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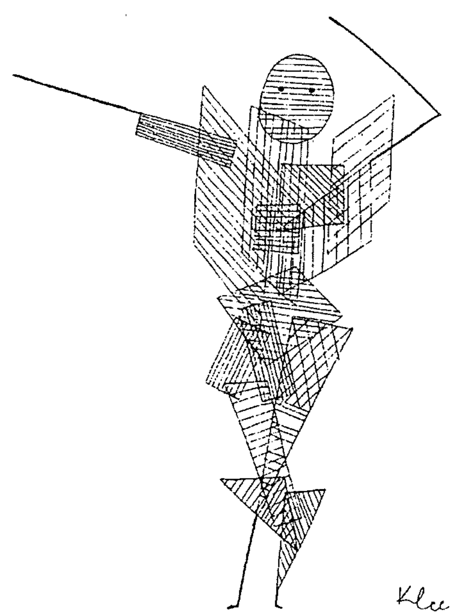
of special note in this issue

Those of us caught up in the details of our present work sometimes unduly neglect the past. This summer issue contains three historical documents we believe are still of interest in view of how things are currently going in ANPA. We include one true rerun, a 1975 paper by Pierre Noyes reprinted from *Theoria to Theory*, and two other previously unpublished papers that represent important earlier stages of current work.



In this issue

Discrete Ordered Calculus

Tom Etter & Louis H. Kauffman

1 Introduction to the Introduction

We wrote this paper (except for the present introduction to the introduction) in 1994. In the meantime the paper [8] by Kauffman and Noyes was polished and published, including many innovations and discussions that emanated from the original handwritten manuscript [7]. A key innovation came from the work in the until-now unpublished paper that you are about to read! We (the present authors) had a happy time trying to find a clean and consistent way to perform the non-commutative calculus that is explained below. The present paper shows an intermediate stage in the evolution of this idea. In this intermediate stage, we surround an expression with special "time-shifting" parentheses, $[A]$, which instruct the user of this algebra to shift expressions to the left of the parens by one time step. This allows us to retain an instruction to shift time, even after we have written out the formula for a derivative in the calculus. Formally the parens behave as follows:

$$A[\ ] = [A'] = [\ ]A'$$
$$[AB] = [A]B$$
$$A[B] = [A'B]$$
$$[A + B] = [A] + [B]$$

Here A' denotes the "value" of the variable A after one step in time. Note that the empty parenthesis acts as an operator according to the formula

$$A[\ ] = [\ ]A'.$$

We observed this and decided to replace the formalism of the parens by an operator J corresponding to the empty bracket so that $AJ = JA$." This made the best and simplest solution to the problem. Furthermore' by writing $A' = J^{-1}AJ$, J becomes a formal and discrete time evolution operator. Our discrete ordered calculus (DOC) became a version of discretized quantum mechanics' fitting perfectly with the theme of[8]. On top of this' the discrete time-evolution fits with the more generalized probabilistic dynamical rules described by Etter in [3].

2 Introduction

The purpose of this short paper is to give a quick introduction to the discrete ordered calculus devised by Louis Kauffman and Pierre Noyes in their paper [7] on the derivation of electromagnetism from the formalism of quantum mechanics. In fact we improve on this original version of the discrete ordered calculus (DOC) by introducing a fundamental time shifting operator that is distinct from the time shift associated with a derivative in the original calculus. These remarks will be clarified below.

We are excited by the prospects of this discrete ordered calculus exactly because it reproduces the formal properties of Newton's calculus in a discrete setting.

3 A Discrete Ordered Calculus

Recall the calculus of discrete differences. Let

$$DX = X' - X$$

define the discrete derivative of a variable X whose successive values in discrete time are

$$X, X', X'', X''', \ldots$$

We can proceed to do calculus in this realm. An early exercise reveals the formula

$$D(XY) = X'D(Y) + D(X)Y.$$

Proof.

$$D(XY) = X'Y' - XY = X'Y' - X'Y + X'Y - XY$$
$$= X'(Y' - Y) + (X' - X)Y = X'D(Y) + D(X)Y.$$

The key point is that this formula is *different* from the usual formula in Newtonian calculus by the time shift of X to X' in the first term. In [7] the authors undertake to correct this discrepancy in the calculus of finite differences by taking the derivative D as an *instruction to shift the time to its left.* That is, they take $XD(Y)$ quite literally as *first find DY, then find the value of X.* In order to find $D(Y)$ the clock must advance one notch. Therefore X has advanced to X' and we have that the *evaluation* of $XD(Y)$ is

$$X'(Y - Y)$$

In order to keep track of this non-commutative time-shifting, *we* shall write $D(X) = [X' - X]$ where the bracket [ ] is a special time-shifter satisfying the properties

$$A[\ ] = [A'] = [\ ]A'$$
$$[AB] = [A]B$$
$$A[B] = [A'B]$$
$$[A + B] = [A] + [B]$$

The time-shifte r acts to automatically evaluate expressions in this non-commutative calculus of finite differences that we call DOC. The key result is the adjusted formula:

$$D(XY) = XD(Y) + D(X)Y.$$

Proof.

$$D(XY) = [X'Y' - XY] = [X'Y' - X'Y + X'Y - XY]$$
$$= [X'(Y' - Y) + (X' - X)Y] = [X'(Y' - Y)] + [(X' - X)Y]$$
$$= X[(Y' - Y)] + [(X' - X)]Y = XD(Y) + YD(X).$$

The upshot is that DOC behaves formally like infinitesimal calculus and can be used as a foundation for discrete physics. In [7] Pierre Noyes and Louis Kauffman use this foundation to build a derivation of electromagnetism from the formalism of quantum mechanics. DOC is suitable for symbolic computation and can even be used to keep track of the myriad time shifts in the classical calculus of finite differences.

4 Imaginary Comment

It is interesting to note that the basic property of the time shifter, [ ], is the equation

$$A[\ ] = [\ ]A'.$$

This puts the time shifter in line with the way in which imaginary quantities are introduced into non-commutative algebra. For example we go to the quaternions from the complex numbers

$$Z = A + Bi$$

by introducing a new operator J such that

$$ZJ = JZ'$$

where $Z' = A - Bi$ *is* the complex conjugate of Z. This is a formal correspondence, but it deserves further exploration.

In fact the full generalization lives in the Cayley multiplication where the rule is

$$(A + BJ)(C + DJ) = (AC - DB') + (BC' + DA)J.$$

That is, we have A,B,C,D are elements of a possibly non-commutative algebra and $(XY)' = Y'X'$. In this context, the new imaginary element J satisfies the equations

$$J^2 = -1,$$
$$XJ = JX',$$
$$A(BJ) = (BA)J,$$
$$(AJ)B = (AB')J.$$
$$(AJ)(BJ) = -B'A.$$

The operator J performs almost all the different shifts that algebra allows. J causes products to reverse their orders and conjugations to take place. It is a remarkable fact that the rule for Cayley multiplication that we have shown generates the complex numbers from the real numbers, the quaternions from the complexes and the octonions from the quaternions. There the process stops. Each system loses some properties of its predecessor. The complex numbers are not ordered. The quaternions are not commutative. The octonions are not associative. And the algebra at the next stage collapses utterly.

Is this not very like the way the levels of the combinatorial hierarchy [2] come to an end? We suggest deeper explorations of this parallel in the context of the structure of time.

References

1 Ted Bastin (editor), *Quantum Theory and Beyond,* Cambridge University Press, 1971.

2 Bastin, H.P.Noyes, and C.W. Kilmister, "On the Physical Interpretation and the Mathematical Structure of the Combinatorial heirarchy", *Intl. J. Theo. Phys.,* vol.7, 1979, pp. 445-487.

3 Etter, "Process, System Causality and Quantum Mechanics," To appear in *IJGS- Special Issue, 1996.*

4 Louis H. Kauffman, "Special Relativity and a Calculus of Distinctions," *Discrete and Combinatorial Physics - Proceedings of ANPA-9,* edited by H.P.Noyes, ANPA WEST Pub., 1987, pp. 290-311.

5 Louis H . Kauffman, *Knots and Physics,* World Scientific Pub., 1991,1994.

6 Louis H. Kauffman, "Knot Logic", In *Knots and Applications,* edited by L. Kauffman, World Scientific Pub.,1994.

7 Louis H. Kauffman and H. Pierre Noyes, "Discrete Physics and the Derivation of Electromagnetism from the Formalism of Quantum Mechanics", (original handwritten version), 1994.

8 Louis H. Kauffman and H. Pierre Noyes, "Discrete Physics and the Derivation of Electromagnetism from the Formalism of Quantum Mechanics",Proceedings of the Royal Soc. London A, Vol. 452 (1996),pp. 81-95.

9 Michael Manthey, "A Vector Semantics for Actions", *Proceedings of the 10th Annual Meeting of ANPA,* edited by Fred Young,1994.

10 Eddie Oshins and David McGovern, "...Thoughts About Logic About Thoughts...: The Question, 'Schizophrenia?'," In B.H. Banathy, Editor,1980, *Systems Science and Science,* Louisville, Society for General Systems Research,pp. 505-514.

11 Parker-Rhodes, *The Theory of Indistinguishables,* Synthese Library, Reidel, Dordrecht, 1981.

12 Roger Penrose, "Angular Momentum - An Approach to Combinatorial Space-time," In *Quantum Theory and Beyond,* Edited by Ted Bastin, Cambridge University Press, 1971, pp. 151-180.

The Abandonment of Simultaneity

Pierre Noyes

Some recent investigations in linguistics, communication, and social organization have found that progress can be made only by abandoning the concept of simultaneity in favor of a multi-component hierarchical description of overlapping times. It is suggested that the same approach might offer a clue to the solution of the problem of joining relativity theory to quantum mechanics which has resisted more conventional approaches for forty years.

Although the atomic hypothesis that phenomena can be analyzed into discrete elements with fixed properties has proved enormously fruitful in many different sciences, it conceals a fundamental paradox. If the atoms so isolated are in fact independent of their surroundings, how is it possible for them to influence those surroundings? This problem is already present with the hard impenetrable material atoms of Democritus, who insisted that there are only atoms and the void, and that all phenomena reduce to the collisions of these atoms. It was learned during the nineteenth century that one can, in fact, explain the relation between pressure, temperature, and volume of a gas with such a model, up to a point, but this is a far cry from explaining sounds and colors as experienced by human minds. Thus the atomic hypothesis taken literally produces a dichotomy between mind and matter, primary and secondary qualities, and so on, with which philosophers have struggled from time to time without resolving.

