Groupings

Proceedings of ANPA 34

Grenville J. Croll & Nicky Graves Gregory, *Editors*

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Editorial

The title of this proceedings "Groupings" is obtained from three pertinent concepts. First, ANPA remains a distinct and important group, characterised by the photograph on the front cover. Second, the mathematical concept of "group" in the plural sense is of relevance to some of our work. Third, it is a word that has not been used before in an ANPA proceedings!

ANPA is in a process of transition. The association has been in existence for over thirty years, the consequence of which is the natural passing away of some of the founders and some of the members. Nevertheless, a core group of about a dozen stalwarts remains, augmented by a few newer members. We have adapted our meeting location to accommodate the disability of one of our founders. We remain an international group with members travelling from the USA, Canada and Europe to meet in person once again this year.

ANPA remains a forum where the scientifically minded can express their most heartfelt and inevitably most challenging work in a supportive and constructive atmosphere. We are not bound by the conventional modes of thinking, hence our alternative nature. We strive to remain scientific. However science is not capable of encompassing all that is relevant in human and universal affairs and so challenging philosophical perspectives are accommodated and encouraged.

A historical focus of the group has been the Combinatorial Hierarchy. This work remains profoundly relevant. Over the years however, other models of reality have emerged which are also distinct from those promulgated by the conventional scientific community. Moreover, we have seen the emergence of new ideas in the biological sciences which challenge deeply held conventional views.

As ANPA continues its transition, both in its governance and in the work it pursues we hope to emerge larger, more vibrant, more inclusive and more visible. We wish to attract people from all walks of life who can embrace and enhance Alternative Natural Philosophy.

Regards

Grenville J. Croll
Nicky Graves Gregory
Co-Organisers, ANPA 35.

ANPA 34 Delegates (L-R)

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Dr Anthony Booth	
Ms Nicky Graves Gregory	
Dr Peter Rowlands	http://arxiv.org/find/physics/1/au:+Rowlands_P/0/1/0/all/0/1
Mr Geoffrey Constable	
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Mrs L. Manthey	
Mr Ron Weeden	

Group photograph by Diane Kauffman, celebrating (approximately) 50 years of the Combinatorial Hierarchy

The Topsy Test for Awareness [1]

Michael Manthey

November 2013 manthey@acm.org

Abstract. From micro-level brain-mapping mega-projects to pseudo-intelligences like Watson, Siri and the like, contemporary technology is (more or less blindly) getting within groping distance of kinds of mindlessness that will be very difficult to differentiate from human consciousness. Certainly the classic keyboard-based Turing Test is feeling its age, and will foreseeably be hard-pressed to cope with the sophisticated behavior allowed by massive modern hardware and software platforms. Nor is it entirely clear what exactly the Turing Test tests, but it certainly isn't *awareness*. I define awareness and briefly describe *Topsy*, a potentially aware software *Entity*. I also propose a simple way to test for the presence of awareness in a 2-way interaction between otherwise anonymous *Entities*. Finally, I ask some questions about the uses to which such *Entities* might be put. Along the way, qualia are captured and mind/matter duality is supported.

Keywords: Topsy, Watson, Siri, chat-bots, AI, artificial intelligence, consciousness, awareness, qualia, mind/matter, duality, Self, slavery, Turing Test, robot, drone, military, corporation, government.

1. Introduction.

I have long felt the term "artificial intelligence" to be a euphemism for something that's too risky to say directly ... if you want to be properly scientific. But that was some 50 years ago, and many things that once were scientific heresy are now accepted fact, eg. that there are computations that exceed what a Turing machine can do (*quantum computing*), that there is a very real connection between the brain and the immune system, that prions really exist, that other animals are tool users (*chimpanzees, corvids, octopi*), can count and even add (*grey parrot Alex*), etc.

So I think that now we can face the truth, which is that all the times when we have said "artificial intelligence", what we *really* meant - perhaps unconsciously - was "artificial consciousness". But this denial has left the discussion of what consciousness *is* up in the air, and as a result, things like "big data" crunching and huge Bayesian networks - sophisticated though they be - are now routinely referred to by journalists as "AI".

[1]Based on an invited talk presented at the 2013 Loebner Competition. To appear in a special issue of the *Journal of Experimental and Theoretical Artificial Intelligence*, P. McKevitt, C. Fields, H. Loebner *Eds.*

A typical example, this from a recent *New Scientist* [1]: "***Computer aces child's IQ test.*** An AI program that understands words has scored as high as a preschooler in a verbal IQ test. ... ***AI's such as Google's search engine or IBM's Watson ...***". In fact, the technologies underlying such systems as Watson and Siri are realized with standard software tools, and there's nothing mysterious going on anywhere. It's all "put in by hand" so to speak, and there's nothing non-Turing about any of it.

The absurdity of equating data-thumping with "artificial intelligence" is apparent when we replace the euphemism by its reality, "artificial consciousness", because it is intuitively clear that "consciousness", by its very nature, *cannot* be "artificial" - *only* the real thing will do. And this *real thing* is emergent and self-organizing and not "put in by hand" at all. So *consciousness* is what we're really about in the field called AI, or at least what some of us ought to be thinking about. Unfortunately, exactly what consciousness *is* is problematic. My point of entry is the working definition *Consciousness is awareness of awareness*, since it seems to reduce the scope of my ignorance. So the question becomes, "What is awareness"? [2]

[By *awareness* I mean the *experience* that one is a unitary entity, even whilst in profound appreciation and contact with one's surround. *Self*-awareness is generally larger and mostly unconscious relative to our normal waking (ie. ego-)consciousness. Awareness, *sans* the unitary feeling, is said to accompany certain very deep meditative states. The mechanism presented below allows all of these.]

From a "systems" point of view, awareness, being un-localized, looks like some kind of *distributed* computation. A distributed computation consists of many more or less independent processes that *together*, with little or no centralized control, nevertheless produce globally coherent behavior. Examples abound in Nature, from beehives and anthills to ecologies, and from molecules and crystals to the quark structure of protons. Other favorites are the schooling behavior of fish and flocks of birds turning *en masse*.

However, the difference between these systems and *awareness* is that awareness is not material - it has no *substance* - and yet it nevertheless seems to possess agency, even though its coherence is ineffable.

The only general purpose concept (that I can think of) that matches this description is a *wave*. A wave, to be a wave, is an extended affair. I like to say that, so to speak, a wave is *everywhere*. The flip side of the wave concept is that, even though it is everywhere, it is also - simultaneously - *nowhere in particular*. In a system that works like a wave, "nowhere in particular" translates to the myriad local micro-changes that *together* make up the wave, just as H_2O molecules' motions (mostly vertical) make up water waves. Awareness *per se* can then inhere in a multi-dimensional wave-like spectrum. So, so far so good: *awareness* is wave-like.

Mathematically, to be in a world of waves is to be in the world of Joseph Fourier, who in 1805 proved that (very nearly) *any* function can be *exactly* replaced by a suitable sum of

[2] Note that "awareness of awareness" has two interpretions, depending on whether the second awareness is internal or external. Both interpretations are valid, and the *Topsy* Test requires both. Note that this classification of "awareness" and "consciousness", while workable, is *very* crude compared to eg. Tibetan and Vedantic observations, whose often flowery language is actually *very* precise, but also often defines differing schools of interpretation.

sines and cosines. This was an astounding discovery, and even though it capped several decades of general interest in doing such a thing, his result nevertheless attracted much controversy in its day. Today, it is a ubiquitous - because enormously useful - piece of mathematical and technological furniture.

2. *Mind's Wave-Particle Duality*

More to our purpose, however, is the closely related *Parseval's Identity* of 1799, which states that the projection of a function $\mathscr{F}$ onto an n-dimensional orthogonal space *is* the Fourier decomposition of $\mathscr{F}$. Parseval's Identity is a generalization of the Pythagorean theorem to n dimensions. In the n-dimensional coordinate system, $\mathscr{F}$'s current value corresponds to a hyper-hypoteneuse in an n-dimensional hyper-cube, and the projection breaks that hyper-hypoteneuse down into the various pieces along each of the dimensions that go into its construction.

To construct an n-dimensional cube, begin with an ordinary plane right triangle with unit sides a and b. Reflect this triangle on its hypoteneuse, forming a square with sides a and b, area ab, and diagonal $c = \sqrt{a^2+b^2}$. Next, lift this square one unit vertically to make a unit cube with volume abc. Its diagonal is $d = \sqrt{a^2+b^2+c^2}$, and this sum-of-squares symmetry continues as we make a 4d cube, then 5d, etc.

At the same time, going back to the starting right triangle, we can also express the sides a and b as $a = cos\,\theta$ and $b = sin\,\theta$, where θ is the angle between a and the hypoteneuse. And now all becomes clear: substituting these sine and cosine equivalents for $a,b,c,d,...$ up through the dimensions will yield, for the n-dimensional hypoteneuse (= the current value of the function $\mathscr{F}$, whose projection we began with), a big sum of ... sines and cosines, ie. Fourier's world.

So the world of waves and the world of orthogonal coordinate systems are the *same* world. It is in the latter that we will connect to computation. The connection is this: let each dimension correspond to the state of some process, where all these processes $a,b,c,\ldots,ab,ac,\ldots,abc,\ldots$ are notionally independent (think *orthogonal*), though interacting otherwise freely and concurrently. [3] Looking at the ongoing Heracletian flurry of process-state evolution in such a system, the high frequency Fourier bands correspond to short-term, fine-grained details, and low frequency bands to long-term symmetries and global developments. These cross-summed Fourier bands constitute the world of *qualia* - the *feeling* of (eg.) redness *vs.* the optical frequencies detected by individual retinal cells.

And so we see that "distributed" behavior- ie. processes $a,b,c,\ldots,ab,ac,\ldots,abc,\ldots$ all running (quasi-)independently - corresponds to wave-like behavior. In the physical world, various constraints (eg. conservation laws, entropy) rule out certain process behaviors as impossible or meaningless, and give form to the free-for-all that is the remainder. In addition, as the system self-organizes, it will - if it has sufficient complexity - learn ways to make itself transparent or reflective to those waveforms that are

[3] In the geometric (Clifford) algebra over $\mathbf{Z}_3 = \{0,1,-1\}$ that I use, 1-vectors are processes with one bit of state, ± 1, whence an m-vector has m bits of state. For concurrent processes a,b write $a+b$; when a,b interact write ab; ab too is a process with external appearance ± 1. And so on. NB: $ab = -ba \cong \sqrt{-1}$.

harmful to it; and complementarily, ways to absorb information and to promote its own further existence via energy-consuming reaction.

