therefore represent another possible technology for the implementation of the Cybernetic Machines we envisage.[25]

A KNOWN PRACTICAL APPLICATION OF THE THEORY IN AERODYNAMICS [27]

We now give a totally different description in this control theory[26] which concerns the stability of an aircraft wing which was discovered by Meirovitch and Silverberg and has been experimentally confirmed by them.

This concerns direct control of a wing, where no spatial discretization as would be required in digital models for such control is required.

Consider a wing where P_c represents the state of the wing

Y is the state vector associated with the control of the system, say the ailerons of the wing (a virtual wing.)

and Z is the operator that will allow us to enforce the stable control we desire so that the ELS is,

$$\begin{tikzcd} P_c \arrow[r, "Z"] \arrow[d, "\Lambda_c + SZ"'] & Y \arrow[d, "R"] \\ \dot{P}_c \arrow[r, "Z"] & \dot{Y} \end{tikzcd}$$

global

local

Now when the wing is autonomous ie without any external action, we have the dynamic equation

$$\frac{d}{dt} P_c = \Lambda_c P_c$$

where Λ_c is linked with the dynamic modes of vibration c, and so when subject to external control S it becomes

$$\frac{d}{dt} P_c = \Lambda_c P_c + SY = (\Lambda_c + SZ) P_c \quad *$$

and further we can choose the dynamical control in such a way using the property of the ELS, to link a new property, STABILITY, to the system by imposing a predetermined differential equation for STABILITY on the ailerons Y, ie

$$\frac{d}{dt} Y = R\,Y \quad \text{using an operator } R$$

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* ie we propose in effect to amplify the stability so that it becomes extended to the whole wing.

And now the ELS tells that by making $\Lambda_c + SZ$ equivalent R $(Y = ZP_c)$*, we can always make both the wing and the aileron have the same property, stability.

So although it seems that the output R is changing with the input $\Lambda_c + SZ$ in reality, it is Z that changes due to the feedback because Z is in the input and in the internal operator. An ELS is a structure therefore that can be used to limit both the internal and external operators of the system.

Now the equivalence of $\Lambda_c + SZ$ and R implies

$$Z(\Lambda_c + SZ) = RZ$$

which is simply an algebraic equation or relation and need be solved only once, whereas in the digital control of such a wing, the output signals must be computed at regular intervals throughout the motion, from the input signals.

Finally from the theory of Lie products as it applies to networks of ELS we know that in principle therefore for a Logical system or LS, Diagram II of such aerodynamic surfaces as represented by a single ELS ie a set of jointed continuous surfaces or wing such as a bird has, we can again impose the overall condition of stability through a single line of aileron tips to the wing surfaces, so that in every configuration the wing remains stable. It is clear that for a flexible jointed wing, feathers represent an ideal technology for achieving such control. We note that the surface of surface is a curve and that the solution of the algebraic equation is only readily accomplished for an equation of up to the 5th degree, and that this could then limit the no of joints acceptable in the design of such a wing.

CONCLUSION

All the basic factors necessary for the implementation of a technology of stochastic machines capable of utilising experiential knowledge via generalised holochory can now be assembled. In particular, holochory by optical means or generalised dynamic holographic processing seems the most promising, although the technology of SQUIDS is another alternative.

It is postulated that the brains use holochoric computation and that neurons are holochors. If this is the case then man-holochoric machine interfaces via natural languages which must also be based on holochoric theory, should enable a simple class of voiced controlled robots capable of intelligently performing simple tasks in ill-determined environments to be designed and constructed.

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* ie we propose in effect to amplify the stability so that it becomes extended to the whole wing.

Since the holochoric theory demonstrated, applies in principle to any physical field including quantum mechanical fields, the theory represents a grand unified field theory of intelligent stochastic machines which work by active control in ill-determined environments. It leads me to conclude paraphasing Juan Mascaro, the Penguin Books translator of the Bhagavad Gita, that "the whole of creation is a mathematical equation for the mind ie the grand unified theory and a song of love for the soul."