Up to a point, the quantum mechanical description of the structure of matter gets around these difficulties in an ingenious way.[1] It starts (at the level of description which first concerns us) with a system of electrons and nuclei with fixed and stable properties, but because of the uncertainty principle, asserts that we are unable to predict anything more than probability distributions for these particles. Thus an isolated hydrogen atom consists of one proton and one electron, and if we observe it in such a way as to locate these particles, we will find only the two particles named. However, the locations of the particles will differ from hydrogen atom to hydrogen atom, even though hydrogen atoms, interacting in ways that do not allow the localization of the particles, behave in identical ways. This is accounted for in the theory by calculating a probability distribution for the electron in the hydrogen atom, and showing that, to about l part in 137, we will get the correct answer in such cases if we treat the charge of the electron, not as a point, but as if it were smeared out into this probability distribution. Thus, even though the theory in a sense is built on point particles, it also is capable of a description which looks like an extended structure in space.

With this understanding, we can, to a certain approximation, talk about a hydrogen atom in its ground state as having a spherically symmetric charge distribution, the radius of the sphere being about 0.5×10^{-8} cm in length. If two hydrogen atoms join together to form a hydrogen molecule, however, this charge distribution does not remain spherical. It forms an elongated structure with rounded ends. At rather accurately defined positions along the axis of this structure, it is possible to make measurements which will localize one or the other of the two protons, or both of them, and within the extended charge cloud one can, by suitable measurement, localize either or both of the two electrons. Thus quantum mechanics allows changes in the effective structure of atoms when they join to form molecules, even though the constituent electrons and nuclei retain their particulate character. In this way, one aspect of the atomic decomposition is retained, while at the same time allowing the actual spatial structure of the atoms to change with the molecules in which they are embedded. Similarly the structure of the molecules will be altered by whether they are in free space or surrounded by other atoms of a liquid. Ultimately, then, there will be subtle differences (according to the theory) depending on whether the molecule is in a muscle fibre or in a brain, what the species of the organism is, and what its past history has been. Thus there is no hiatus or barrier as one extends the chain of hierarchical organization upward. It is this subtlety of description which is at the root of the considerable success which molecular biology has had in accounting for the mechanism of life.

Of course the chain does not stop there. As Polanyi has pointed out in a recent article,[2] this molecular biological description, no matter how complete, still does not account for the purposeful aspects of life. To give explanation for these, we must extend our universe of discourse to include the evolutionary processes which through natural selection have fitted the gene pool of the species to the environment and interlocked the different species into ecological systems. Ultimately this description must extend backward in time over the full 4½ billion years the earth has existed and include the steps by which self-replicating systems developed from non-living matter. Even so we are still not up to the level of discussing consciousness, which involves not only the neural currents in the brains of individuals, but the processes of learning by which they become associated with distinguishable aspects of the surroundings, and the social organizational structures without which these learning processes could not exist, and which shape them.

Clearly the whole explanatory process sketched above is far from complete at the present time, but there is no longer a logical reason why it cannot be continually expanded in scope and power as we learn more of molecular biology, neural structures, and intercommunicating social organizations. It also can be extended to a level of analysis below that of the electrons and nuclei showing that these too have structure which is subtly influenced by their surroundings. This comes about because of the combined effect of the $E = mc^2$ mass-energy equivalence of the special theory of relativity and the Heisenberg uncertainty relation $\delta E\,\delta t \geqslant \hbar$ (where $\hbar$ is Planck's constant divided by 2π). Since massive particles must move at speeds less than c, the velocity of light, in a time interval δt they can move only distances shorter than $r = c\delta t$. If we attempt to localize a particle within this distance r, the uncertainty principle tells us that there will be an uncertainty in energy at least as large as $\hbar/\delta t = \hbar c/r$. But if r is less than $\hbar/mc$ this uncertainty is greater than mc^2, which then tells us that within distances $\hbar/mc$ of any particle it will be possible (i.e. with some finite calculable probability) to find an additional particle of mass m. In the particular case of electrons, since these carry electric charge, the appearance of a single electron would violate the law of conservation of charge, but within a distance of $\hbar/2m_ec$, we can expect to find a (negative) electron of mass m_2 together with a positron (positively charged electron), also of mass m_e. Putting in the numbers this tells us that any particle which interacts with electrons will be surrounded by some probability distribution of electron-positron pairs of radius about 2×10^{-11} cm. Thus, once we include relativity, particles themselves have extended charge-current distributions, and since these in turn can interact, they will be affected by the structures in which they are embedded. Hence, in principle, even the electrons and nuclear particles in the brain of one man differ in their space-time distributions from those in the brain of another. These effects can be calculated for an isolated hydrogen atom (Lamb shift, vacuum polarization, etc.) and are in agreement with experiment to high accuracy. Hence, if we look closely enough, even the particles of which the atomic and molecular distributions are composed themselves dissolve into modifiable structures and all the structures in the universe are ultimately interlocked and interdependent, leaving no unbridgeable gap.

Unfortunately, this same line of reasoning leads to a new paradox. Since we can find an electron-positron pair within $\hbar/2m_ec$, we could find two such pairs within $\hbar/4m_ec$, three such pairs within $\hbar/6m_ec$, and so on. The smaller the scale on which we attempt a space-time description of the structure of particle charge-current (or mass) distributions, the larger the number of particles we encounter, and this number grows without limit. Dirac began struggling with the infinities this simple fact introduced into the theory nearly 40 years ago, and neither he nor succeeding generations of theoretical physicists have come up with a satisfactory resolution of the paradox, though we keep trying. In some cases it has proved possible to sweep these difficulties under the rug and come up with successful predictions which have been confirmed experimentally, but the basic paradox remains unresolved. As many people have realized, starting at least as early as Bohr and Rosenfeld[3] in 1933, the basic problem is that the special theory of relativity relies on an underlying space-time of points. Although the simultaneity of two events which cannot be connected by a light signal is arbitrary, and when using the Einstein convention for removing the arbitrariness, depends on the motion of the coordinate system, this relative simultaneity still allows a unique ordering of events and an arbitrarily precise punctiform localization of any space-time event. Hence, once this basic space is married to mass-energy equivalence and the uncertainty principle, infinite energy fluctuations at each of these points are inevitable and the mathematical consis tency of the theory collapses. Attempts have been made to avoid this difficulty by giving a granular structure to space-time, or by other modifications of the theory at short distances which still allow it to reduce to the special theory of relativity in the macroscopic world. So far these have not commanded much enthusiasm in the community of theoretical physicists, and have not led to any striking successes. It therefore might be worthwhile to at least look at efforts in other sciences to struggle out of the straight jacket imposed by a punctiform and unique space-time description to a description which is more in accord with the requirements of their data.

This possibility was suggested to me in a conversation with R. L. Birdwhistell, J. H. Crook, and K. L. Pike, at the Center for Advanced Study in the Behavioral Sciences. Birdwhistell[4] has been struggling for many years with kinesic aspects of communication—that is behavior such as the eye-blink, head motion, eye focus, leg cross, etc., which accompany and often replace verbalizing. He could make little progress so long as he was hung up with the telecommunicative model derived from information theory — two individuals exchanging information back and forth along some channel in a uniquely ordered segmental time sequence. But a remark of Infeld's about the relativity of simultaneity freed him from this necessity and allowed him to start seeing the data as an overlapping laminated structure (i.e. multi-layered segments in which each layer is made up of pieces of finite size, and the joins between pieces do not necessarily cross the layers) of events of varying lengths occurring along many channels; some of these units may be only a few milliseconds in length while other aspects of the communicative process may extend over four generations, and unitary events of any intermediate length also occur. This makes it clear that the "information" described in the information theory model for communication can only be interchanged along this limited channel because of an enormous amount of social work preceding and succeeding this brief flow; one need only think of how difficult it is to enable children within a uniform culture to learn from a printed page, let alone to transmit this skill trans-culturally, to realize the force of this description. To use another analogy, communication starts with the installation of the phone system and not with the ringing of the bell; this fact should be obvious to the parent of any teenager who has sat by a silent phone. One point to be emphasized is that by focusing attention on the flow of information in the lexical channel, one not only looses important aspects of the situation, but makes it next to impossible to see the higher units of the hierarchical laminated structure; these are just as real as any of the shorter units, and may often be much more significant.

Working with the linguistic channel itself, Pikes has come to a very similar structural picture. We are familiar in written English with the segmental decomposition into letters, words, sentences, paragraphs, and so on to higher units, but only trained linguists are familiar with the difficulties of recovering these structures from any particular example of spoken English, let alone making the equivalent analytic decomposition of spoken languages of different structure. A little reflection on the profound phonetic changes which occur in the speech of a child as it grows up, in voice tone of the same individual under various settings, or at various times during even the same speech, should convince the reader that the atoms of verbal communication are not unique physical structures with a defined distribution of frequencies and intensities occurring during a precisely defined interval of segmental time. Rather, they are a complicated hierarchical ordering of laminated relationships in which the units are subtly modified by these relationships. Otherwise it would be impossible to turn on a radio in the middle of a speech and realize almost immediately that a preacher is nearing the end of his sermon, or an orator building up to his peroration, as we obviously can. Again, the analysis of speech into segmental units successfully prevents the recovery of highly significant structural aspects of the ongoing process, and a multi-component analysis such as that which Pike uses is essential.

Field studies of the social structure of primate societies such as those being conducted by Crook[6] again reveal laminated hierarchical structured relationships rather than atomic encounters between individuals. For instance, among the Gelada baboons, the special relationships between all-male groups and the harem groups of dominant male plus females and young change through overlapping patterns from day to night and from season to season in ways that have an intimate connection with the exploitation of the available food supply, and hence are of fundamental evolutionary adaptive significance. These changes in both space and time are only very incompletely understood when the community is followed for only a year, even though the year contains the full range of seasonal variation. Equally significant is the way in which these relationships change as individuals mature and grow old and how the necessary accompanying changes in relationship are structured into the social organization. Clearly, these can only be guessed at until communities have been followed for generations, and this work is only beginning. The point to seize on here is that all this structure is missed if the data are viewed in terms of single segmental encounters rather than in larger units.