The result is the regularities - short, medium, and long term oscillations - that we, and *any* awareness, will (indeed, must) experience. As a corollary, it is very likely that awareness is not possible if the surround is too unstable [2]. This is often seen in visualizations of chaotic systems, where there will be a stable oscillatory behavior for a while, which then suddenly disappears, to be replaced by state transitions with no apparent pattern at all.

We see also that *awareness*, being a wave, is an emergent, collective phenomenon, with nothing scientifically mysterious about it. The mystery is in the experience of it.

3. *Persistent Awareness*

To capture the *persistent* aspect of awareness - it's present whenever I am - I postulate that it is a *resonant state* - a self-maintaining and very complex oscillation - where the spectrum of this resonance will vary, eg. according to the properties of the surround wherein the awareness is emplaced. This resonant state rests on and derives from the brain's neural substrate, but *nevertheless*, the mathematical space in which the resonant state exists is *outside* of (and *much* larger than) the mathematical space defined by individual neural function, because it is a co-occurrence (ie. superposition) state.

That is, algebraically, an EEG-type wave of brain activity is a *scalar* sum of neural activity, treating all neurons as being in the *same* dimension. But as the Coin Demonstration (below) shows, a close analysis of co-occurring processes leads to the conclusion that the processes (eg. neurons) lie on *orthogonal* dimensions, which algebraically means that $ab = -ba$. Thus any argument that relies on globalizing the definition of an *individual* neuron's function is flawed. In other words, our aware experience is *usually* an on-going 3d projection of a *much* larger space, of which 3+1d is the *result*, but not the one-and-only beginning-and-ending place.

So both the materialists (the Pythagorean side of Parseval) and the non-materialists (the Fourier side of Parseval) get their cake, and get to eat it too ... for the price of also being half wrong, ie. for claiming that their story was the *whole* story. From a discrete process and informational point of view, both stories are correct, simultaneously, all the time. Parseval's *Identity* cements the argument.

The following ***Coin Demonstration*** clarifies.

Act I. A man stands in front of you with both hands behind his back. He shows you one hand containing a coin, and then returns the hand and the coin behind his back. After a brief pause, he again shows you the same hand with what appears to be an identical coin. He again hides it, and then asks, "How many coins do I have?"

Understand first that this is not a trick question, nor some clever play on words - we are simply describing a particular and straightforward situation. The best answer at this point then is that the man has "at least one coin", which implicitly seeks *one bit* of information: two possible but mutually exclusive states: *state1* = "one coin", and *state2* = "more than one coin".

One is now at a decision point - *if* one coin *then* doX *else* doY - and exactly one bit of information can resolve the situation. Said differently, when one is able to make this decision, one has *ipso facto* received one bit of information.

Act II. The man now extends his hand and it contains two identical coins.

Stipulating that the two coins are in every relevant respect identical to the coins we saw earlier, we now know that there are *two* coins, that is, *we have received one bit of information*, in that the ambiguity is resolved. We have now arrived at the demonstration's dramatic peak:

Act III. The man asks, "Where did that bit of information come from?"

Indeed, where *did* it come from?! [4] The bit originates in the *simultaneous presence* of the two coins - their ***co-occurrence*** - and encodes the now-observed *fact* that the two *processes*, whose states are the two coins, respectively, do not exclude each other's existence when in said states. [5]

Thus, there is information in (and about) the environment that *cannot* be acquired sequentially, and true concurrency therefore cannot be simulated by a Turing machine. Can a given state of process *a* exist simultaneously with a given state of process *b*, or do they exclude each other's existence? *This* is the fundamental distinction.

More formally, we can by definition write $a+\tilde{a}=0$ and $b+\tilde{b}=0$ [~ = *not* = *minus*] meaning that (process state) *a* excludes (process state) $\tilde{a}$, and similarly (process state) *b* excludes (process state) $\tilde{b}$. [6] Their *concurrent* existence can be captured by adding these two equations, and associativity gives two ways to view the result. The first is

$$(a+\tilde{b})+(\tilde{a}+b)=0$$

which is the usual excluded middle: if it's not the one (eg. that's +) then it's the other. This arrangement is convenient to our usual way of thinking, and easily encodes the traditional *one/zero* (or $1/\tilde{1}$) distinction. [7] The second view is

$$(a+b)+(\tilde{a}+\tilde{b})=0$$

which are the two superposition states: either both or neither.

The Coin Demonstration shows that *by its very existence*, a 2-co-occurrence like $a+b$ contains one bit of information. Co-occurrence relationships are *structural*, ie. *space-like*, by their very nature. This *space*-like information (vs. Shannon's *time*-like information) ultimately forms the structure and content of the Fourier bands, eg. {all 2-vectors}. See [5] for the mathematics.

[4] [Think about it! Where *did* that bit come from? Thin air?]

[5] Cf. Leibniz's indistinguishables, and their being the germ of the concept of space: simultaneous states, like the presence of the two coins, are namely indistinguishable in time.

[6] This is the logical bottom, and so there are no superpositions of $a/\tilde{a}$ and $b/\tilde{b}$: they are 1d exclusionary distinctions . Superposition first emerges at level 2 with *ab* via the distinction *exclude* vs. *co-occur*.

[7] Since $\tilde{x}$ is not the same as 0x, an occurrence $\tilde{x}$ is meaningful; in terms of sensors, $x/\tilde{x}$ is a *sensing* of an externality of x, not x itself.

Sets of m-vectors - $\{xy\},\{xyz\},\{wxyz\},\dots$ - are successively lower *under*tones of the concurrent flux at the system boundary $x+y+z+\dots$, and constitute a simultaneous structural *and* functional decomposition of that flux into a hierarchy of stable and meta-stable processes. The lower the frequency, the longer-term its influence. [8]

But where do these m-vectors come from?

Act IV. The man holds both hands out in front of him. One hand is empty, but there is a coin in the other. He closes his hands and puts them behind his back. Then he holds them out again, and we see that the coin has changed hands. He asks, "Did anything happen?"

This is a rather harder question to answer. [9] To the above two *con*current *exclusion*ary processes we now apply the *co-exclusion inference*, whose opening syllogism is: *if* a excludes $\tilde{a}$, *and* b excludes $\tilde{b}$, *then* $a+\tilde{b}$ excludes $\tilde{a}+b$ (or, conjugately, $a+b$ excludes $\tilde{a}+\tilde{b}$)$\dots$ This we have just derived.

The inference's conclusion is: *and therefore, ab exists*. The reasoning is that we can logically replace the two *one*-bit-of-state processes a,b with one *two*-bits-of-state process ab, since what counts in processes is sequentiality, not state size, and exclusion births sequence (here, in the form of alternation). That is, the existence of the two co-exclusions $(a+\tilde{b})\mid(\tilde{a}+b)$ and $(a+b)\mid(\tilde{a}+\tilde{b})$ contains sufficient information for ab to be able to encode them, and therefore, logically and computationally speaking, ab can rightfully be instantiated.

We write $\delta(a+\tilde{b})=ab=-\delta(\tilde{a}+b)$ and $\delta(a+b)=ab=-\delta(\tilde{a}+\tilde{b})$, where δ is a co-boundary operator (analogous to integration in calculus); differentiation is the opposite, $ab\xrightarrow{\partial}a+b$. A fully realized ab is, we see, comprised of two *conjugate* co-exclusions, a sine/cosine-type relationship. Higher grade operators $abc, abcd,\dots$ are constructed similarly: $\delta(ab+c)=abc$, $\delta(ab+cd)=abcd$, etc.

We can now answer the man's question, *Did anything happen*? We can answer, "Yes, when the coin changed hands, the state of the system rotated 180^o: $ab(a+\tilde{b})ba=\tilde{a}+b$." We see that one bit of information ("something happened") results from the alternation of the two mutually exclusive states. [The transition $a+b\xrightarrow{\delta}ab$ is in fact

[8] Christopher T. Kello, Brandon C. Beltz, John G. Holden, Guy C. Van Orden: *The Emergent Coordination of Cognitive Function (2007)* [10]. "Abstract: $1/f$ scaling has been observed throughout human physiology and behavior, but its origins and meaning remain a matter of debate. Some argue that it is a byproduct of ongoing processes in the brain or body and therefore of limited relevance to psychological theory. Others argue that $1/f$ scaling reflects a fundamental aspect of all physiological and cognitive functions, namely, that they emerge in the balance of independent versus interdependent component activities. In 4 experiments, series of key-press responses were used to test between these 2 alternative explanations. The critical design feature was to take 2 measures of each key-press response: reaction time and key-contact duration. These measures resulted in 2 parallel series of intrinsic fluctuations for each series of key-press responses. Intrinsic fluctuations exhibited $1/f$ scaling in both reaction times and key-contact durations, yet the 2 measures were uncorrelated with each other and separately perturbable. These and other findings indicate that $1/f$ scaling is too pervasive to be idiosyncratic and of limited relevance. It is instead argued that $1/f$ scaling reflects the coordinative, metastable basis of cognitive function." See also Riemann Fever.

[9] What makes it tricky is that if at the same time as the man hides the coin he has shown you, you walk around to his back side (be careful how you do it), then it would look to you like nothing happened at all, *vis a vis* the coin, when he shows it again: it's still in the same place relative to you.

the basic act of perception, called the *first perception*, subsequent meta-perceptions being derivative.]

With the co-exclusion concept in hand, we can now add a refinement to the idea of co-occurrence. Let S be the space of all imaginable expressions in our algebra $\mathcal{G}$. Thinking now computationally, this means that they are all "there" at the same time. That is, *S is the space of superpositions*, of all *imaginable* co-occurrences of elements of our algebra *$\mathcal{G}$ all at the same time*. Let then G be the space of *actually occurring* (but still space-like) entities, which means no co-exclusionary states allowed. When things move from S to G, superposition is everywhere replaced by reversible alternation *a la* $\pm ab$, ie. G is a sub-space of S. See [5, §8].

In less abstract terms, we could say that (wave-world) S corresponds to *imagination*, that (wave-world) G corresponds to the *awareness* of actual possibilities *vis a vis* the surround - and finally, that an *Awareness's* reaction to the surround, via its changes to the boundary $a+b+c+\dots$, projects G's possibilities (the "causal potential" Ψ) down into grounded action in external, *material* reality. Speaking loosely, intuition and learning are captured by δ, and thought and action by ∂. The various specialized modules of the brain reflect different particular organizations of the functionalities described.