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CONCLUDING REMARKS

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ANPA 8 ended with an apparent convergence of ideas amongst the members of the loosely organized research group. I suggested then, in my closing remarks, that we were rapidly coming together, going off in a common direction.

One year later, this seems a prediction that has both true and false aspects. The group of long-standing ANPA members has continued to work in substantially the same direction: divergence here seems unimportant compared with the common intention. I wisely refrain from gilding the lily by attempting to summarize just what this common position is. It is evidenced by the papers of Amson, Bastin, Deakin, Gefwert, Marcer, McGoveran and Noyes. Two other papers that should be mentioned here because they are directly physics, are those of Maria Burgos (left from ANPA 8) and of Chatterjee. The sixteen queries listed by Noyes at the end of Chapter 5 certainly define a common research program for the year leading up to ANPA 10. The exciting thing about this year at ANPA 9 is the way in which the Old Guard have been joined by a number of people who see, I hope, the general spirit of ANPA as applicable in their fields.

All are welcome, but I begin—because it is a major step away from our concentration on theoretical physics—with the Life Sciences. Clement and Comfort have both provided new notions here and Heather's paper links biological ideas of consciousness with previously unrecognised mathematical ones. Next, although John Amson has been with ANPA from the beginning, his appearance as the first Frederick Parker-Rhodes Memorial Lecturer cannot be omitted here. As well as giving us the opportunity to express our regret at the loss of a friend and founder member in the most appropriate way, he has provided an insight into the connection between indistinguishables, set theory and multisets (about which we also had the benefit of expert advice from Blizard).

One of the consolations for having to make concluding remarks is that one can spend some of them on a personal and idiosyncratic account. So, with apologies in advance to anyone who thinks I have done scant justice to his contribution, I want to say a little in detail about the papers by Kauffman and Manthey. Kaufmann's treatment of the formalism of the Lorentz group brings in notions of an ANPA kind, but with some striking new ideas as well. His realization of the 1+1 dimensional Lorentz group in terms of transformations $[A, B] \to [KA, K^{-1}B]$ is home ground for older ANPA members; they (especially David McGoveran) would only cavil at the way in which A, B are assumed to be real or complex numbers. The step ahead which appeals most to me is the way in which the analysis of $[A, B]$ into temperance and polarity leads onto the generalization to a number of polarities, and so to the 3+1 Lorentz group, via Clifford algebras. Since in his earlier version[1] of this, he relates it back to the quadratic group, I feel very much back in the days of my youth, when Ted Bastin and I tried to go (with only partial success) in the opposite direction along the very same path[2]. Mike Manthey's title points, for me, to one of the pressing problems about the combinatorial hierarchy (CH). He neatly isolates the distinction between the bits in program universe (PU) and those in his computational metaphor (CM). PU is taken by him to be a homomorph of CH. The validity of this homomorphism is something that needs further investigation, but certainly what Manthey says about PU bits applies to CH bits, too. I feel that CM has a great deal to offer PU, and so probably CH, as well, if only we can start off from Manthey's paper and tighten the links. My own line, however, is that the bits in CH are not prior, but are a derived representation of the elements. It is proving extraordinarily difficult to get this fully worked out, but I offer a hostage to fortune by promising that, at ANPA 10, I will institute a new custom, that of the retiring* President giving a presidential address, and I will try to set out in it my thoughts on what the elements of CH mean. So I hope to see you all at ANPA 10 in August 1988.

The Committee is very grateful to Professor Michael Redhead for making available the use of the facilities of his Department of History and Philosophy of Science on Free School Lane.

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⋆ Editor's note: Clive should have said "receding;" he is committed to serve as President for one more year, concurrently with the newly elected vice–President.

ALTERNATIVE NATURAL PHILOSOPHY ASSOCIATION

Statement of Purpose

1. The primary purpose of the Association is to consider coherent models based on a minimal number of assumptions to bring together major areas of thought and experience within a natural philosophy alternative to the prevailing scientific attitude. The combinatorial hierarchy, as such a model, will form an initial focus of our discussions.