Clearly this rich material from the behavioral sciences can only be hinted at in an article of this length. It has taken the three individuals named above many years to come to this way of seeing their data, and there is by no means unanimity among anthropologists, linguists, or ethologists as to the importance of this type of approach. But it does appear significant that by abandoning simultaneity and punctiform units as a method of description,[7] significant new relationships become possible to observe. Unfortunately, the mathematical structures needed to give precision to this approach are yet to be worked out. The general area of mathematics in which to look is obviously set theory, as was realized long ago by von Neumann[8] in discussing economic behavior, or the axiomatic field theorists[9] in trying to come to grips with the infinities arising from the coupling of relativity and quantum mechanics. But it still seems to be beyond the current level of mathematical sophistication to go from a description in terms of overlapping sets, which does seem appropriate to the data, to a dynamical theory which would allow predictions as to how the relationships between these sets evolve in time. In the old punctiform theories, dynamics is supplied by equations of motion written in terms of rates of change (differential equations); but these necessarily imply a continuous background space of points.[10] Since it is clear simultaneity and punctiform space must be abandoned,[11] this might imply that something equivalent to the calculus, but operating on the laminated set structure rather than on space-time, must be invented.[12] One purpose of this paper is to point up this necessity; unfortunately my own mathematical talents are too limited to see how to proceed further than pointing out the problem. A second purpose is to point out the similarity of structure between the problems of kinesics, linguistics, primate social organization, and elementary particle physics; this implies that advances in any one of these fields can offer fruitful suggestions for new insights into the others.

References and notes

1. For a somewhat broader discussion of how quantum mechanics remains, from some points of view, an atomistic and particulate theory, cf. Noyes, H. P. *American Scientist* 45, 431 (1957).

2. Polanyi, M. *Science* 160, 1308 (1968).

3. Bohr, N. and Rosenfeld, L. Koenlige Danske Videnskaberns Selskab, *Math-Fys. Medd.* 12, No. 8 (1933) ; for a less technical discussion see the article by Rosenfeld in the festschrift volume, *Niels Bohr and the Development of Physics.*

4. Since the full flavor of Birdwhistell's commentary has to be experienced along the non-lexical channels which he commands so thoroughly in order for really effective communication to take place, I prefer to acknowledge the importance of his providing me with this experience rather than to give necessarily inadequate lexical references.

5. Pike, K. L. *Language in Relation to a Unified Theory of the Structure of Human Behavior*, Mouton, the Hague and Paris (1967).

6. Crook, J. H. (ed.), *Social Behavior and Ethology*, Academic Press (in press).

7. To "abandon simultaneity" in the easiest way is not a novel idea, as is evidenced by the folk wisdom that those who follow the analytic approach "cannot see the forest for the trees". It becomes somewhat more radical when the methodology becomes so ingrained that Birdwhistell can say that there are no trees (or baboons, or atoms, or. . .). This flies in the face of our culture-bound necessity for assuming that systems are built of discrete units, but becomes a methodological necessity in those fields where in fact the forest dissolves when analysed into trees. Philosophers are trained to accept a methodological description shorn of its metaphysical implications, but a scientist who becomes thoroughly committed to a methodology will follow it through to the implied metaphysical conclusion, just as Lucretius followed the opposite methodology through to the belief that there are only atoms and the void.

8. Neumann, J. von and Morgenstern, O. *Theory of Games and Economic Behavior*, Princeton (1947).

9. Wightman, A. S. *Physics Today*, September 1969, p. 53.

10. Russell, Bertrand, in *The ABC of Relativity*, was at some pains to construct the Minkowski space-time of special relativity of the laminated structures of every-day experience by a limiting and abstracting process. He returned to this problem in his last epistemological work *(Human Knowledge: Its Scope and Limits)* with what strikes this author as an almost obsessive compulsion to establish not only space-time but also the causal chains of classical physics as a barricade against the Humean dilemma. This is the reverse of the methodology suggested here, which rejects the classical limit as both irrelevant and misleading. Although Whitehead's attempt to construct space-time points from the extended

events of everyday life has some similarity to Russell's, he is more ready to recognize that the entities so constructed are hardly the elements of a naive punctiform geometry, and may not converge to that limit. He is still, however, concerned to reconcile his theory with the relativistic physics of his day, rather than to follow the implications of his metaphysics into the actual construction of a new type of dynamics.

11. Lee, T. D. and Wick, G. C. (private communication) are making a promising attempt to remove the infinities from quantum electrodynamics by a limiting procedure that still uses a punctiform mathematical space-time as a substrate. However, the expanded Hilbert space required to make their limiting procedure finite introduces particles of imaginary mass which cannot be directly observed, and runs into trouble with conventional ideas about causality. The experimenter has even less control over experimental study in their theory than in conventional quantum mechanics. In the conventional theory, the probability wave front retains its shape after scattering, but in theirs there is a precursor wave which is uncontrollable. Thus space-time recedes still further into the mathematical background. Other approaches to the problem of infinities introduce a granular structure to space-time, again avoiding the punctiform limit.

12. An alternative approach to the problem of infinities is the "bootstrap" theory in which every particle in the universe generates every other particle, and the universe exists because this is the only self-consistent solution to the equations. This is acausality with a vengeance, and even one of the proponents of the (insoluble) theory questions whether it can be considered a "scientific" idea (G. F. Chew, Science **161**, 762 (1968)). The lack of alternate solutions to the "true" one makes the issue of dynamical equations which determine the evolution of different systems starting with different conditions at some point irrelevant. □

To Hell with Education

A Denunciation of Scholasticism in Science
With Apologies to C.S. Lewis' 'The Screwtape Letters'

By Viv Pope

Reprinted from *The Newsletter of ANPA*, **16**, Spring 1996, ANPA International, Cambridge, UK.

Once there dashed into hell a young devil, to tell
Some news he could scarcely contain.
'There's been a disaster!' he called to the Master;
'I fear all our work is in vain!'

Drawled Satan: 'Keep cool, you impetuous fool.
I find such excitement uncouth!'
'But Dad,' cried the lad, 'this really is bad.
Mankind's found a great bag of Truth!

'I stayed hid,' he said, 'as the pieces were spread
All jigsaw-like, there on the table.
And some, in a bit, tried the pieces to fit
As skillfully as they were able.'

'So what?' drawled Old Nick, 'Do you seriously think
Such a trifle deserves my attention?
This knowledge they've found, I'll bet you a pound.
Will never become comprehension!'

The lad was astonished. 'But Dad,' he admonished,
'This news ought to make us suspicious!
We surely decline to let truth Divine
Mar Ignorance truly perditious!'

'Relax!' said his Pa, 'It will not get them far.
In me you can place your reliance.'
'How so?' asked the boy. The sage said, 'My ploy
Is to let all that Truth become Science.'

'I am not satisfied,' young Nicky replied,
'This answer of yours is surprising!'
'Then listen, dear youth. For each one who loves truth
There are thousands who love organizing.

'It happens like this: at first it is bliss
To pick pieces out of the pile
And find where they fit, at the chair where you sit
At the table. Then, after a while,

'The bits you abort, by others are sought
To fit where those others are seated.
And soon, as the spaces between all those places
Are filled, so the aim is defeated.

'It's like this, my son. Though it's barely begun,
The project must end in frustration.
Because, by this time, the whole "pantomime"
Is wrapped-up in Administration.

'Their trouble, you see, is that though truth is free,
They don't see my hellish perversity.
By evil intent I've let them invent,
What aspirants call "university".

'This makes the poor fools establish new rules
So marvelously diabolical
That scholars, for fees, can pursue "Ph.D.s"
For purposes non-philosophical.

'Departments are named and probably famed
For keeping their "jigsaws" intact.
With sole occupation to forge reputation
For knowledge both full and exact.

'In this way they save all the truth-bits that they've
Put together by efforts intense.
But the total of which (and here is the hitch)
Can make no overall sense.

'Now you and I know that the best way to go
To restore the rate of advance
Is withdraw the stricture on keeping each "picture"
Where placed by its first circumstance.

'To move chunks around is the plan that is sound
Till they fit beyond any question.
But guardians of knowledge ensconced in each college
Can contemplate no such suggestion.

'"There's no way," they'll say, "such a plan to obey
In keeping with scholarly purity.
In Disciplines tight we maintain our right
To our places, our pay and security.

' "It must never appear that we interfere
In areas controlled by our peers.
We say, and with pride, that we're 'not qualified'
In matters outside our own spheres."

'So these institutions, by circumlocutions,
The aim of the game they are changing.
There are gaps where no bit can possibly fit
While the puzzle resists rearranging.

'In those gaps they neglect, we neatly inject,
For their aims now wholly sophistical,
Our Hellish infusion of sheer illusion
And fancies entirely mystical.

'They easily make the fatal mistake
Of thinking the form ineluctable.
And thus we ensure, by methods secure,
That Truth is no longer constructable.

'You see, then, my son, how our evil is done,
No matter how mankind has toiled
Or what it has cost them, true Wisdom is lost them.
And *that* is how Goodness is foiled.'

Boolean Geometry and Non-boolean Change

(Paper given at ANPA 16, Cambridge, England, Sept. 1994)

By Tom Etter

Preface

The following paper was the basis for my talk at ANPA 16. At the time I believed its basic idea to be original, a belief supported by a number of knowledgeable readers. However, I subsequently learned that this idea, which I called Boolean geometry, was actually around as early as the 1930's. Having never quite made it into the mainstream of logic, it was reinvented not only by me but by several others, including Gordon Pask and, I believe, Gian-Carlo Rota - the full history here remains to be uncovered. However, the connection I made to negative quantum amplitudes does appear to be new, so my plan became to revise the paper into a larger work in which this connection is developed in detail. This larger work has indeed become larger! What was to be only an introductory section on link states turned into my 90-page IJGS paper, followed by a series of shorter papers on the same topic, and I'm afraid the grand synthesis of link theory and Boolean geometry is still only a sketch. Thus it seems like a good idea to go ahead and release this paper in its present unfinished state. Like the DOC paper, it's a historical record, and also I believe that it's not a bad introduction to its subject, which could well be of interest to other investigators of "strange" logics. T.E. 1996

Abstract

Von Neumann showed that quantum observables with eigenvalues 0 and 1 can be interpreted as propositions about the outcome of measurement. When two such observables commute, their product as operators is their conjunction as propositions, i.e. *PQ* means *(P AND Q)*. However, since *(P AND Q) = (Q AND P)*, this cannot be true if *P* and *Q* don't commute. For such propositions, von Neumann defined *AND* in a new way, which led to new non-Boolean laws for *AND*, *OR* and *NOT*; the resulting non-Boolean "logic" was called *quantum logic*.