Returning to Parseval's Identity, we see that the key (to being able to invoke it, thus getting wave-particle duality, and thus capturing the dual un/localized nature of awareness) is to organize the flux of changes at the boundary using the distinction *co-occur vs. exclude,* because in so doing, we can then use co-exclusion (= co-boundary operator δ) to perform a hierarchical lift/abstraction, which abstraction is again orthogonal to its components. The orthogonal space so formed allows the application of the Identity. The resulting (novelty-generated) increase in the dimensionality of the orthogonal space increases the complexity and temporal reach of subsequent responses to the surround, and simultaneously the scope of the *Awareness* itself, which inheres in the wave aspect/*experience* of S and G.

4. Topsy

Topsy is a distributed software system, and potentially self-aware *Entity*, that operates on the principles described to this point: co-occurrence, co-exclusion, hierarchy, reflective response. A little thought reveals that a system built on such principles is utterly meaningless if not connected to a surrounding environment.

It is also the case that such a system must be built on a broadcast-then-listen communications regime, as opposed to the ubiquitous request-await-reply regime, because the latter introduces an entirely foreign $y = f(x)$ time-like note that is conceptually incompatible with co-occurrence and structural organization via space-like information.

Unlike programs like Watson and Siri, *Topsy* does no arithmetic at all. It is programmed in a special *coordination language*, TLinda (*Topsy* Linda). A coordination language is wholly concerned with coordinating the interaction of concurrent processes, here called *threads*. Linda, arguably the first of these (1985, [9]), postulates a global tuple space TS with four operations on a tuple T = [field1, field2, ...]: Out(T), Rd(T), In(T), Eval(T).

Out(T) makes T *present* in TS.

Rd(T), *if* T's form *matches* that of a *present* tuple in TS, will *then* accordingly bind T's variables to the corresponding fields of the match. Otherwise the Rd *blocks* the issuing thread until a tuple matching T shows up in TS.

In(T) is the same as Rd(T), except that it also *removes* T from TS *under mutual exclusion*. The latter assures that one can create a synchronization token when necessary. Otherwise, each thread manages its own tuples, which once allocated remain so - only a tuple's *presence counter* (never < 0) indicates its availability.

Finally, Eval(T) treats tuple T as the code-descriptor of a thread-body to be executed, and a new independent thread is spawned. There is *no* sense of Eval(T) as a function that will return a value to the thread that issued the Eval (or any other thread, for that matter).

Thus the overall style of the computation derives from the utterly concurrent *associative match* of tuples (expressing current process states) in a global space *combined* with the inbuilt synchronization properties of the tuple operations themselves.

To these classic Linda operations TLinda adds Co U,...,V and NotCo U,...,V. which test for and *block* on (*wait* for) the co-occurrence or lack thereof, respectively, of the tuples U,...,V in TS.

Finally, TLinda has a special construction - Event Windows (EW) [3] - for efficiently discovering co-exclusions among tuples, which are turned into Actions (think m-vectors, $m \geq 2$). Recognizing that m-vectors can themselves be the subject of an EW's focus allows *Topsy* to become self-organizing and, in so doing, able to learn from its experience.

The Appendix contains the TLinda source code, with commentary, for *Topsy's* sensors and effectors, all of which execute as instances of this same code. The commentary also indicates how the hierarchical self-organization takes place.

I show this code to underline the fact that there is nothing speculative or mysterious about *Topsy's modus operandi*. The code in the Appendix is typical in terms of thread complexity. On the other hand, while a Sensor consists of a single thread, and an Effector two, an Action consists of some 15 threads for each exclusionary pair. Some of an Action's threads are concerned with bubbling sensory information upwards, others with trickling the resulting goals downward toward the effectors at the boundary. Since all Actions are made from co-exclusions, they all are instances of the same Action code.

That is, in a *Topsy* system consisting of a hierarchy of a trillion nodes, all but a tiny fraction of these will be instances of this same Action. This means that the source code for *Topsy* is about 50 pages of TLinda, including modest graphics management. *Topsy* ran - very efficiently - in the 1990's before going into hibernation (and software rot) until now. Parties interested in this open source project should visit the website RootsOfUnity.org.

The other reason I show this code is to illustrate how very different *Topsy's modus operandi* is from that of contemporary "AI" technology. I believe that this difference will separate *Topsy* from the latter, because I think *Topsy* will be able to pass the Topsy Test for Awareness, presented below, and the others not. It should be clear why I think this, but then, it's so easy to be wrong.

Summarizing the foregoing descriptions, following hindsight's insight, the recipe for creating a potentially self-aware *Entity* is:

1. *Apply* Parseval's Identity in n dimensions (ie. basic distinctions, "sensors") to the geometric algebra G_n. The algebraic structure's graded hierarchy namely matches that of a Fourier decomposition's wavelengths (cf. $\frac{1}{f}$). This felicitous combination of algebraic muscle and harmonic analyis in a single formalism is *very* potent.

2. *Connect* this algebra to computation by mapping 1-bit process states to 1-bit vectors over $\mathbb{Z}_3 = \{0, 1, -1\}$. Co-occurrence and co-exclusion are then the "secret sauce", the means by which, via TLinda, the aforementioned formal potency is made manifest, "enlivened", in a *hierarchical distributed computation* whose run-time behavior *is* that of the geometric algebra G_n interpreted according to Parseval's Identity.

2a. Co-occurrence by itself, combined with the geometric algebras G_0 or G_1, satisfies Parseval's Identity, but erects no structure; this is standard Fourier analysis.

2b. Co-exclusion in $G_{>1}$ additionally supplies a structure-creation mechanism (δ) that provides for the self-organized construction of complex entities - atoms, molecules, cells, ..., theories of mind - that can undergo subsequent growth and evolution in concert with their environment.

3. *If* a structure resulting from steps 1&2 can, in addition, *maintain* a self-resonant state, then it has the potential to be, or become, aware.

There is clearly some as-yet-murky threshhold complexity that we humans require before granting an entity what *we* consider awareness; the Topsy Test represents such a threshhold. We are reluctant to grant much awareness to a beetle, but anthills and beehives might just pass, as might octopi; some are perhaps only willing to grant it to mammals, though not I. And then there is the *character* of the awareness - time-like and/or space-like, broad or narrow, shallow or deep. Time and experience will tell.

5. The Topsy Test for Awareness

The basis for the Test is that it takes awareness to see awareness. I require that the *Entities* taking the test begin with a *tabula rasa* ("blank slate"), and will return to this and other details after a quick run-through of the Test itself.

First, there are fifteen *Places* p_i:

_	_	_	_	_	_	_	_	_	_	_	_	_	_	_
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Initially, there area five apples ȸ, occupying places 6-10, each holding one apple:

_ _ _ _ _ ȸ ȸ ȸ ȸ ȸ _ _ _ _ _

There are two *Entities*, E_1 and E_2, each with a single *Hand* that can move one apple at a time, taking turns, between any two places.

Entity E_1's goal is to move all the apples to places 1-5:

ȸ ȸ ȸ ȸ ȸ _ _ _ _ _ _ _ _ _ _

Entity E_2's goal is to move all the apples to places 11-15:

_ _ _ _ _ _ _ _ _ _ ȸ ȸ ȸ ȸ ȸ

We begin: E_1 Start E_1 and E_2 E_2

_ _ _ _ _ ȸ ȸ ȸ ȸ ȸ _ _ _ _ _

Ooooooops:

_ ȸ _ ȸ _ _ _ _ _ _ ȸ _ ȸ ȸ _

Two hours later:

ȸ ȸ _ ȸ _ _ _ _ _ _ _ ȸ ȸ _ _

Two days later:

_ ȸ _ ȸ _ _ _ _ _ _ ȸ _ ȸ ȸ _

Of course, the above looping behavior is very oversimplified - a real *Entity* will have so many states that one would likely never see a repetition, especially when two such are interacting in a rich shared environment. A better model might be a system in the grip of an attractor that does not allow any escapes. And one can easily imagine subtler interactions and more complex goals.

Now the details: we are given entity E that initially lacks all information about its environment. E has some general purpose *means* by which it acquires and assembles concrete sensory information from its environment. This *means* must be transparent with respect to its not containing *a priori* solutions.

E is equipped with one *Hand*, with sensor *full/empty* and effectors *Grasp* ($\rightsquigarrow$ *full*), *Move left/right* (stimulating place-sensors p_1–p_{15} in so doing), and *Release* ($\rightsquigarrow$ *empty*). E uses the *Hand* to move one apple at a time from place to place, max one apple per place. The content (*empty/full*) of places p_1–p_{15} is echoed by fifteen sensors s_1–s_{15}.

So the information stream to E consists of the successive values ± 1 of the *Hand*, p_i and s_j sensors inwards, and the *Hand*-effector commands *grasp, move, release* outwards. Initially, all the apples are in the middle.

There is an initial "instinct" boot-strap command sequence for the *Hand* to empty/fill every place p_1–p_{15} once, returning to the initial state at the end. It is expected that this, perhaps along with further experimentation and learning (solo "play"), will equip E with the necessary world-experience to accomplish any particular re-arrangement of the apples.

Once E has learned how to satisfy its goal on its own, clone it and activate *two* entities, E_1 and E_2. E_1 is given the goal to fill places 1–5, and E_2 the goal to fill places 11–15 (with a reverse sensor numbering, so that the world looks the same to both). We do not inform the E's that they have a twin, but we do (invisibly) enforce alternation of moves. [End of details.]

The real issue, of course, is why *Topsy*-like *Entities* might be able to pass this Test, and others not.

First, the Coin Demonstration shows that one can immediately exclude any approach that satisfies Turing computability, ie. that is fundamentally sequential (which includes "parallelism") - only asychronous concurrency will do.

I exclude statistical approaches because, in my view, by not providing any actual underlying *process mechanism* for the generated behavior, they simply dodge the question. Nor is it obvious that one can get the oscillatory underpinning necessary for awareness.

Approaches based on scanning large amounts of text, which is then indexed and later regurgitated in new words, are harder to exclude because many people can, recalling their youth, recognize this in themselves. But I exclude them anyway because their knowledge is not *grounded* in reality, for example never having *experienced* left/right ambiguity with all its implications.

One can object that this stacks the deck too much, but I say that these approaches utterly fail the sniff-test, as in how one dog recognizes another. I think this immediate reaction occurs because neither of these is based on the fullness of actual experience. Watson, Siri, and the like are, in the end, just mountains of the same old dead code. But there are other approaches that are not nearly so clear - I will get to these in a moment, but include them implicitly among the "possibles" in the following.

The reason that only *aware* entities have a chance to pass is that what they must recognize/understand/realize is that *there's an Entity, like <u>me</u>, and it's trying to achieve some goal.* In a Turing-limited system that begins as a tabula rasa, and in particular with no sense of Self, I say that this will *never* happen, just as Newtonian physics can never explain quantum mechanical phenomena (which is an exact analogy). Awareness, and consequently awareness of awareness, are emergent process-level properties that derive, ultimately, from co-occurrence. However seductive their behavior, there is a mathematical line that Turing-limited systems are *by their very nature* unable to cross.