2. This purpose will be pursued by research, conferences, publications and any other appropriate means including the foundation of subsidiary organizations and the support of individuals and groups with the same objective.

3. The Association will remain open to new ideas and modes of action, however suggested, which might serve the primary purpose.

4. The Association will seek ways to use its knowledge and facilities for the benefit of humanity and will try to prevent such knowledge and facilities being used to the detriment of humanity.

Organization

1. The Executive Council is the governing body of the Association. The Coordinator, Secretary and Treasurer serve at the pleasure of the Executive Council, in consultation with the membership. Decisions of the Executive Council are subject to majority vote of the Council as a whole.

2. Members of the Association are (a) members of the Executive Council and (b) others nominated by the members and approved by the Executive Council.

3. Vice-Presidential candidates are nominated by the Membership and the Executive Council during the first year of the President's term of office. Any nomination must be accompanied by a statement from the nominee that he will serve a full term if elected. If more than one nominee exists, selection will be made by mail ballot of the Membership decided by plurality of votes. The vice-President is elected biannually to serve for one year concurently with the then current President, and the two following years as President. The vice-President becomes a member of the Executive Council on election, and retiring Presidents remain on the Council for five years. Presidents must serve for one full term (1 year as vice-President and 2 years as President) and cannot run for re-election until three years after their initial term has elapsed.

4. The President is the official representative of the Association in external affairs, and has the responsibility for calling meetings of the membership, at least annually, for the determination of overall policy.

5. The Treasurer is the responsible financial officer of the Association for the receipt and disbursement of funds and shall maintain appropriate records of Association activities, membership, mailing lists, etc.

6. The Secretary is responsible for keeping minutes of the membership and Executive Council Meetings, production of a newsletter to keep members of the Association informed of its activities, and such other duties as may be assigned.

7. President, Secretary and Treasurer will not be paid for their services but may, as appropriate, receive funds for travel expenses, secretarial help, etc.

8. The Coordinator may be paid an appropriate salary for his services, funds permitting. These services will include the organization of meetings and the editing of the proceedings of such meetings for publication, coordination of and participation in the research activities of the Association, preparation when appropriate of research reports and publication of such reports, and such other duties as may be assigned.

9. The Executive Council has selected an independent Advisory Board. It may adopt its own rules for operation and replacement of members. The Executive Council may nominate candidates to the Board. Any member of the Board, or the Board collectively, may make recommendations to the Executive Council, or directly to the Membership. Action taken on such recommendations must be promptly reported by the Executive Council back to the Board in writing. (Current Advisory Board: G.F.Chew, C.Isham, M.Redhead, P.Suppes)

10. Dues are currently 10 pounds UK ($ 15.00 US) per annum.

Executive Council: Dr. Faruq Abdullah, Dr. John Amson, Dr. Ted Bastin, Dr.A.M.Deakin, Prof. Clive W. Kilmister, Mr. David McGoveran, Dr. Michael J. Manthey, Prof. H. Pierre Noyes.

President (1987-1989): Clive W. Kilmister; 11, Vanbrugh Hill, Blackheath, London SE3 7UE, ENGLAND; *vice-President (1988-1989):* Michael J. Manthey; Institute for Electronic Systems, Aalborg Universitetscenter, Strondejen 19, 9000 Aalborg, DENMARK.

Coordinator: Ted Bastin, Pond Meadow, West Wickham, Cambridgeshire CB1 6RY, ENGLAND [0223-029-848]

Secretary: Faruq Abdullah, Room E517, The City University, Northampton Square, London EC1V 0HB, ENGLAND [01-253-4399 x 4417]

Treasurer: A. M. Deakin, 75, Clatterford Road, Carisbrooke, Isle of Wight PO30 1NZ, ENGLAND [(0983)524120]

Advisory Board: G.F.Chew (Berkeley), C.Isham (Imperial College), M.Redhead (Cambridge), P.Suppes (Stanford).