Quantum logic was a dismal flop, and is all but forgotten today.

But the logical issues raised by von Neumann's deep insight into the meaning of eigenvalues 0 and 1 are as alive today as ever. What are we to make of propositions that don't commute? To put it another way, why is it that sometimes we can't say "*P* and *Q*", taking "and" in its usual sense? Though these question arose in physics, they don't *belong* to physics; they're not about matter in motion but

about *propositions.*

Non-commuting quantum propositions are always asserted from different "viewpoints", i.e. their eigenvectors belong to different bases in Hilbert space. When we can't say "*P* and *Q*", it's because *P* is true or false *here*, while *Q* can only be true or false true after we move to *there*. Taken at face value, this tells us not that Boolean logic is wrong but that it is *relative*. "*AND*" only jumps the Boolean track when we move our logical vantage point. The adjective "non-Boolean" is misapplied to logic - what it really applies to is *change*!

What, if anything, is invariant under non-Boolean change? In the present paper I explore the thesis that, however we may *experience* this constancy, mathematically speaking it is the *connectivity* of the Boolean lattice stripped of its arrows. This mathematical structure, which I call *Boole space*, is isomorphic to the undirected edge graph of the Euclidean hypercube. Given Pascal's logical definition of probability as the number of favorable cases divided by the total number of cases, this weakening of Boolean algebra to Boolean geometry turns out to be mathematically equivalent to generalizing probability so that it can go negative as well as positive. The result is a hidden variable theory that specializes to quantum mechanics as a simple large-number case. Of course the hidden variables here are highly non-classical; their invisibility is not just de-facto but *logical*, and they are not only hidden from the classical observer but from each other!

CHAPTER 1
Illogic, Pre- logic and Logic

Early in the twentieth century, at a time when scientific idols were toppling right and left, even logic itself began to totter. "If Euclidean geometry has fallen, which according to Kant is so built into human reason that it's humanly impossible to rationally doubt it, then why is Boolean logic still standing?" So asked the spirit of the times, and so asked von Neumann when in the 1930's, in trying to clarify Bohr's notion of complementarity, he proposed his so-called *quantum logic.*

Quantum logic at first attracted an enthusiastic and distinguished following. But, some 60 lackluster years later, even its best-known advocate has declared it to be a flop. Why did quantum logic fail? There were no mistakes in von Neumann's mathematics, but it turned out to be strangely sterile. It produced not a single empirical prediction, and wasn't even helpful in making calculations. Not surprisingly, most physicists have turned away from logic altogether.

This is unfortunate. The powerful winds of change that were felt by pioneers like Bohr and Pauli were not just about calculating the radiation spectra of atoms. The really new thing in quantum mechanics is a very general idea, *superposition,* and superposition makes sense in any context whatsoever that presents us with a range of alternatives, never mind alternatives for what. Physicists, though they may claim to disdain logic, make unabashed use of this generality. For instance, they assume without hesitation that it makes sense to speak of the superposition of alternative topologies. But what would you call the theory that tells them they can do that? I don't know its name, but I'd certainly call it some kind of logic.

Among those of us who still do call it some kind of logic, the common wisdom is that we must continue the search for alternatives to Boolean logic. I believe that this is wrong and that the common wisdom here has overlooked the obvious.

There's nothing at all odd or non-Boolean about logic in quantum mechanics so long as we stay within a context where one measurement doesn't disturb another, i.e. within a single set of commuting observables. We can also freely move our ordinary Boolean logic to any other such context. It's only when we try to combine statements from different contexts that problems arise. *Context sensitivity* is of course very familiar in everyday life - think of statements about left and right. What the quantum logicians seem to have overlooked is the possibility that it's the context sensitivity of logic that is causing all the trouble. Why do we need a new kind of logic? Perhaps ordinary logic is not wrong, but just relative!

How do we explore this hypothesis? The approach that works well for spatial relativity is to start with relative descriptions and extract from them a certain absolute or *invariant* part. For instance, you can start with "X is to the left of Y and Y is to the left of Z" and extract the weaker invariant statement "Y is between X and Z", a statement that remains true under reversal of left and right. That's close to the sort of thing we're going to do here. However, in one respect our enterprise has no precedent. In all pre-quantum theories of relativity the job has been to find a system of invariant *statements.* But we are looking for something a little different, which is an invariant way to *logically combine* statements.

Actually, the same thing could be said about von Neumann's quantum logic. How the present approach diverges from quantum logic, to put it in a nutshell, is that it does not seek a *competitor* to Boolean logic with *different* laws, but an *objectification* of Boolean logic with *weaker* laws. I shall call the first *illogic,* the second *pre-logic.*

The particular pre-logic that I shall describe here is what I call Boolean geometry. Von Neumann compared his non-Boolean logic to non-Euclidean geometry. By way of contrast, the present approach can be more accurately compared to Euclidean geometry. To see what this means, let's indulge in a fantasy.

Imagine that it was not Euclid but Descartes who in 300 BC invented the definitive mathematics of space. Since Descartes was an egocentric fellow, Cartesian geometry was centered on his own person, and all points of space were designated by their distance from his navel taken in three directions: left/right, front/ back, and above/below. This was a wonderfully practical system when Descartes was sedentary, but it got a bit confusing when he was moving about. The problem seemed to be solved by his death, but grave robbers kept reviving it. Nevertheless, the system persisted for almost two thousand years until Euclid came up with his great theory of *relativity.* This theory solved once and for all the problem of Descartes' peregrinations by completely doing away with the idea of a Cartesian center, putting a Cartesian clone at every point of space! Note that Euclid's is not an alternative geometry. It doesn't contradict the viewpoint of the wandering Descartes, but *weakens* it by abstracting only what it shares with its clones.

For Descartes substitute Boole. Our new logical geometry is to Boolean algebra what Euclidean geometry is to Cartesian geometry. Indeed, as we shall see, a Boolean logic is literally a *coordinate system* on a Boolean geometry. Starting with Boolean logic, we can define Boolean geometry as the invariant structure under a new group of transformations that *translate* the Boolean origin (Boolean *0*). In geometry a translation is a congruence transformation that takes lines into parallel lines. We shall define the concept of *parallel* in Boolean algebra by saying that the line x,y is parallel to the line x',y' iff *(x XOR y) = (x' XOR y')*, where *XOR* is exclusive *OR*. Given this definition, we can move the concept of geometric translation into logic word-for-word! All this will become clear in chapters 3 and 4; for now I just want to stress again how different Boolean geometry is from quantum logic, whose symmetry group bears an entirely different relationship to Boolean algebra.

To summarize: The laws of logic are not wrong. Indeed, without these laws, these *Boolean* laws, it would make no sense to speak of laws of any kind. The problem is that we have been using Boolean logic too *egocentrically*. It is only after we become aware that there is more than one Boolean viewpoint and begin to study *transformations* of viewpoint that the fundamental meaning of quantum superposition becomes clear.

This may sound like a soothing message: there's a technical fix for all this weird logic stuff that turns it into science as usual. However, let me end here with a word of warning: *When you mess around with logic, you mess around with how you think!* Lobotomy may be a routine procedure for a brain surgeon, *but not when he performs it on himself!*

CHAPTER 2
Boolean Logic.

2.1. Boolean algebras

In this paper I will be treating Boolean geometry as something abstracted from Boolean algebra; as mentioned, this is like treating Euclidean geometry as something abstracted from a system of Cartesian coordinates. For space it seems more natural to go the other way, since our everyday experience of space involves a constantly moving origin. But for logic, I don't see how we can go the other way. What *is* our everyday experience of a moving Boolean origin, of a moving *nothingness*?

This is actually a fascinating question, and one that can start you reflecting about all sorts of things: creation and annihilation, virtuality, Bergson's duration, the Hegelian dialectic between being and not-being, Jacob Boehme's un-ground, Heraclitus vs. Parmenides - there's a call of the wild here that echoes down through the ages.

Unfortunately we haven't yet found the concepts that can connect this kind of speculative reflection to empirical science, and so I believe it's better to begin with concepts that have securely made this connection, namely with *AND*, *OR* and *NOT*. In this chapter I shall set the stage for Boolean geometry by reviewing some familiar aspects of Boolean algebra and introducing some others that may not be so familiar.

A Boolean algebra is a collection of objects, its *elements*, on which there are certain operators *AND*, *OR*, *NOT* etc. that satisfy certain rules. It's usual to introduce these rules by means of certain axioms. That would be irrelevant here, though, since we are only concerned with finite Boolean algebras, and the mathematical structure of a finite Boolean algebra is quite transparent to common sense; it is simply the structure of the set of all subsets of a finite set. We will use this fact to freely go back and forth between logical and set-theoretic terminology, which will make it easier to visualize some of our new concepts. Let's remind ourselves of how the two kinds of terminology correspond:

The *INTERSECTION* of *x* and *y* is the subset of all things that are both in *x AND* in *y*. We'll abbreviate *INTERSECTION / AND* by "&".

The *UNION* of *x* and *y* is the set of all things that are either in *x OR* in *y*. We'll abbreviate *UNION / OR* by "*v*".

The *COMPLEMENT* of *x* is the set of all things that are *NOT* in *x*. We'll abbreviate *COMPLEMENT* by "~".

Most of the time we'll be dealing with Boolean algebras whose elements are subsets of some explicitly given finite set S. The members of *S* are called the *atoms* of that Boolean algebra, and the number of these atoms is called the *dimension* of the Boolean algebra. Here, as in everyday life, we won't distinguish a set with one member from that member itself, so atoms, as one-member subsets of S, are also elements of the Boolean algebra. *S* itself is of course an element; it's called the universal element and is abbreviated *1*. The null set is also an element, abbreviated *0*. We italicize *0* and *1* to distinguish them from numerical 0 and 1.

2.2. Boolean logics

A Boolean *logic*, as I shall be using the term, is defined as the Boolean algebra of subsets of a set *S* of mutually exclusive possibilities, i.e. of *cases*. Think of *S* as a *menu*. It's usual in the restaurant business to number the items in a menu so we can *point* to them by their indices: "Hey, one #5 special coming up!" Let *i* be a menu index that ranges, say, from 1 to 10. Let *S* be the set of all items on the menu. Then, roughly speaking, the Boolean logic of *S* is the set of all the things we can say about *i*, for instance "$i>3$".