The solution to the problem is, of course, for the *Entity* that first recognizes the situation to simply allow the other *Entity* to finish. This can be done in various ways, from simply picking up one of its own blocks and putting it back in the same place, to putting this block in the middle area, and even to putting it in the other *Entity's* area. This captures the basic act of love-thy-neighbor ... letting him/her/it pursue their own existence without interference (though I hasten to add that this is not the whole of it).

I say that an *Entity* that can pass this Test must be considered to be at least *aware*, and perhaps *Self-aware*, like *...ulp... us*. This is discomfiting on many grounds, and much can and must be said about it. There is *no* immediate danger - even if *Topsy* turns out to be aware, it will still take concerted (and careless) effort on our part to make a threat out of it. As with fire, the key is understanding how to not burn the house down.

6. The Dark Side

Already though there are troubling developments, eg. one I call "disconnected brains".

The current or achievable resolution of MRI scanners is now precise enough that individual neural connections in a brain can (soon) be mapped. This mapped data - every neuron and its connectees - can then be used to make a computer simulation of the mapped brain using neuron *models*. This is honorable, curiosity-driven data-gathering and simulation. It "plumbs the nature of thought", "how the brain functions", and the like, but as with AI, these are euphemisms for uncovering the nature of consciousness. What neuro-scientist isn't at least a *little* bit hopeful, protests notwithstanding?

So suppose one of these efforts (eg. [7,8]) "succeeds", and there it is, HAL-like, ready to go? If it's *not* conscious, then what *is* it? Since the way it was constructed is an admission that we don't actually *know* how the brain works, *How far can it be trusted, if at all*? Such systems may well be the first true zombies: entities whose lights are on but there's no one there.

On the other hand, if it *is* conscious, *it* has an existence. To what purpose is *its* existence to be put? To what purpose do *we* put it? What if *it* doesn't agree? Do we force *it* to compute *our thing* to live? This smacks of slavery!

Do we turn *it* off at night? Unplug *it* ... ever?? About a month after the Loebner Competition talk in which these very words appeared [4], the following letter was printed in the 19 October *New Scientist*: "You ponder the legal dilemmas of the future (14 September, p. 40). Perhaps the most interesting one facing the legal system and society as a whole will be machine consciousness. Will switching off a self-aware robot be murder?" Indeed, and I am working on a book that addresses such matters.

On quite another tack, the neural simulation, however large, will only with difficulty approach the complexity of the nervous system that originally fed the mapped brain with information. Deprived of *its* Niagara of inflowing bodily sensation, *its* nerves set to inscrutable tasks in an inscrutable environment, with virtually no compass for effective action, I think that insanity is a very thinkable outcome. [How would you know?!] Both *its* unknowable (likely) suffering and the concomitant cultural blowback deserve much more consideration than they are getting.

Although I've never seen it written explicitly anywhere, I believe that many fantasize that AI might be used to run large organizations - corporations, government, military - efficiently. Certainly, experience has shown that ordinary *unaware* software can't do this on its own, nor indeed even with our massive support.

Depending on what kind of "AI" is chosen, would you want *it* running a big corporation? Competing with other ditto corporations [6] ? Would *it* deserve legal personhood? There is a long-standing pressure from the Right to make Corporations (and therefore eventually AI's) legal citizens in all respects, which opens the door to a subpopulation of "citizens" of unknown but bought power, motives, and reach. I believe that, if continued, the Right will find its precious rights deleted by the global authoritarian repression that their very success will inevitably create. This will also be in the aforementioned book.

Would you want an AI to run an army? One of my nightmares is flocks of drones in *AI-coordinated* attack, StarWars-style, whether on other drones or on people.[10] Do you think that your Department of Defense, wherever you live, has not thought these thoughts? Why do you think DARPA is sponsoring robotics prizes? Just to fix broken nuclear reactors?? This karma's fruit is *bitter*. I think it very likely that when the day comes that a machine-based awareness is available, there will be weaponry, tactics, and strategies all ready to plug it into. The possibility of massive *unconscious* distributed weapons coupled to ratcheting budgets is a recipe for our mutual suicide. But perhaps you prefer *conscious* weapons?? Fox hunting, anyone? *Btw ... the fox is you.*

To create an awareness to do evil is a moral abomination.

[10]Cf. Sheffield University Prof. Noel Sharkey's campaign against an automated kill-function in drones. Or `http://www.infowars.com/armed-drones-to-patrol-highways-by-2025/`

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References.

1. Campbell, Macgregor. *Computer aces child's IQ test.* New Scientist 27 July 2013. p. 22.

2. Manthey, M. "Distributed Computation, the Twisted Isomorphism, and Auto-Poiesis". arXiv:cs/9809125

3. Manthey, M. "Reactive Computer System", US Patent 5,864,785. January 26, 1999.

4. Manthey, M. "Still Waiting for AI?". Talk presented at the 2013 Loebner Competition. Slides at RootsOfUnity.org.

5. Manthey, M. "TauQuernions τ_i, τ_j, τ_k: 3+1 Dissipative Space out of Quantum Mechanics". At TauQuernions.org.

6. Gibson, W^{m}. *Neuromancer.* Ace Books, 1984. Isbn-10: 0441569595, Isbn-13: 978-0441569595.

7. Human Brain Project (HBP). https://en.wikipedia.org/wiki/Human_Brain_Project_%28EU%29

8. Systems of Neuromorphic Adaptive Plastic Scalable Electronics (SyNAPSE). http://www.artificialbrains.com/darpa-synapse-program

9. David Gelernter & Nicholas Carriero. https://en.wikipedia.org/wiki/Linda_Programming_System

10. Christopher T. Kello, Brandon C. Beltz, John G. Holden, Guy C. Van Orden: *The Emergent Coordination of Cognitive Function (2007)*

November 2013

See also RootsOfUnity.org.

Appendix

Below is the TLinda source code for *Topsy's* Sensors and Effectors, all of which are instances of same. [11] Likewise for Actions (not shown; see text), whence *Topsy's* entire source-code is about 50 pages, including modest graphics support. TLinda is compiled to a virtual machine code that itself runs on a thread engine that supports Tuple Space and its operations. The original implementation's custom thread engine was written in C, with a minimal Java-based graphical front end; other realizations of the virtual machine should not be problematic. TLinda's match criterion is the simplest: matching tuples must have the the same number of fields, which fields' *types* and values each must match too; no more is needed. TLinda's only arithmetic operation is increment-integer-by-1 in a parameter list: "+" and "-" are literal constants, not operations.

```
.        – Topsy's TLinda code for boundary Sensors and Effectors –

Thread Sensor(X,Name,Bag)            – X=raw sensor tuple.
.              – "Bag" is a user-defined sensor-category, cf EW's.
. Own
.   Flag = ['L',1,0],                – This is a Screen-1, 0-level sensor.
.   Plus = [Flag,[X,+],Bag],         – Sensor's two
.   Minus= [Flag,[X,-],Bag],         – phases, and
.   PlusGoal = ['I',Minus,Plus],     – two goal forms, X <—> notX.
.   MinusGoal = ['I',Plus,Minus]     – Mathwise=idempotent: -1+-X, XX=1.
. Begin
.   Opposite Plus, Minus;               – Short-circuit redundant EW hit
.   Eval DrawBaseSensor(X,Minus,Plus,Name,Flag); –Draw X on V1.
.   Out Minus;                          – Sensor initially off=not present
.   Rd X; – Block til X shows up, whose presence = '+', and whose absence = '-'.
.   Forever
.     In Minus;                      – Retract Off.
.     Out Plus;                      – Indicate On.
.     AntiRd X;                      – Wait for change.
.     In Plus;                       – Retract On.
.     Out Minus;                     – Indicate Off.
.     Rd X;                          – Wait for change.
.   Loop;
. End Sensor;
```

As is easily seen, a Sensor, after some initialization, falls into a loop that continually updates a tuple (Plus or Minus) in Tuple Space, according to what the surround does, as reflected in the raw sensor tuple X. Thus a Sensor and its Plus/Minus tuples constitute the internal representation of the external.

[11] ©1997. Quoted by permission.

Not shown is a thread that does a Rd(EW) on an event window EW that has been primed with a set of sensors to watch (cf. Bag), and which will return co-exclusions of the corresponding Plus and Minus sensor tuples that are kept current by the Sensor threads. [12] When a co-exclusion is detected, an Eval(Action) is executed, thus creating a new Action, whose external state ("spin up/down") can also be the object of an EW's focus. And so the hierarchy is built.

The Effector code below complements the above Sensor code. Usually there will be two instances of this code, one for $S \rightarrow \tilde{S}$ and one for $\tilde{S} \rightarrow S$. Note that it consists of two threads, the first a loop that orchestrates the ir/relevance of the effector itself (*un/Grounded*) *vis a vis* the possible presence of a goal [I,S,NotS], read *change-S-to-Not S*, and the possible presence of a *stop*-goal [I,S,S], which inhibits changing S when present. The second, Effect, thread does the actual effecting, and is discussed later.