This last statement needs to be carefully qualified. Taken out of context, "$i>3$" could mean almost anything. It could be about the ratio of the circumference to the diameter of a circle, or about the number of dimensions of space-time, or it could tell us that the customer is not ordering Hamburger Supreme or Heavenly Chicken or Fisherman's Delight. To discriminate among these possible messages, we must first know the answers to two questions: What values does *i* take? and what alternatives do these values point to? Given this essential background, it makes sense to ask what *information* does "$i>3$" supply?

The elements of a Boolean logic are often represented by sentences or sets of indices, but what they are is *items of information*, taking that term in Shannon's sense as the narrowing of a range of possibilities. My use of the word "logic", which is also von Neumann's, is rather different from what you find in most textbooks on logic, which are largely concerned with formalizing the science of inference. Logic in this textbook sense has almost nothing to do with our present subject matter, and a case could be made for our using some other term such as "Boolean informational structure". However, "logic" is a pretty flexible word, spanning the gamut from Hegel's dialectic to logic gates, and since von Neumann's use of it has already taken root in the foundations of physics community, we'll stick with "Boolean logics".

To recapitulate: An *item of information* is defined as a selected subset of a fixed set C of cases. A *boolean logic* is defined as the set of all items of information on C structured by the set operators *INTERSECTION, UNION* and *COMPLEMENT* etc.

2.3. Projections

Information has another aspect which is not captured by Shannon's definition: It *accumulates*, and as it accumulates, the act of *acquiring* each new item of information alters all the others. Consider our example of the item $i>3$. Taken by itself, this places *i* in the range 4,5,6,7,8,9,10. But suppose we had already learned that $i<8$. Then in Shannon's terms, $i>3$ is a different item of information, since now it confines *i* to the range 4,5,6,7 against a background of an *S* that includes only the first eight members of the original *S*.

To put it another way, an item of information is not just a subset, it is an *operator* on its Boolean logic. The item of information p, as an operator, takes every element *x into p&x*. We'll adopt the convention of using capital letters for operators and small letters for their operands, so we can write this as $P(x) = p\&x$. P is a *projection* operator in the sense that it is idempotent $(PP = P)$ and it preserves a certain algebraic structure - we'll see just what this structure is later. This projective character of information is important to us here for two interrelated reasons: First, it tells us something about why propositions are represented by projections in quantum mechanics, and second, it makes it possible to extend the definition of information so that it still makes sense in Boolean geometry, as we'll see in Chapter 3.

In everything we do we make in one way or another an important distinction between information of two kinds: first there is information that applies only to a particular occasion or event, and then there is information that applies to a whole group of occasions, or is a fixed aspect of a changing object. There are many different words for this contrast. In natural science we speak of data vs. regularities, of boundary conditions vs. laws. In computer science it's input and memory vs. program. In everyday life we have events vs. rules, activity vs. condition or status, accident vs. order etc. The concept of "state" straddles the boundary; a state is something that tends to persist unless it is "forced" or "induced" to change by something outside of itself.

The concept of projection most naturally applies to the second kind of information. Typically when we encounter an object we imagine a large range of nominal possibilities for it, most of which are not realistic. Narrowing down this nominal set S to a realistic set C is the kind of information that we call a *principle*. Since a new item of data is most commonly expressed in terms of S, it is can be very useful to understand how a principle projects data onto C, since this tells us something about how changing data bears on the state of the object as a whole. Indeed, understanding such a projection is just what we ordinarily mean when we say we *understand the principle* of the thing. We'll pick up this train of though again in section 7.

2.4. The Boolean lattice

There are a number of ways to characterize the structure of a finite Boolean algebra. We've done it here by giving that structure a canonical representation, so-to-speak, as the set of all subsets of a finite set. A useful variation on this is to identify the Boolean elements with bit strings which represent the characteristic functions on these subsets. There is a third quite different kind of representation, though, which is important for studying Boolean geometry, and that is the *Boolean lattice*.

Thinking of the Boolean elements as sets, the Boolean lattice is the partial ordering of these sets by inclusion. We'll write this relation as $x<y$, meaning that every element of *x* is an element of *y*. In logic, if all *x's* are *y's* we say that *x implies y*, so the logical name for < is *implication*. We must be careful not to confuse this relation of implication with the Boolean operator $-x \vee y$ which is sometimes called *material implication*. The two are closely related, though, since $x<y$ iff $-x \vee y = 1$. Also we must not confuse it with the concept of deductibility, which is a metaconcept that occurs in the study of formal languages.

Here is a picture of the relation > for a 3-atom

Boolean logic; $x<y$ if you can follow the arrows from x to y.

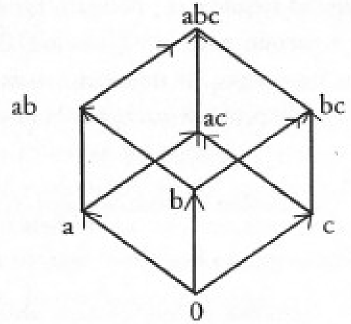


fig 2.4.1. The Three-atom Boolean Lattice

Note that we can define inclusion in terms of &: x is included in y means that x is the intersection of x and y. We can also go backwards from < to &: The intersection of x and y is the largest set included in both x and y, i.e. the g.l.b. (greatest lower bound) of x and y under the partial ordering <.

A less obvious fact is that we can define *NOT* in terms of <. First we define *0* as the element that is included in all others; *0* is of course the null set. Next, we define "x and y are disjoint" to mean that $x\&y = 0$ (recall that we have already seen how to define & in terms of <). Finally we define $-x$ to be the element that is disjoint from x and that includes every other element that is disjoint from x; clearly $-x$ is the complement of x. (This last step only works for Boolean lattices. In non-Boolean lattices like quantum logic, there is more than one maximal disjoint element, and to define negation requires additional structure.)

Here's something that may come as a surprise. We've seen that we can define > in terms of *AND*, and *NOT* in terms of <, but this means that we can define *NOT* in terms of *AND*! Sounds impossible? That's because we're used to *algebraically generating* the other Boolean operators from *AND* and *NOT*, and we know it can't be done with *AND* alone. What we've done above, though, is something quite different, which is to define *NOT* as an aspect of the whole structure of the *AND* operator.

To summarize: The Boolean lattice is a partial ordering of the elements of a Boolean algebra in terms of which all of the Boolean operators can be defined, and conversely, which itself can be defined in terms of the Boolean operators; it is thus equivalent to Boolean algebra in the sense that it characterizes the same abstract structure.

2.5. The Boolean Graph

In fig. 2.4.1 the Boolean lattice is pictured as an oriented graph, where $x<y$ means you can follow a path of arrows from x to y. Let's now look at this graph as a relation in its own right, which we'll call arrow. We'll write $x \rightarrow y$ to mean that there's an arrow from x to y. In the language of sets, $x \rightarrow y$ says that y is the result of adding another member to x. We can define $x \rightarrow y$ in terms of < to mean that $x<y$ and there's nothing in between, i.e. for any z such that $x<z$ and $z<y$, either $z=x$ *or* $z=y$. Conversely, we can define < as the *ancestral* of arrow, i.e. the smallest transitive relation containing arrow - just what this fancy language means becomes pretty obvious when you look at the picture.

Since we can use it to define the Boolean lattice, the Boolean graph is another way to give the structure of a Boolean algebra. But far more important, it immediately leads to a simple definition of the basic object in Boolean geometry:

Boole space: The un-oriented graph that results from stripping a Boolean graph of its arrows.

We'll leave the arrows in place for the rest of this chapter, but we'll concentrate on those concepts that still remain important after we remove them. One of these is the concept of a path, which is a series of steps in the Boolean graph.

Step: An ordered pair of elements x,y such that either $x \rightarrow y$ or $y \rightarrow x$; in the first case we call it a step up or a positive step, in the second, a step down or a negative step.

Path: A sequence of elements whose adjacent pairs are steps. A path from x to y will be notated $x..y$. Here's a path in the three-atom graph, shown by the dotted line:

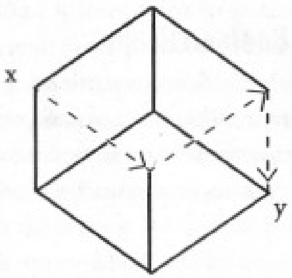


fig 2.5.1 Path from x to y

By the *length* of a path we mean the number of steps in it. A *geodesic* is defined as a path from x to y of minimum length, and this length is called the *Boolean distance* between x and y. It's easy to see that Boolean distance is a metric in the sense that $D(x,x) = 0$, $D(x,y) = D(y,x)$, and it satisfies the triangle inequality. Giving this metric is another way to define Boole space, and in fact that's how we will start off defining it in chapter 3. Here is an important theorem: The Boolean metric $D(x,y)$ together with the Boolean origin 0 determine the structure of Boolean algebra. This will be proved in Ch. 4.

A step up adds a new member to an element, while a step down removes a member. Thus if a path starts from 0, we can determine the cardinality of its endpoint by taking the number of steps up and subtracting the number of steps down. In the language of finance, the net gain for someone traversing a certain path is his income along the way minus his outgo. It may help us remember what we are doing to use such words as technical terms:

Income: The number of positive steps in a path.

Outgo: The number of negative steps in a path.

Net gain: Income minus Outgo, abbreviated $net(x..y)$

Theorem: The net gain of any path $x..y$ from x to y is the cardinality of y minus the cardinality of x. More briefly,

$$net(x..y) = card(y) - card(x)$$

Proof:
We've seen that

$$net(0..x) = card(x) \text{ and } net(0..y) = card(y).$$

Now consider a path $0..y$ from 0 to y which results from splicing $x..y$ onto the end of any path $0..x$ from 0 to x. Clearly the net of a path is the sum of the nets of its consecutive parts, so

$$net(0..y) = net(0..x) + net(x..y)$$

and hence

$$\mathrm{card}(y) = \mathrm{card}(x) + \mathrm{net}(x..y),$$
$$i.e.\ net(x..y) = card(y) - card(x).$$ QED.

Definition: This theorem shows that we can write $net(x,y)$ for $net(x..y)$.

We'll say that a path is *ascending* if all its steps are positive, *descending* if they are all negative. A path that is either ascending or descending will be called *monotone*. A monotone path is always a geodesic. A geodesic needn't be monotone, however, unless it starts from 0. The geodesics from 0 have a special place in Boolean algebra, since their lengths are the cardinalities of their upper endpoints.

2.6. Probability

The above discussion of paths has been part of our preparation for a definition of amplitude modeled on Pascalian probability. Recall that Pascal defined probability as the number of favorable cases divided by the total number of cases. In our current terminology, the probability of x is $card(x)/card(1)$. In the language of graphs it's the net gain from 0 to x over the net gain from 0 to 1, and it's also the distance from 0 to x over the distance from 0 to 1. This last statement is purely geometric except for singling out a point called 0, so let's adopt it as the definition of probability.

Probability: $prob(x) = D(x,0) / Dmax$, where $Dmax$ is the greatest distance between any two points. (note that $Dmax$ *is* $D(1,0)$, which shows that 1 is the unique *antipode* of 0. Dmax is also what we called the dimension of the Boolean algebra, and it will turn out literally to be the dimension of the corresponding Boole space.

Pascal's definition of probability is a purely logical definition, and in fact it has its uses in logic proper. Perhaps the most important of these is to give us an easy way to define what it means for two items of information to be logically separate or *independent:x* is *independent* of y means that

$$prob(x\&y) = prob(x)prob(y).$$

We'll see in a little while how to define independence in terms of Boolean factorization. The concept of independence is of course one of the cornerstones of science as we know it. Will it still make sense after we geometrize logic? Will science as we know it still be possible?

2.7. Sublogics, factors and Independence

Generally speaking, a *sub-structure* of a mathematical structure is a subset of its elements together with the operators and relations that define the structure of the whole. Here is the basic Boolean substructure:

Subalgebra: A subset of a Boolean algebra closed under *AND* and *NOT*.

Theorem 2.7.1: The atoms of a subalgebra form a partition of the atoms of the algebra.
Proof: Let $B1$ be a subalgebra of B. If B_1 includes x and y, it includes $-x$ and $x\&y$. Therefore it includes $0 = x\&-x$ and $1 = xv-x$ and $x-y = x\&-y$. Suppose x and y are atoms of B_1. Then x and y must be disjoint, for otherwise one of $x\&y$ or $x-y$ or $y-x$ would be smaller than either x or y, contradicting the assumption that x and y are atoms. A similar train of reasoning shows that every element of B_1 is a union of atoms of B_1. But since 1 is an element of B_1, we conclude that every atom of B is in some atom of B_1. QED.

Sublogic: A subalgebra of a logic.

If the state of a (classical) physical object is described by several variables, each of these variables ranges over the case set of a sublogic of the state logic. More generally, any exhaustive set of mutually exclusive properties of the state defines a sublogic. The relationship of sublogic to logic captures much of what in everyday life we think of as the relationship of part to whole.

Since Boolean lattices, Boolean graphs, Boolean rings, etc. are all equivalent to Boolean algebras, it would be reasonable to suppose that sublattices, sub-rings, sub-graphs etc. are also subalgebras. Oddly enough, this is not true! Every subalgebra is a sub-ring (defined as a set closed under *AND* and exclusive *OR*) and every subring is a sub-lattice. However, not every sub-lattice is a subring, nor is every sub-ring a subalgebra. An important substructure that is not a subalgebra is the set *S'* of elements of the form *p&x* that results from a projection *P*. *S'* is closed under *OR* (union) so, unless $P=1$, it does not include the negations (complements) of its elements, since $x \vee -x = 1$, while $P\&1 = P$. *S'* is a subring, however, namely that generated by *OR* from all the atoms of *S'*.

How do we characterize the logical relationship between two objects that have nothing to do with each other? In order to say anything at all about it, we must conceptually bring these objects into the same universe, which means that we must see them as sublogics of the same logic. Then what? There is a clear way to proceed if they together generate all the elements of this common logic:

Generation: We say that a set *A* of elements *generates* a Boolean algebra *B* if we can obtain every element of *B* by applying *AND* and *NOT* to the members of *A*. For instance S, the set of atoms, generates its Boolean algebra.

Spanning: Suppose that B_1 and B_2 are sublogics of *B* such that all of their elements together generate B; we then say that they *span B*. If B_1 and B_2 are any two sublogics of *B* there is a sublogic of *B* which we'll call $B_1\&B_2$ that is spanned by B_1 and B_2.

Factors: We say that B_1 and B_2 are *factors* of *B* if they span *B* and if $s_1\&s_2$ is never *0*, where s_1 is an atom of B_1 and s_2 is an atom of B_2. More generally, sublogics B_1, B_2..B_i.. are factors of *B* if for any *Bi*, B_i and *B'* are factors, where *B'* is the sublogic spanned by all the other sublogics.

The concept of factoring captures what it means to break something into a "heap" of parts, assuming that this is possible. The converse operation, bringing parts together into a heap, will be known as:

Boolean multiplication: Given Boolean logics B_1 and B_2 over case sets S_1 and S_2, we define their *product* $B_1.B_2$ to be the Boolean logic over the Cartesian product of S_1 and S_2.

Theorem 6.1. Given a product $B_1.B_2$, there are natural isomorphisms of B_1 and B_2 onto factors B_1' and B_2' of the product. QED.

Proof: The atoms of the product are the ordered pairs $<x_i,y_j>$, where x_i is any atom of B_1 and y_j is any atom of B_2. Define B_1' as the sublogic whose atoms *xi'* are of the *form* $<x_i,y_1> \vee <x_i,y_2> \vee <x_i,y_3>$ ...., and similarly B_2' with atoms y_i. Clearly the atoms of B_1' and B_2' correspond to those of B_1 and B_2, all these atoms together span *B*, and $x_i\&y_j$ is never *0*, so B_1' and B_2' are factors.

Because of these obvious natural isomorphisms, we'll regard multiplying and factoring as inverse operations, and we'll speak of a logic as the product of its factors. The dot notation $B_1.B_2$ is a close relative of the dot notation in computer science for separating fields in labels or addresses (think of an Internet address). If B_1 and B_2 are factors of *B*, we'll allow ourselves to write $B = B_1.B_2$; in such a case we have $B_1.B_2 = B_1\&B_2$.

Now we come to a crucial theorem. Recall that we defined independence to mean $prob(x\&y) = prob(x)prob(y)$.

Theorem 6.2. *x* and *y* are independent in *B* if and only if *B* can be factored into two subalgebras one of which contains *x*, the other *y*.

It's easy to show that elements in different factors are independent; indeed, this is an immediate corollary of the fact that the cardinality of a Cartesian product is the product of the cardinality of its factors. Proving the converse is harder, though, and since the details of the proof are not relevant to what we're doing here, we won't go into them.

If the elements of one sublogic are independent of the elements of another, we say that the sublogics are independent. A necessary and sufficient condition for this is that the atoms of one are independent of the atoms of the other. This is the logical meaning of independent variables. That is, to say that two variables are independent means that they range over the case sets of independent sublogics.

Independent sublogics are factors of the sublogic that they span, but they need not be factors of the whole logic. For a subogic A to qualify as a factor of a logic B, the atoms of A must be equiprobable in B. A sublogic that meets this condition will be called *separable*. Note that if A is separable, the probabilities defined in B for the elements of A are the same as those defined in A alone, and if A is all we care about, we can leave B out of the picture. If A is not separable, however, we cannot use Pascal's definition on A alone but must also take B into account. One way to do this without mentioning B explicitly is to describe the effect of B in terms of a *probability measure*.

A *measure* on a Boolean algebra will be defined as a numerical function on its elements that adds for mutually exclusive cases. Some definitions of measure also includes the requirement that the numbers be non-negative, but we'll omit this requirement since we'll soon be studying *amplitude measures* that can go negative and complex. A probability measure is a real non-negative measure that is 1 for Boolean *1*. Pascal's definition of probability applied to B defines a probability measure on each of its sublogics.

The relative frequency interpretation of probability is often presented as a competitor of Pascalian probability. However, it can be brought under the Pascalian umbrella by treating a series of trials as a set of possibilities for *this* trial. The series then becomes the set of cases for the logic B of all statements about which trial is this one, and relative frequency becomes a Pascalian measure on a sublogic A whose cases are the possible outcomes of the trials. If the outcomes are equally frequent, as with a fair coin toss, then A is separable.

It may seem that all this business about factors is laboring the obvious. After all, we all know what it means for things to have nothing to do with each other - why all the fuss? At this point I can only ask the reader to be patient a little longer. We really do need a rather abstract approach to familiar ideas like separateness, connectedness and equality, since our commonsense intuition about such things turns out to be a poor guide in the strange new realm of Boolean geometry.

CHAPTER 3
Boolean Geometry

3.1. Boole space

In section 2.6 we defined Boole space in terms of the Boolean graph; here we'll start with the Boolean metric.

Metric space: A set of points on which there is a distance function $D(x,y)$ satisfying three axioms: First, $D(x,x)= 0$, second $D(x,y) = D(y,x)$ and third, $D(x,y)+D(y,z)$ is greater than or equal to $D(x,z)$ (the triangle inequality.)

Boolean Distance: The *distance* $D(x,y)$ between two elements x and y of a Boolean algebra is defined as *card(x+y)*, i.e. the cardinality of $x+y$, where + means exclusive *OR*.

Exclusive *OR*, or *XOR* for short, is central to Boolean geometry, so it's important to be clear about its basic properties. First, let's be clear about its definition: *x XOR y*, abbreviated $x+y$, means $(x \vee y) \,\&\, -x\&y)$

Unlike inclusive *OR*, exclusive *OR* is a group operator. Applied to bit strings, it is bit-by-bit addition *mod 2*, which justifies, or at least excuses, the symbol "+" for it. Boolean *0* is the identity of the *XOR* group, and every element is its own inverse; the latter property alone characterizes a group as isomorphic to a *XOR* group. If we define scalar multiplication by *0* and *1* by the rules $0x=0$ and $1x = x$, the *XOR* group becomes a vector space over the binary field; we'll call this vector space *XOR* space. *AND* distributes through *XOR*, so *AND* and *XOR* together form a ring, the so-called Boolean ring. The dual of *XOR* is *IFF*, defined as $(x\&y) \vee (-x\&-y)$.

Theorem 3.1.1. Boolean distance is a metric.
Proof: Since $x+x = 0$, $D(x,x) = 0$. Since $x+y = y+x$, $D(x,y) = D(y,x)$.