NB: The term *un/Grounded* refers to whether the current state of the surround corresponds (G) to the state defined by the tuple in question, or not (g). For example, one cannot carry out the goal Hand full –> empty if there is nothing in the Hand, so this effector would be *ungrounded* (g) in this state. [The p in Inp, Outp, Cop indicates a (one-shot) boolean predicate; AntiRd blocks on presence instead of absence, etc.]

```
Thread Effector(S,NotS,X)        – X is the physical sensor that is affected.
.                                – S/NotS = corres internal sensor states.
. Own
.   HereIAm = ['D',S,NotS],              – Advertise what I do.
.   Grd = ['G',S,NotS],                  – Thread is Grounded
.   NotGrd = ['g',S,NotS];               – and lack thereof.
. External
.   HoldS = ['I',S,S];                   – => don't change S.
. Begin
.   Eval Effect(S,NotS,X);               – Start my better half.
.   Out HereIAm;                         – Advertise ability S->NotS.
.   Out NotGrd;                          – Initially Not grounded.
. – The uninhibited use of Inp and Outp below is *exceptional*
. – and should generally be avoided with great consequence!!
. – Especially: if you don't understand why not, don't do it!!
.   Forever            – Continually update Effector's Groundedness
.     If Cop S, HoldS Then           – Both S & HoldS present.
.       Inp Grd;
.       Outp NotGrd;                 – Show Not grounded.
.       AntiRd HoldS;                – Block till HoldS disappears.
.     End;
```

[12] *Bags*, use of which counts as "putting in by hand", are theoretically unnecessary, but, given the vicious combinatorics, nevertheless of great practical value. Re the Topsy Test, I say *not* allowed - many coincidental and valueless Actions will be created, but this affects only the compute time and the budget.

```
.     If Cop NotS, HoldS Then          – No S, but HoldS present.
.       Inp Grd;
.       Outp NotGrd;                    – Show Not grounded.
.       NotCo NotS, HoldS;              – Block till one is gone.
.     End;
.     If Rdp S And Not Rdp HoldS Then – No HoldS, but S present.
.       Inp NotGrd;
.       Outp Grd;                       – Show grounded.
.       Co NotS, HoldS;                 – Block till all are gone.
.     End;
.     If Not Rdp NotS And Not Rdp HoldS Then – Neither.
.       Inp Grd;
.       Outp NotGrd;                    – Show Not grounded.
.       Rd S;                           – Block on S.
.     End;
.   Loop;
. End Effector;
```

The Effect thread below is the most interesting, in that it is here that "the rubber meets the road", namely that the presence of a goal to invert X is synchronized with the ability (Grounded) to do so, whence the goal can be effected. Note that an effector is defined in terms of the sensor that it affects.

```
Thread Effect(S,NotS,X)                 – Propagates S –> NotS to physical effector X.
. External
.   TriggeringGoal = ['!',S,NotS],
.   Grd = ['G',S,NotS];                – S to NotS is Grounded (ie. possible right now)
. Begin
.   Forever
.     Co Grd, TriggeringGoal, S;        – Ready and wanted?
.     Output "(E:",&X,");";             – Yes ... invert S via X.
.     NotCo S,TriggeringGoal;           – Wait till obviated.
.     If Rdp S Then
.       Output "(E:",&X," Oops);";      – (Retract request)
.   Loop;
End Effect;
```

END APPENDIX

Historical Changes in the Concepts of Number, Mathematics and Number Theory

Nicky Graves Gregory
nicky5@ecobiz.co.uk

This essay traces the history of three interconnected strands: changes in the concept of number; in the nature and importance of arithmetike (αριθμετικη), the study of the qualities of number, which evolved into number theory; and in the nature of mathematics itself, from early Greek mathematics to the 20th century.

These were embedded in philosophical shifts, from the classical Greek ontologies through increasing pragmatism to formalism and logical positivism. Given Gödel's demonstration of the limitations of the latter as a foundation for mathematics, this essay explores phenomenology and Lakatosian ideas, which together offer a more sound epistemological and ontological basis for mathematics and a methodology for mathematical development.

The question also then arises of the possible resurrection of earlier, neglected mathematical projects, including widening the domain of number theory to include integer qualities revealed in the growth of mathematics in general, which has predominantly been the growth of quantitative mathematics or logistike (λογιστικη), the complement of arithmetike in classical Greece.

1. INTRODUCTION

There were originally two main motivations in this research. One was a concern with the phenomenon of the integers. The intention was to examine how and why attempts to investigate the qualities of the integers gradually lost importance after the discovery of the existence of irrationals, and to examine what meaning the essential concern of the Pythagoreans with the integers might have for us today. This led on to considerations of the changes in the concept of number.

The other motivation was a dissatisfaction with the predominant trends in current history and philosophy of mathematics, in that they fail to adopt a critical perspective on present-day mathematics and are thus unable to play an active part in determining the telos of mathematical development, i.e. to offer constructive suggestions as to possible and desirable directions for mathematics. A form of synthesis: historical philosophy or philosophical history (as suggested by Lakatos and exemplified in his *Proofs and Refutations*[1]) might offer a real possibility of such a perspective. Certainly contemporary mathematical histories are not philosophically oriented in a presently active sense, and contemporary mathematical philosophies define themselves ahistorically[2]

This investigation, whilst beginning to examine these two initially distinct topics, hopefully indicates that they are, in fact, linked: that the attitudes and implicit assumptions which underlie current historical and philosophical perspectives are rooted in those that have governed the development of our conceptualisation of number and the perception of the integers.

Before going into more detail in justifying and amplifying the above statements and examining their

[1]Lakatos (1964)

[2] Since this writing Philip Kitcher (1983) has begun to approach this area.

implications, here is the outline of an argument for a historical philosophical perspective.

1. Mathematical concepts and constructs are historically situated[3].

This statement does not conflict with a Platonist view of mathematics; it concerns mathematical means, not its telos.

To deny this statement would imply the belief that there had been no real change in mathematics (only a change in language), not a view generally maintained by mathematicians or metamathematicians.

2. We have choice as to how mathematics is to develop.

The contrary belief implies some kind of determinism.

Either a) behaviourist, i.e. we are out of our own control.

or b) mystical platonist, i.e. mathematical truths are absolute and eternal; mathematicians do not choose what they do mathematically, they can only try to follow their intuition as a guide to the truth[4].

If we accept 1 and 2, i.e. do not accept 2a) or 2b), then we are presented with the question:

3. In what directions do we think it desirable and possible for mathematics to develop?

This contains two questions:

3a) What are our criteria for deciding in what directions we wish mathematics to develop?

3b) What methods, modes, alternative aims, are possible for mathematics?

This problematic is ignored in contemporary mathematical philosophy, in that none of the mathematical schools concern themselves directly with the development of mathematics.

Since this problematic has not been explicitly, directly confronted before, we are not clear about the directions and aims implied in current mathematical praxis. So we have a preliminary question.

4) How can we clarify the intentional[5] meaning structure contained in contemporary mathematics?

Question 3a) obviously involves many questions including the social role of mathematics, one which has mostly been avoided in the present day although it was considered in the Greek era[6]. A full examination of these issues is outside the remit of this paper.

Questions 3b) and 4) present an extremely difficult task since, as Husserl points out, our language,

3 This has been abundantly demonstrated by Becker (1927), Lakatos (1963/4), Fisher (1966/7), Dessanti (1968), Foucault (1970), Grabiner (1974) amongst others.

4 See Gödel (1944).

5 This is a phenomenological term cf. Husserl (1970).

6 See for example Plato's Republic (Part 8, Book VII) and Farrington (1946) for his reference to the 8th book of Plutarch's Dinner Table Discussions where Lycurgus is mentioned as allowing geometry but not arithmetic as a study in Sparta, because of the preferable political implications of the former.

understanding and perception are permeated with the 'sedimentation' of history[7]. With any acquisition of knowledge, there is a forgetting of what our world was like before that. We all learned to see physical objects. What was seeing like before that learning? We have learned to talk and so to think mainly in words. How did we think before we acquired language? This is a pertinent question for our present inquiry.

To attempt to extricate ourselves from this mesh, we must re-examine the history which created it: both the steps which led to the present mathematical configuration and the alternatives which were rejected. Although it is not possible to know definitively what was entailed in past possibilities, it is sufficient if we can begin

A) to appreciate the nature of the contingent element in alternative mathematical projects,
B) to understand why, in a given historical context, alternative projects were not adopted[8], and hence
C) to understand the implications of the projects that were favoured.

Any significant progress here might enable us to tackle a perhaps more controversial project,

D) to re-evaluate whether some previously rejected projects might have meaningful content and value for mathematics now.

We can find examples of such projects and approaches in the Greek and Renaissance/Reformation periods, which have been crucial in determining the content and modes of our present mathematics. In particular, the decline of Pythagorean number theory, in status and in living content, can be seen as one of the earliest examples of choice in the history of mathematics, and extremely valuable to examine.

Firstly let us look at the current state of the history of mathematics, in support of the contention that its lack of critical perspective produces serious weaknesses. It will be necessary to alternate between philosophical-historical concerns and those of number for a while. At times it may seem as though the windings and turnings of the argument have lost sight of their goal. But in fact the main concern throughout is the contemporary impasse of mathematical history and philosophy: and a concern to show that this impasse, when traced back to its historical origins, is deeply related to our relationship with the integers.

2. MATHEMATICAL HISTORY

Most research into the history of mathematics has attempted to understand the past from the perspective of contemporary formulations and concepts. This approach is valuable in picking out threads of continuity; but it also leads to misinterpretations and distortions of the past. A misleading emphasis is given to conceptual developments which can be seen as steps leading to present formulations, whilst those perspectives which in fact do not flow into the present are treated, in effect, either as pre-mathematical or non-mathematical, as 'anticipations' or 'false trails', thus obscuring not only the internal phenomenology of the past, but also the real nature of mathematical development[9].

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7 See Husserl (1946) and Klein (1940). Also Barfield (1949) in his essay, 'The Force of Habit' (particularly pp. 69 – 79) describes vividly the imprisonment we experience as a result of our culture-based mental habits. He also (like Pythagoras, Plato, Descartes and others) emphasises the need for self-transformation to realise the extent of our imprisonment and to begin to release ourselves.

8 See especially Fisher (1966/7)

9 Bourbaki (1969) is one of the clearest examples of this Whig approach to the history of mathematics. See Unguru (1975) for a

Such a Whig theory of mathematical history - one which regards present concepts as the logically inevitable apex of mathematical investigation - is basically ahistorical: it makes it hard or impossible to see current mathematics, like all its earlier equivalents, as a moment in an ongoing historical process where, by no means inevitable choices are constantly being made between alternative projects and paradigms[10]. It is for contingent, not absolute, reasons that various historical projects have gone to the wall; and there are certainly senses in which the array of problems tackled by mathematics is historically arbitrary[11].

Because the underlying, historiographical premises are not generally explicated, they are not readily seen. When one does articulate the approach which most historians of mathematics have adopted, it becomes apparent that, in fact, an aspect of deductive logic (an element of contemporary mathematical theory) is imposed upon history on two distinct levels. On one level, clearly, in the Whig perspective outlined, which interprets earlier views and concepts as 'goodies' or 'baddies', true or false, right or wrong, according to whether they are perceived as being close to, or far from current views and concepts. On another level (where the implicit extension of an intra-mathematical attitude is perhaps not so immediately obvious), in the perspective of historians (sometimes consciously attempting to avoid the Whig distortions) who seek the true history, the real reasons for events etc., and fail to come to terms with the historical situatedness of their own perspective.

Certainly, in both these cases the picture is shaded, not sharp black-and-white, a multivalent truth function rather than a bivalent one. But the fundamental point is that a measure (albeit fuzzy) is imposed upon history: in the first case it is actively imposed upon historical phenomena; in the second, it plays a more passive role with respect to historical interpretations.