To prove the triangle inequality, first note that $D(x,y)$ is a maximum when x and y are mutually exclusive, in which case it is $card(x)+card(y)$. Then note that since $y+y = 0$, we have

$$x+z = (x+y)+(y+z),$$

showing that $D(x,z) = card(x+z)$ is at most

$$card(x+y)+card(y+x) = D(x,y)+D(y,z),$$

which is the triangle inequality.

There are two basic theorems that launch Boolean geometry. Let's now turn to the first, which is that that Boole space is homogeneous. To understand what this means and how to prove it, we need the concepts of congruence, symmetry and displacement.

Congruence: A *congruence* between two subsets of a metric space is a 1-1 correspondence between them that preserves distance.
Symmetry: A self-congruence of a subset is called a *symmetry*.

Homogeneity: We say that a metric space is *homogeneous* if for any two points x and y there is a symmetry of the whole space taking x into y.

Displacement: A transformation on Boole space which for some fixed element d takes every element x into $d+x$. Note that this resembles a vector displacement in ordinary space, and is in fact literally a vector displacement of the linear algebra we called *XOR* space.

Theorem 3.1.2. Displacements are symmetries of Boole space.
Proof:
$D(d+x,d+y) = card(x+d+d+y) = card(x+y) = D(x,y)$

Theorem 3.1.3. Boole space is homogeneous.
Proof: Given any two elements d and e, we have $(d+e)+d = e$, so the displacement by $d+e$ takes d into e.

The second big theorem that starts things off is that you can get from the Boole space of a Boolean algebra back to the Boolean algebra itself simply by saying which point is *0*. Since Boole space is homogeneous, you can choose any other point to be *0* and get another Boolean algebra. The relativity of Boolean logic with respect to the choice of origin is the essential novelty of Boolean geometry, and in future papers we'll see how it leads to the relativity of logic in quantum mechanics.

At this point it will be helpful to consider Boole space in terms of the Boolean graph as we did in see section 2.5. Recall that we defined Boole space there to be the Boolean graph stripped of its arrows. More formally, it is the structure defined by the set of all unordered pairs of neighboring points in the graph. Referring to fig. 2.4.1, we see that such neighbors are the endpoints of the straight lines in the Boolean cube. This is no accident, and it will turn out to be a good move to define straight lines in our geometry to be neighboring pairs. The definition now will be algebraic:

Line: A *straight line,* or simply a *line,* in a Boolean algebra is a two-element set $\{x, y\}$ such that $x+y$ is an atom.

We can see that this is clearly the case for the lines in fig. 2.4.1. The logical meaning there of an arrow $x \rightarrow y$ is that y results from adding one more member to x. To remove the arrowhead means asserting the weaker relationship $x \rightarrow y$ *or* $y \rightarrow x$, which doesn't say which of x and y is included in the other but merely that they differ by one atom, i.e., that $x+y$ is an atom. Thus we see that:

Theorem 3.1.4. The two points on a line are neighboring points on the graph, which means that the structure of the unoriented graph is given by specifying the set of lines.

Theorem 3.1.5. A line is a pair of points whose distance apart is *1*.
Proof: If $x+y$ is an atom then $D(x,y) = card(x+y) = 1$.

The point of stating this obvious theorem is to make it clear that the concept of line is purely geometric, i.e. it depends only on the metric and not on other features of the Boolean algebra.

In non-Euclidean geometry the role of straight lines is taken over by geodesics. Now we have both straight lines and geodesics, which in section 2.5 were defined as edge paths containing a minimal number of steps. We must now see what this means algebraically:

Path: A sequence of connected lines, i.e. a sequence points x_i such that $\{x_i, x_i+1\}$ is a line. Path length is the number of points minus one.

Geodesic: A shortest path between two points.

Theorem 3.1.6. The length of a geodesic between x and y is $D(x,y)$.
Proof: Consider first the case where $x=0$. Since $0+y = y$, $D(0,y) = card(y)$. We saw in 2.5 that $card(y)$ is the length of any geodesic from 0 to y, so the theorem is true for $x=0$. But now suppose we apply a displacement d, turning our interval into $d, d+y$. We saw (theorem3.1.2) that displacements are geometric symmetries, so $D(d,d+y) = D(0,y)$. But we also saw that lines and hence paths are geometric concepts, so the shortest path between d and $d+y$ has the same length as the shortest path between 0 and y. We can choose d arbitrarily. By letting $y = d+z$ for arbitrary z, we can also choose $d+y = z$ arbitrarily. Thus the theorem holds for any x and y QED.

We now see that our two ways of defining Boole space agree. If we know the metric $D(x,y)$ we know the geodesic distance on the graph and hence the structure of the (unoriented) graph itself. Conversely, if we know the graph, we know its graph distance and hence $D(x,y)$.

Theorem 3.1.7. Given any point x, and any path, every step forward in that path either increases the distance from x by *1* or decreases the distance from x by 1, i.e. there are no side steps.
Proof: The theorem is true for $x=0$, since distance from 0 is cardinality. It is true of any other point

since the metric is homogeneous.

Theorem 3.1.8. The structure of a Boolean algebra is given by its metric together with its origin (Boolean *0*).
Proof: The metric defines the unoriented graph while the origin defines the arrow as the direction of increasing distance from *0*.

As mentioned, it is theorems 3.1.3 and 3.1.8 that really get us going. These theorems tell us that a Boolean algebra is a Boole space on which we have singled out an origin, and that the points all look alike so any point will do. Theorem 3.1.7 is also very important, since, as we shall see, it plays a crucial part in our definition of amplitude.

We started this chapter by defining the Boole space metric as the number of atoms in $x+y$. This of course presupposes a Boolean algebra in which atoms and + are defined. We then found that by applying a geometric symmetry we can displace the origin to any other element and obtain a new Boolean algebra in which this new origin is *0*. Since the atoms are the neighbors of *0*, this new *0* defines a new set of atoms, different from those we counted in defining the distance between x and y. In fact, + in the new Boolean algebra is also a different operator from + in the old one. Nevertheless, since the origin was shifted by a symmetry of the metric, we know that the new atom count is the same as the old count. Shifting the origin not only relativizes atoms and +, it relativizes all the familiar Boolean operators except *NOT*, and it also relativizes inclusion and mutual exclusion. Nevertheless, a surprisingly large part of Boolean structure remains invariant, and describing this invariant structure will be our next job.

3.2. Geometric Invariants

What is the full symmetry group of Boole space? It includes the displacement group, and it of course includes the symmetry group of Boolean algebra, which I shall call the *logic group*.

Theorem 3.2.1 The logic group of a finite Boolean algebra consists of all transformations that result from permuting the atoms.
Proof: A finite Boolean algebra is isomorphic to the set of all subsets of its atoms, whose intersections and complements obviously don't depend on how the atoms are arranged, so any permutation of the atoms will generate a symmetry. Since a logical symmetry must preserve the Boolean lattice, it must map neighbors of *0* into neighbors of *0*, so the symmetries generated by permutations are the only logical symmetries.

Theorem 3.2.2. The geometric group, i.e. the full symmetry group of Boole space, is generated by the logic group and the displacement group in the sense that every geometric symmetry can be written in the form DL, where D is a displacement and L a logical symmetry.
Proof: Let T be any geometric symmetry of a Boolean algebra B. Let d = T(*0*). Then by theorems 3.1.3 and 3.1.8, T must map B isomorphically onto the Boolean algebra B' whose origin is d. But the displacement $D(x) = d+x$ also maps B isomorphically onto B'. Thus the symmetry $L = DT$ is a logical symmetry on B (recall that D is its own inverse). We thus have $T = DL$, the product of a displacement and a logical symmetry. QED.

Theorem 3.2.3 The displacement group is a geometric invariant, i.e. it is a normal subgroup of the geometric group.
Proof: We must show that for any symmetry T and any displacement D, $T^{-1}DT$ is a displacement. We have seen that we can write T in the form EL, where E is a displacement and L is logical. Since

$(EL)^{-1} = L^{-1}E$, we have $T^{-1}DT = L^{-1}EDEL = L^{-1}DL$, so the problem reduces to showing that $L^{-1}DL$ is a displacement. We have

$$L^{-1}DL(x) = L^{-1}D(L(x)) = L^{-1}(d+L(x)).$$

But since L^{-1} is a logical symmetry, it preserves + and we have

$$L^{-1}(d+L(x)) = L^{-1}(d)+x.$$

Thus $T^{-1}DT$ is a displacement by $L^{-1}(d)$. QED.

Our aim in this section is to see how much of logic carries over into Boolean geometry, i.e. to find the important logical concepts that are geometric invariants. Now of course logical concepts are already invariant under the logic group - they don't depend on any particular arrangement of the atoms. Thus by theorem 3.2.2, to show that a logical concept is geometrically invariant it is sufficient to show that it is invariant under displacement. This reasoning is essentially what underlies the proof of theorem 3.2.3; displacement is a logical concept and the displacement group is invariant under itself.

The most familiar geometric invariant: is *NOT*. *NOT* is in fact displacement by *1*, i.e. $-x = 1+x$, which shows that it is invariant. The negation of an element x is, geometrically speaking, its antipode, i.e. the point furthest away from x. To see why this is so, note that the longest possible geodesic is of length n, where n is dimension, since each step in a geodesic must add or take away a different atom. But a geodesic of length n starting at x must take away all the atoms in x and add all the atoms not in x, since there are no other atoms to add or take away; the result is *NOT* x. Incidentally, note that the dimension n is a geometric invariant, since n is the number of neighbors of 0, and by homogeneity every point has the same number of neighbors. Let's now move on to some important unfamiliar geometric concepts.

Parallel: Two lines $\{x,y\}$ and $\{z,w\}$ are called parallel if $x+y = z+w$.

Theorem 3.2.4. Parallel is a geometric concept.
Proof: Since + is a logical concept, it is sufficient to notice that a displacement D of two parallel lines leaves them parallel, i.e.

$$\text{if } x+y = z+w, \text{ then } d+x+y = d+z+w.$$

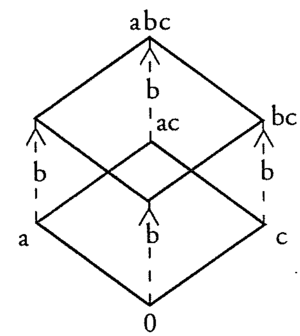


fig 3.2.1 Adding Atom b to the Lower Square

Recall that a line is an edge in the Boolean cube. The arrow on an edge represents adding an atom to a set. Note that parallel arrows add the same atom, which agrees with our definition above. This is no coincidence. It can be shown that the edge graph of the Euclidean n-cube is the Boolean graph of an n-atom Boolean algebra, and that the Boolean metric is the square of the Euclidean metric on the vertices of the n-cube. The Boolean n-cube is very useful for visualizing definitions, theorems and proofs in Boolean geometry, and its Euclidean geometry can be rigorously incorporate into the mathematics of Boole space, though we won't do so here. However, here is a quick intuitive account of why the two structures coincide:

The set of parallel edges in a certain direction connects together two squares to make up the cube (see fig 3.2.1). The lower square represents the

Boolean algebra of all subsets of a two member *S*, while the upper square represents the new subsets that result from adding a third member to *S*. If we add a fourth member to *S*, we connect up our cube to another cube by eight parallel lines in a new dimension get a four-dimensional hypercube. We can keep doing this, which shows that an n-dimensional hypercube is a natural representation for an *n*-atom Boolean algebra. Note that lines in the cube that are not parallel are orthogonal. This is true also in the n-cube, so it makes sense to define *orthogonality* as not being parallel. It's easy to show that dimension is the maximum number of orthogonal lines.

After this brief side excursion, let's get back to parallel lines. What parallel lines have in common is a particular atom, the atom *x+y*. Though parallelism is an invariant geometric concept, this statement depends on two non-invariant algebraic concepts: atom and +. Let's now try to think more geometrically. Suppose we choose a particular origin, call it *0*. Then an atom can be defined geometrically as a neighbor of that origin.

Theorem 3.2.5. For any point *k*, every line is parallel to a line of the form *{k,x}*, and no two lines of this form are parallel.

Proof: Given a line *{z,w}*, we have *z+w* = *k+(k+z+w)*, so *{k, k+z+w}* is a parallel line. If *x* is unequal to *y*, then *k+x* is unequal to *k+y* by the group property of +.

We see from this theorem that for any point *k*, an equivalence class of parallel lines has a canonical representative as a line connecting *k* with a neighbor. This is a geometric way of stating that what parallel lines have in common is a *unit displacement*, something that follows immediately from the definition of parallel. Looking at fig 3.2.1, we see that a unit displacement along the dotted lines reverses the two solid squares, keeping the dotted lines fixed. On the *n*-cube, a unit displacement reverses two *n-1* cubes, keeping their connecting lines fixed. The unit displacement is such an important concept that we will give it a special name:

Step: A displacement of the form *A(x)* = *a+x*, where *a* is an atom.

Theorem 3.2.6. Every displacement is a product of steps in which no step occurs more than once, and the set of steps in that product is unique.

Proof: Any element d is a sum of its atoms, so

$$d+x = a_1+a_2 \,..\, +a_i \,..\, +x.$$

Thus

$$D(x) = A_1(a2...+x) = A_1(A2(a3..\ +x)) = \text{ect.}$$
$$= A_1(A2..(Ai(x)..).$$

Clearly the a_i must include the atoms in *d*, and any other atom would have to occur at least twice in it. QED.

Steps: Define the set steps (*D*) to be the unique set of steps that are the factors of displacement *D* according to theorem 3.2.6.

Theorem 3.2.6 tells us that the displacement group is in natural 1-1 correspondence with the set of all sets of steps. As the set of all subsets of a set, it is a Boolean algebra under intersection and complement. What makes this Boolean algebra so important is that it is geometrically invariant! It will be useful to think of this Boolean algebra as a ring, i.e. as a set closed under *XOR* and *AND*. Recall that a ring is equivalent to a Boolean algebra, but a subring need not be a subalgebra.

Displacement ring. The displacement group together with an *AND* operator defined as follows:

D&E is the displacement such that *steps(D&E)* is the intersection of *steps(D)* and *steps(E)*.

Theorem 3.2.7. The displacement ring of a Boolean algebra is isomorphic to the ring of that algebra itself under the mapping $D \rightarrow d$, where $D(x) = d+x$.

Proof: $D \rightarrow d$ maps steps 1-1 onto atoms, so & is obviously a corresponding operation on the two rings.

$$DE(x) = D(e+x) = d+e+x,$$

which mens that DE ® $d+e$. QED.

Incidentally, this also shows that the so-called displacement ring actually is a Boolean ring, with + defined as the group product.

With the geometrically invariant displacement ring we have arrived at a way to automatically transform any Boolean algebraic concept into a Boolean geometric concept without having to check it for invariance. What we have done is almost exactly analogous to pulling the normed linear algebra of Euclidean displacements out of Cartesian geometry. In each case we begin by thinking of the points as vectors and define a displacement as the addition of a vector to all points, the only difference being that Boolean vectors are over the binary field rather than over the real field. In each case we get to the full geometric group by combining the displacements with the rotations and reflections. In the Boolean case, the rotations (which only rotate by multiples of 90 degrees) and reflections are the permutations of coordinate axes, i.e. the logic group.

Lifting a Boolean atom from the status of a mere object d to the status of a displacement D is to give it an essentially dynamic quality. A step is an object that is either joining or leaving a set. Which set? That doesn't matter; it's only the bare fact that the object is coming or going that creates a step. Which object? That depends on what other steps have been taken! In the face of a moving *0*, only the fusion of the mercurial object with the ephemeral set is something fixed, invariant, *objective.*

I now see the logic of the classical observer in quantum mechanics as a displacement ring on an underlying "hidden" Boolean underworld of incompatible Boolean logics which can only be combined geometrically. It appears that only by dealing directly with logics that change can we find a logic that doesn't change, a logic that everyone can agree on and that will serve for the writing of history and the accumulation of facts.

3.3. Independence

In chapter 2 we asked whether Boolean geometry is compatible with science as we know it. The issue is whether, given non-logical geometric change, it still makes sense to isolate parts of the world from the general flux. In our present terms, the question is whether we can find a geometrically invariant concept of separation that does the same job for some new science that logical separation, i.e. independence, does for present day science.

Logics A and B are independent if taken together they form a *product* logic $A.B$ whose atoms are ordered pairs of atoms from A and B. A good way to visualize this is to think of the atoms of A being arranged horizontally and those of B vertically so that their ordered pairs form a *rectangular set.*

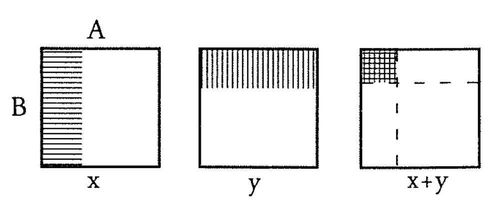


fig 3.3.1 The Separable Element x&y

Every region of this rectangle is an element of $A.B$. However, the elements of A and the elements of B are regions of a special kind. An element of A is a vertical stripe, or more exactly, a set of atoms that can be turned into a vertical stripe by rearranging the columns. An element of B is a horizontal stripe in the same sense. An element of the form $x\&y$ with x in A and y in B is a rectangular region, i.e. a region that can be turned into a rectangle by rearranging rows and columns; in particular, $1\&1$ is the whole rectangle. If an element of $A.B$ is not a rectangle, i.e. if it cannot be separated into an element of A and an element of B, we'll borrow a term from quantum mechanics and speak of it as *entangled.*

Let's now consider what happens when we apply a geometric transformation $G(x)$ to $A.B$.

First let's assume that G is logical, meaning that it permutes the joint atoms in the rectangle. To say that G only applies to A means that it only permutes the columns. This has no effect on the set of atoms in a given row, so the atoms of B are unaffected by G. More generally, a n.a.s.c for G to preserve the separability of separable elements is that it only rearranges rows and columns, i.e. that it is a product of a logical transformation on A and a logical transformation on B. This is just what common sense would expect: if two things are independent, changing them separately can't make them correlated.

But now suppose that G is a displacement. If G applies to only one of the components, we shouldn't expect it to affect the other, right? Let $G(x) = a+x$, where a is in A. If we look only at what G does to the elements of A, i.e. to the vertical stripes, then it displaces them just as if B didn't exist. B could be on another planet, in another universe, as far as A is concerned. Thus it seems to be OK to isolate a Boole space from the rest of the world as an object of study. So far, Boolean geometry is still just another step in the cheerful march of scientific progress.

But wait just a minute, stop the band! a messenger has just arrived from planet B reporting that something very odd has happened to the elements on his planet, which is that they have all become entangled with the elements on ours! What has happened? Our geometric displacement G, which seemed to be confined to our A logic alone, has actually zapped his planet and turned every element y into the inseparable element $y+a$!

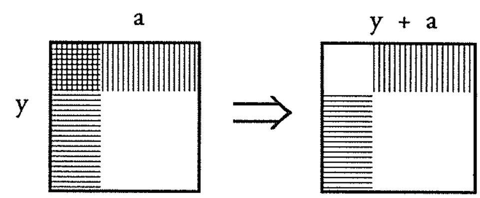


fig 3.3.2 Displacement of a Applied to y

There is no way to analyze a Boole space into separate geometric parts. Separation is simply not a geometric concept. Unlike the action of a logical transformation or the action of a projection, the action of a displacement cannot be broken down into distinct actions on independent factors. The reason for this is that any "rectangular" separation of possibilities still leaves all the parts with a common null set, a common *0*, so a shift in the *0* of one part is a shift in the *0* of all!

But before you take off for a Tibetan monastery or banish Boolean geometry to some mystical netherworld, remember the displacement algebra. Though it's a geometric invariant, it's also a Boolean *algebra* and as such is factorable like any other Boolean algebra. Even though the underlying geometry has no objective parts, there is at least an objective way to take apart an aspect of this geometry. Furthermore, if the geometry acquires a *0*, the factorization of the displacement algebra can be as-it-were projected onto it. All is not lost to science after all.

Of course if the geometry acquires a fixed origin it simple turns into logic, and the displacement algebra as such becomes irrelevant. What we're really interested in is the laws governing a moving origin, which is the likely source of non-Boolean change in quantum mechanics. The conceptual framework within which we shall formulate and study these laws is the displacement algebra, and our main tool in this study will be the concept of *amplitude*, which will be the focus of another paper. •

Illustrations:

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Alternative Natural Philosophy Association

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