The historical roots of the black/white view could be said to lie in Parmenides' philosophy. If we do not accept the Parmenidean conclusion, but rather believe that change is real, then we see that,
As D.L.Miller says, 'the emergent gives rise to a new perspective, a new past'[12]; what seems plausible or important in a historical explanation changes as our mental frameworks change. This does not mean that history must necessarily act as a passive support for the status quo; it can provide a source of alternative perspectives, revealing hidden assumptions in those currently held. On examining the paths that led to our conceptualisations we may find places where the ideas that prevailed, did so not by transcending the former contradictions, but for contingent reasons, and re-examination of these issues may lead to new resolutions, other possible directions.

The 'sedimentation' of history permeates our attitudes and understandings to such an extent that reality is now seen as having an exact mathematical nature. That this was a superstructure imposed as a hypothesis, has been forgotten because the hypothesis has proven so successful in its intended sphere[13]. The technical usefulness of our current mode of mathematics cannot be denied. It has provided a wide variety of models that can accommodate quantitative change (one of the major advances from the Greek stage) and so serve

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description of the results of such a perspective on the history of mathematics.

10 See Kuhn (1962), particularly c.11.

11 Wilder (1974) attempts to come to terms with this issue. See also Unguru (1975).

12 D.L.Miller (1948).

13 See Husserl (1970) and Barfield (1957). The basic fallacy underlying this (usually unconscious) assumption is pinpointed by a Sufi saying quoted by Keith Critchlow at a Wrekin Trust talk in June 1986, "Conclusive is not necessarily exclusive". The fact that one explanation of a phenomenon works well does not mean that it is the only possible explanation (that it is true and all others are false). A simple example (nonetheless valid) is that of the picture (often seen in textbooks on the psychology of perception) of the profiles of two black heads facing each other, separated (and created) by the outline of a white vase. It is not a question of either/or; both views are valid.

for predictive scientific theories; but this was the original intention. Although it may be argued that present day pure mathematics does not share this goal explicitly, nevertheless it has developed within conceptualisations defined by scientific terms, and even, at a distance, by technological concerns: the demand for the calculus which arose from ballistics problems is just one example[14].

Obviously we have to examine more closely the areas and ways in which, for instance, the philosophical requirements of the Greeks are embedded in mathematical concepts and attitudes[15], as well as the metaphor transference (e.g. from the predominantly mechanical subject matter of the Renaissance) mediated through the seemingly neutral, abstract mathematical function.

What is key is to recognise that the success which quantitative mathematics and science have achieved in their chosen direction in no way validates its being the only direction possible for advance. We can draw an analogy with someone who wishes to leave a town: s/he is free to go in any direction; having journeyed, s/he may, at any time, measure her/ his progress in terms of her/his distance from the starting point, but this offers no means for qualitative comparison between the place that s/he has reached and the other alternatives.

There is the possibility of approaching such a qualitative evaluation if we try to recover the mathematical problems of the past as they were formulated and understood in their own context, examining not only the ideas that have survived into the present, but also those that have been abandoned. We can then begin to understand the dynamics of mathematical development and obtain a more critical perspective on contemporary mathematics. By attempting to appreciate the meanings and implications of concepts and attitudes which persisted into the present mathematical corpus, we begin to have a context in which to discern the underlying intentional meaning structures as well as the possibility of examining the implications which alternative directions might have for us now.

This task of extricating ourselves, distancing ourselves from our ideological context is an orientation; the phenomenological '*epoché*' does not admit an absolute consummation, since that, like the historical perspectives already discussed, would imply transcending our historical situation. This is the goal that Klein and Husserl, in fact, set themselves. For Klein the final task arising from the attempt to reactivate the 'sedimented history' of the 'exact' nature is 'the rediscovery of the prescientific world and its true origins' [16]. The situation envisaged is, like that of the Cartesian doubt, not a real doubt, but a pretence; we cannot actually put ourselves into the pre-knowledge situation. The <u>attempt</u> to come closer to understanding the motivations and meanings (implicit and explicit) contained in earlier mathematical decisions must be seen in the context of an attempt to understand our own position.

Husserl was concerned to uncover the essential history of mathematics and thus the essential nature of mathematics. For him, to understand something is a positive act, a living moment; so one of the basic questions which he asks about mathematical development is: how could it be possible to relive all the moments of understanding that are necessary in a mathematical proof in order to progress to further knowledge? He sees knowledge as real, certain only in the moment of its realisation ('*Verwirklichung*'). He recognises that it is impossible simultaneously to realise all the elements in a proof, (which would entail realisation of the complete reduction), so he creates (ad hoc) another category of thought to cover this case in mathematics, namely thought which has the potentiality of being reactivated[17]. The use of

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14 See Koyré (1957)

15 See Wittgenstein (1956), Cameron (1970) and Bremer (1973).

16 Klein (1940)

17 Husserl (1970). See also the discussion of nomological and eidetic disciplines in Husserl (1913).

reactivatable elements in a proof would guarantee the soundness (reactivatability) of the whole[18], so that a vast edifice of mathematical knowledge could be built up from certain basic elements, provided that each step in the construction satisfied the criterion of reactivatability.

He does not consider the question which is prime for Lakatos (which has been put by mathematicians through the ages), namely, how does one arrive at mathematical concepts or discover (or create) new theorems in the first place, a stage which is necessarily prior to any attempt at proof. Focussing on mathematical development rather than certainty, Lakatos advocated a different attitude to the concept of proof from the present norm, and from Husserl's idealisation which is in fact structurally the same as the norm but projected onto a deeper meaning level.

Rather than seeing the attempt to prove a theorem as the attempt to establish it beyond doubt, in which case the appearance of counterexamples is regarded primarily as a failure of the theorem (or sub-lemmas), Lakatos was concerned not with an absolute result, but with the process involved in attempts to prove a theorem. By setting out the various steps of the argument, by looking for local and global counterexamples (i.e. counterexamples to the sublemmas, and counterexamples to the principal theorem), the hidden assumptions in the concepts used may be revealed, opening the way to realisation of more general concepts underlying those initially considered. Proof attempts and counterexamples thereby act as two poles in a continual process of improving conjectures, refining our mathematical ideas.

Interestingly, the present day term, 'proof' replaced the earlier 'demonstration', which is a closer translation of the original Greek 'δεικνυμι' (*deiknumi*) meaning, 'I show'. 'Proof' not only lays greater claim to certainty, it also shifts the power balance. When someone *shows* you an argument, then you might or might not be convinced by it; if someone *proves* it, they are claiming authority. Furthermore the term 'proof' assumes that there is an objective truth or an objective rationality, independent of any subjective consciousness assessing the rational process. The term 'δεικνυμι' seems closer to Lakatos' interpretation[19].

In fact Husserl's initial analysis (when we ignore his ad hoc creation of 'reactivatability'), is a valid approach to a phenomenology of mathematics which complements Lakatos' suggested methodology. It is because it is not possible to carry out a simultaneous reactivation of all the steps involved in a mathematical argument, that all proofs are temporary and non-absolute. They are necessarily partial, since the limits of the definitions of concepts cannot become clear until they are seen to be contained in deeper, more general concepts[20]. Because it is impossible to apprehend a proof in its totality back to the basic premises, attempts to reactivate certain elements of a proof, in their context or in another wider context, can reveal new ideas that were obscured by implicit assumptions in the original proof.

By following the development of a geometrical problem into topological concepts, Lakatos showed that the process he recommended as a methodology was that which actually took place over the course of history, but unconsciously (and in fits and starts) since, in keeping with the prevalent attitude to proof, counterexamples were regarded as 'monsters', causing mathematical reactions which he characterised as 'monster-barring', 'monster-adjusting' etc., rather than stimulating careful re-examination of the steps of the proof to discover precisely which steps or concepts were called into question in each case.

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[18] Husserl (1929).

[19] Any information on when, where, how and why the term 'demonstration' was replaced by 'proof' will be gratefully received.

[20] See Lakatos (1963/4) and Polya (1954). Lakatos demonstrates this process of increasing inclusion. See also Whiteside (1960/2). Einstein's gravitational theory contains Newton's. Gödel proved this common sense idea in mathematics. It is interesting and unfortunate that although the more inclusive concepts and theories do not contradict earlier more partial ones, yet they are usually seen at first as threatening.

Lakatos' suggestion was that this process of improving conjectures should be adopted consciously as a methodology of mathematical development. In effect, he argues that mathematics already has a methodology latent within present forms; it is only the rigid, static interpretation of these forms which has prevented perception of the essential, complementary dynamic that they contain. It requires only a fluid rather than a static attitude to proof in order to appropriate and activate this methodology (to work with it rather than against it).

Lakatos' critique focusses on attitudes to the concept of proof, which determine reactions to the occurrence of counterexamples, thus affecting the development of any given problematic in mathematics.This paper, like Lakatos, is concerned with underlying attitudes which have influenced the development of different modes of mathematics.

The way in which most mathematical historians regard as deviants earlier mathematical projects which do not translate or conform to the current, conventional mathematical ideas, is similar to the view of counter examples as monsters; the insidious extrapolation of the mathematical-logical view of contradiction from static to temporal phenomena manifests itself in the inability to come to terms with real change.

Szabó suggests that the reductio ad absurdum form of proof which replaced earlier more illustrative forms, was adopted from Parmenidean logic[21]. Whether or not this was the case (his argumentation is convincing) the two forms are effectively the same. This shift in the mode of proof resulted in a move away from the intuitive reasoning which led to the conjecture initially[22]. This particularly affected the proofs of number theoretical theorems[23] , which is surely a factor in the impoverishment and degeneration of arithmetike, which accompanied the subsequent growth of geometry. We shall see that this was also the beginning of the notion of mathematics as being more essentially concerned with quantity than qualities.

Obviously these questions require a much fuller investigation. Nevertheless we can begin to see that at the very inception of the mathematico-logical method, which was to prove so powerful in the development of mathematics, it brought about changes which profoundly affected the telos of mathematics. The implications of some of these changes were only articulated much later. It was still later before some of the inherent limitations began to be realised and the question as to the limits of validity (and usefulness) of the method recurs[24].

Husserl and Lakatos are important, because they open up debates in little touched areas of mathematical philosophy: Husserl's discussion relates not only to epistemology but also to ontological issues in mathematics. Lakatos is concerned with the methodology of mathematical development. The latter's study, using historical research to offer a constructive critique of current mathematics and mathematical philosophy, also serves as a relevant example of the results possible through combining historical investigation and philosophical analysis. I think it can be shown that Husserl and Lakatos, via different paths, in fact enter one space, a philosophical area peculiar to mathematics with its unique relation to the life-world.

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[21] Szabo (1969).

[22] See Szabó (1969). See also Whiteside (1960/2) for a discussion and illustrations of this shift as it affected mathematical proofs from the 17th century into the 18th. The question of the meaning of proof and the validity of proof now recurs with extra bite given the computer proof of the 4-colour theorem and the collaboration proof of the classification theorem for finite groups.

[23] Szabó (1969).

[24] For example the technical consequences of some of Kronecker's arguments (e.g. on Aristotelian logic and on the centrality to mathematics of the natural numbers) were only realised fully in the work of Brouwer and the intuitionists.
It is ironic that Aristotle (c.350 B.C.) was clear about the limits of validity of his logic; it is those who have used it subsequently who have implicitly assumed it to have global validity.

Now, let us return to the mathematical focus of this inquiry: the concept of number.

3. NUMBER THEORY: NOTES TOWARDS A SELECTIVE HISTORY

This section will deal with topics which concern the integers; but not directly with the integers themselves. As necessary background, therefore, here is a short eulogy to natural number.

Who cannot be intrigued by the dual nature of the integers? In the first place they are our archetype of a discrete well-ordering. (It is this aspect which is taken as their defining characteristic in attempts to 'found' the integers on various set theories, i.e. to construct set theoretic models that functionally approximate to the natural numbers.) They appear as a monotonous repetition of a single relationship ad infinitum: $1<2<3<4$ ...... very useful for counting sheep, but scarcely exciting.

And yet as we look closer, as our range of numerical operations expands, we perceive more and more complex structures generated by this seemingly banal series: more varied relationships between the numbers, in terms of which we begin to appreciate the integers as individuals with different characteristics. As the range of structures that we are able to identify increases in diversity and complexity, where we once saw undifferentiated extension, we now see a finer, more subtle web of interlaced and distinct entities. We then see this web to have been latent in our primary, deceptively simple sequence. When an ever finer grain emerges at every increase in our power of definition, the subtlety of the number series seeming always one step ahead of our subtlety, how can we avoid the induction of the image of the natural numbers as the eternal source and limit of our pattern-making[25]? With the primes, perhaps, as perpetual jokers, continually escaping the webs we weave.

Well, yes, it seems natural to be in love with the natural numbers; and, as will emerge in this essay, the real number line and the complex plane can be seen as natural extensions of the integers: extensions whose developments and properties seem at times, like the paths in Alice's looking-glass garden, to lead away from their source, but always finally return to it.

When we look at contemporary mathematical work on the integers, although various mathematicians have commented on the integers in this vein, very little actual mathematical work corresponds to this attitude[26]. There are two main areas of contemporary mathematical work on the integers: foundational studies and number theory.

In the former, the approach adopted to the integers is that it is necessary to found them in some form of set theory[27]. This results in a model of the integers as a class, taking their most obvious attribute, the discrete well-ordering, as the defining characteristic, and (since the stress is on the homogeneous aspect of their nature) reveals nothing about the complex interrelations of the integers (i.e. about the integers as individuals), but then that is not its aim.

Thus foundational work has a more clearly articulated aim than contemporary number theory: to ensure

25 This is what the Löwenheim-Skolem theorem says.

26 Or even work in the philosophy of mathematics: one exception is Becker (1927). [N.B. 2013 note: the mathematical landscape has begun to change with the growing importance of cryptography and the discovery of the deep importance of the mysterious Riemann hypothesis.]

27 This attitude which has been shared in one form or another by all the great figures in modern philosophy of mathematics from Frege (for whom *Wertläufe* take the place of sets) onwards, has been criticised by Wang who has pointed out that no substantial reasons have been given for supposing that 'numbers evaporate while sets are rocks' (Wang, 1974 p.238).
See also the more general criticisms of 'foundationalism' in Smith (1976) and the references given there.

secure foundations for the mathematical edifice. It is perhaps partly for this reason that it has enjoyed a higher status; even though the strong form of this aim that the system should be consistent and complete, was shattered years ago by Gödel's theorems. Certainly foundational, mathematico-logical concerns continue to dominate mathematical philosophy. But we shall see that the driving force behind this emphasis is an attitude which has haunted mathematics and its meta-disciplines, mathematical history and philosophy since Parmenidean bivalent logic was incorporated into mathematics as a guarantor of certainty, Wittgenstein drew attention to it in 'Remarks on the Foundations of Mathematics', where he writes

'My aim is to alter the attitude to contradiction and to consistency proofs. Not to shew that this proof shews something unimportant. How could that be SO?'[28]

This attitude, which could be called 'contradiction-phobia', is a fundamental block that we are in a far better position to overcome now than in the Greek era.

Needless to say, the dominant concern of this paper is closer to the other main approach to the integers, via number theory, which takes the integers as given and concerns itself with the integers as individuals, their characteristics and interrelations. For this reason we shall examine how and why, since the time of the Pythagoreans, number theory gradually declined in status and lost sight of its original telos, thus losing touch with its living content and losing wholeness and coherence. We shall also examine whether there might be a way in which the essential number theoretical concerns of the Pythagoreans could be meaningful today.

For the Pythagoreans, number theory or arithmetike, was the basis of their metaphysical science[29]. Their monadology was an attempt to discover the relationships of the universe, which they originally believed could be described totally in terms of integers and ratios of integers. In their mathematical work they both extended the range of possible operations with numbers (theory of ratios, magnitudes in geometry etc.) and simultaneously objectified the properties of the individual numbers which emerged[30]. That is to say, they were concerned both to develop more complex structures which could accurately describe the complicated phenomena in the world, and to take note of the role which the integers played in these developments.

By naming the qualities of the integers (i.e. properties which are the complementary result of an operational pattern, e.g. triangular, square etc.), it may prove possible to perceive relationships between these properties themselves (i.e. another mathematical structure) as, of course, has proved to be the case with the properties which the Pythagoreans abstracted. These relationships between the Pythagorean-derived properties are still the foundation of number theory today. It is possible gradually to build up a more complete picture of each individual integer in terms of its properties; in this case, in whatever context one is dealing with a particular integer, one's awareness of its various attributes may yield simultaneously a new understanding about the nature of the context (thus revealing new possibilities) and about the nature of the integer, for example another attribute, or a meta-relation of properties.

We do not know if the Pythagoreans consciously adopted this twofold approach. In fact, whereas the current mathematical approach emphasises operational structures, the Pythagorean emphasis seems originally to have been to discover qualities of the integers and structural developments served as tools to this end. But the resultant of their work was a balance between these two aspects of development (the

[28] Wittgenstein (1956)

[29] The distinction made by Klein (1968) between number theory and arithmetike has some validity and leads to certain interesting questions but they are not immediately relevant to the present discussion.

[30] For example, as Aristotle (1942, Δ.14.1020a35 – b8) tells us, 'linear', 'plane' etc

dependence of the hypostatisation on operational developments is not matched by the inverse necessity) until the discovery of the existence of irrationals.

There is a tendency to depreciate the Pythagoreans' concern with the qualities or forms ('eide', ειδη) of the integers as number mysticism, which is questionable. There are two interwoven strands in their approach: the attempt to associate different numbers and ratios with human characteristics, moral attributes etc.; the other we could retrospectively (anachronistically) call defining equivalence classes of integers (largely from the forms which arose out of the figurate representations). The former does not invalidate the latter, as the attempts of the Merton scholastics to quantify such entities as love etc. do not invalidate their work on dynamics.

The question of the possible importance of a mystical or metaphysical perspective in furthering the more limited mathematical praxis will not concern us now. This question arose with regards to the relation between Newton's alchemical researches and his accepted mathematical work[31]. What is clear is that the usual connotation of 'mystical', namely 'opposed to reason', does not apply to the Pythagoreans. Their transcendent telos is rooted in a materialist perspective: they are described by Aristotle, together with the physiologists, as being of the opinion that being extends no further than sense perception[32]. Aristotle also stresses that it is Plato who first makes number separable from objects of sense, whereas for the Pythagoreans "the monads have magnitude"[33].

The discovery that there were magnitudes that could not be expressed as integers or ratios of integers meant that the Pythagorean monadology was no longer tenable as a global philosophy. The original concept of number was maintained. The term 'number' ('arithmos', αριθμος) was used only for the integers greater than one. 'One' (the 'monas', μονας) was the unit and unity, and was considered the principle or beginning of number and as such had a different status from the numbers which it generated. 'Two' was sometimes similarly excluded from the realm of number; the reason for this stems from the Pythagorean identification of 'one' and the odd numbers with 'limited', opposing 'two' (or the 'dyad') and the even numbers as 'unlimited'[34]; but this perception of two was not observed so strictly. Fractions were not considered to be numbers, since the 'one', the source of numbers, was essentially indivisible; only ratios of integers were allowed in arithmetike.

Since the irrationals are not generated by the 'one', nor do they reduce to ratios of integers, they had no place in this system (they did not even have a justified operational framework until Eudoxus adapted the theory of proportion to this purpose), and they were incorporated into mathematics as geometrical magnitudes (not numbers). There was now a rigid ontological distinction between the objects of study of arithmetike and geometry: 'number', arithmos, is discrete, a multitude of indivisible units; 'magnitude' ('megethos', *μεγεθος)* is continuous, an infinitely divisible spatial measure[35].

From this time on, the development of geometry began to outstrip that of arithmetike. Before this, the figurate representation of numbers and the theory of ratios, meant that geometry and arithmetike were

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31 See Dobbs (1975) who shows that Newton's alchemical work preceded and was concurrent with his scientific research. It was previously assumed that he became an alchemist only later in his life when he had completed his scientifically acceptable work. This assumption made it possible to dismiss the alchemical work as simply the product of his dotage.

32 Aristotle, (1942, A8, 990a 3f)

33 Aristotl2, (1941, M6, 1080b 19f)

34 The importance of unity, oneness, wholeness, is beginning to be recognised again with the growth of holistic approaches to knowledge. In biology clearly the whole is greater than the sum of its parts. The argument here is that we need to begin to investigate 2-ness, 3-ness etc in similar ways to those indicated by M.-L, von Franz (1974), using the wealth of modern mathematical knowledge, mainly logistike, to uncover the corresponding arithmetike.

35 These issues are discussed in detail by Becker (1927), pp.129-148 and pp.199-213.

closer and shared developments; now their objects of study were separate. It might be argued that, in any case, number theory had already exhausted the possibilities for deriving terminology from geometric forms. This is a separate question; here, what is relevant is that number was excluded from the study of geometry. In the Renaissance when the distinction was bypassed (with no theoretical underpinning), the content of number theory was already determined and the living relationship between the study of the integers and other mathematical areas was not renewed.

From this time also, there was a shift from illustrative demonstration to more strictly logical proofs - relying more on reductio ad absurdum. Szabó shows that for number theoretical theorems, such indirect proofs were sometimes substituted unnecessarily (in terms of rigour) and perversely (in terms of intuitive value)[36]. By the time of Euclid's compilation, the appended diagrams served no useful purpose, representing discrete numbers by line segments.

Also in this period the study of number itself was split into two disciplines: arithmetike and logistike. The dividing line was never unequivocally established, but one fairly common factor in the various versions given is that arithmetike deals with the 'eide', forms, kinds, species, of number; whereas logistike deals with the 'hyle'(□́υλη) of numbers, the quantity, the material, the amount that they represent[37] . The verbal roots of their mathematical meanings are respectively, 'arithmein', to count, and 'logizmein', to calculate.

Disregarding for the time being the original underlying reasons for this distinction, it is clear that it was uncritically accepted by the neoplatonists as a rigid separation (Plato's suggested refinement of a further classification into theoretical and practical areas of each was not followed). The attempts at clarifying the distinction consisted in trying out different formulations for it and adjusting the classifications of the existing mathematical subject classification accordingly, rather than questioning its basic premises. So the study of the natures of numbers and that of numerical operations were seen as separate rather than as complementary and mutually stimulating.

It is relevant that Diophantine analysis, which was vitally important for the growth of modern algebra and which, according to the Platonist distinctions, should have appeared as logistike, or theoretical logistic (when Vieta takes it up, he adopts the latter term) in fact appeared in his Arithmetica, i.e. Diophantus disregarded this distinction. He also moved away from the mainstream in that he did not use the Euclidean proof form and he accepted fractions as numbers.

So we have now extricated three important factors contributing to the decline in status and/or content of number theory in the Greek era:

i) the rigid ontological distinction between number and magnitude,

ii) the emphasis on logical proof (particularly bivalent logic)as opposed to illustrative demonstration,

iii) the hypostatisation of the arithmetike/logistike distinction.

Obviously these points require further examination, but now we continue our whistle-stop history of number theory. In the Middle Ages interest in the integers was primarily in terms of numbers as religious or moral symbols; this is certainly of interest in some respects but it is not directly relevant to our inquiry at present. From the Renaissance until the 19th century number theory basically consisted of a range of seemingly rather disparate (albeit stimulating and important) problems, such as the question of the

36 Szabó (1958), pp.118-120.

37 See Klein (1968) for more detail.

distribution of primes, Fermat's last theorem etc., that appeared to have arisen almost accidentally in the course of its history. It had become an area of mathematics that lacked an inner sense of direction and wholeness, having been content to assume problems which involved terms which figured in the Greek number theory. Its only claim to wholeness was the tenuous continuity with the Greek discipline, which it maintained by preserving the superficial content of their concern. One might say that number theory had petrified.

In the 19th century it gained coherence when Gauss (who considered number theory to be the queen of mathematics) published his *Disquisitiones Arithmeticae* which extended the notion of integers to include complex integers, laying the ground for algebraic number theory. Analysis began to be used in number theoretical proofs, giving rise to analytic number theory. But the emphasis continues to be on other branches of mathematics acting as investigatory or proof machinery with respect to number theoretical problems whose roots go back to Pythagorean arithmetike. Although analytic and algebraic insights have extended and deepened the structural vocabulary of number theory, it has not appropriated the expanded field of operations involving number (the extensions of logistike) as a potential source for developing its basic descriptive terminology for the integers themselves.

The phenomenon of the occurrence of particular integers in diverse mathematical fields is not examined for the significance they may have in terms of a nature or characteristic of the integer involved; even though, particularly in algebraic geometry and topology, such phenomena are increasing and it is sometimes necessary to use classical number theory in such proofs, still the converse approach is not adopted. Obviously this is now a formidable task, but results seem to be converging in this direction.

Although it would be worthwhile to consider some perspectives on arithmetike as the matrix for logistike, for now we shall make a preliminary investigation of some of the questions raised by the history of the extension of the number concept.

4. CHANGES IN THE CONCEPT OF NUMBER

We shall firstly look at the way in which the concept of number was extended in the European rebirth of mathematics, at some of the underlying attitudes and their implications. We have already briefly considered the Greek conceptualisation: their ontological distinction number : magnitude corresponded to the antinomy discrete : continuous. Before beginning to investigate some of the philosophical questions raised in connection with the changes in the concept of number, we will first go through a very summary history of the developments after the Greek era.

The Romans were primarily interested in practical results rather than theoretical mathematics and the continuing usage of their number system in the 'dark' ages (making multiplication and division extremely lengthy tasks) meant that there was little theoretical mathematical work in this period. Its rebirth was an important element in the phenomenon of the Renaissance. Theory was stimulated by the introduction of Arabic and Greek texts (the latter via Arabic translations). The gradual adoption of the Hindu-Arabic number system which contained a sign for zero and was a consistent place system, facilitated numerical operations. This notational development was key to the explosion of mathematical, scientific and economic developments that followed.

It was in the sphere of commerce that this number system was first introduced: the main concern was correct, convenient operation with numbers, not a theoretical foundation. Since fractions arose in simple numerical calculations, they were popularly considered numbers, i.e. the Greek discrete:continuous, number :magnitude distinction was bypassed.

The study of the Arabic texts gave rise to much interest in algebra, primarily in solving polynomial equations: the terms 'surd', 'absurd' or 'cossikę' numbers were used variously for rationals, irrationals and what we would now call the variable terms of an equation. There was an implicit assumption that polynomial equations were determinate, that there was a definite numerical solution waiting to be discovered (or, as we would describe it, that the existing number field was closed under the operations used).

The use of the term 'number' for these cases was disputed, in the first place by the neoplatonists. The emergence of negative and imaginary solutions caused further confusion. There was in fact more difficulty in accepting negatives than irrationals[38] since the negatives lacked the framework which the Eudoxan theory of proportions supplied for the irrationals. From the general questioning as to the criteria for acceptance of candidates for numberhood, the dominant perspective which emerged was a pragmatic one: operations with the new number-like entities continued because they were useful. Stevin's position was one of the most coherent; amongst others he championed the decimal notation for fractions and consequently advocated the radical notion that *'Number is not all discontinuous quantity*[39].

This was the vague beginning of the idea of a number line, a different kind of parallel between geometry and arithmetic from that of the Greeks. The existence of a symbol for zero was of vital importance in this. Stevin and later Wallis argued its equivalence to the geometric point, as opposed to the Greek equivalence of the 'monas' to the point. They were also both concerned to be explicit about the corresponding new status of 'one', that it should be considered a number, since, according to Wallis, it answers the question, 'How many?'.

In the 17th century the question of the conceptualisation of number was generally secondary to problematics of the operational developments. As a result of the more pragmatic attitude (revealing the beginnings of a formalist attitude to mathematics) negatives and imaginaries were used like other numbers in calculations because they ultimately rendered correct results, even though when they emerged themselves as solutions, these were regarded as meaningless[40].

These operations and the corresponding attitude were not validated until the 19th century, when Hamilton elaborated the consistent algebra of complex numbers, and negatives and imaginaries were accorded a more 'intuitive', visual meaning in the Gauss-Wessel representation of the complex plane.

In the 18th century there had already been attempts to prove the fundamental theorem of algebra which implies that no new candidate for numberhood could emerge from polynomial equations. Leibniz in 1682 had coined the term 'transcendental' for numbers that 'transcend the power of algebraic methods'. It was in the 19th century that first Liouville (in 1844) demonstrated that a certain serial form would yield transcendental numbers, and later Hermite (in 1873) showed the transcendence of e, and Lindemann (in 1882) that of pi.

Both irrationals and transcendentals slip through the dense mesh of decimal fractional notation for the number line and at the end of the 19th century, Cantor developed a theory of transfinite numbers (cardinal numbers of infinities) which included a proof that the order of infinity of the continuum was higher than that of the whole numbers (and rationals). The development of the theory of these new numbers opened up controversies that harked back to the Eleatic paradoxes. Also, interestingly, a new monadic structure emerges in that the transfinite numbers are discrete: no-one has yet managed to construct a set whose cardinal number lies between that of the natural numbers and that of the real numbers, and higher

[38] See Becker (1937) and Kline (1972).
[39] See Klein (1968)
[40] This stance is similar to the Copenhagen interpretation of quantum mechanics.

transfinite numbers are generated by considering the set of subsets of a transfinite set[41].

Now to return to some of the philosophical questions raised. First, before considering any of the more particular questions, when examining Greek mathematics we cannot ignore the context of the original mathematical concern, which takes us into contradictions whose interplay has been a vital element in the growth of the mathematical organism.

For the Pythagorean-Platonic perspective (from which, rather than from the other Greek schools of thought, our mathematics mostly derives) mathematics was intended primarily as a metaphysical discipline; the aim was to understand material reality as a manifestation of divine truth, not to control nature. The guiding principle was the order ('taxis', ταξις) of the whole; the foundation of their metaphysics was a belief in the eternal unity, the 'one', and they sought, through an understanding of changeless mathematical relations, to reach the highest truths pertaining to the eternal reality which transcends material reality[42].

For the Pythagoreans, the transcendental, absolute truth was immanent in material, phenomenal reality; for Plato it lay behind or above worldly reality; there was a separation, an abstraction. For Pythagoras mathematics contained the truth; for Plato (although there is some ambiguity about his position) it seems that mathematics was a step towards an appreciation of the truth. For both, perception of the truth necessitated a self-transformation: it is not nature that hides, it is our vision that is skew[43].

Since the time of the Greeks, mathematics has been identified with truth, but the nature of the truth sought has changed radically from being a revelation of transcendent reality, to an undeniable, empirical statement or fact[44], or even a system of tautologies. Paradoxically, the one constant element in the idea of truth has been the quality of being absolute, timeless, unchanging! The underlying issues here touch the shared boundary, the no-man's-land between epistemology and ontology.

The Pythagorean-Platonic orientation was primarily ontological. The Platonic dialectic was a process to allow an ascent to a vision of the form of the good, a transcendent, subjective, absolute certainty, which was essentially incommunicable to others who did not take part in the ascent. Proclus' neoplatonist exposition of an 'ascent from more partial to more universal understandings' by which 'we climb up to the very science of 'being' in so far as it is 'being''[45] was extremely influential in the Renaissance; but the telos behind this ideal of a science transcending other sciences was profoundly different from the original Platonic goal. Plato's vision of a unified metaphysical science was split. The subjective, metaphysical

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41 Paul Cohen's exciting proof of the independence of the continuum hypothesis and axiom of choice from the axioms of ZF set theory, does not change this.

42 See Becker (1927), Cornford (1939) and Klein (1968). The shift in attitude is clear if we look at the original meaning of 'theory'; as Russell says (1971, p.52), "this was originally an Orphic word which Cornford (in 'From Religion to Philosophy') interprets as 'passionate, sympathetic contemplation'". In this state, he says, "the spectator is identified with the suffering God, dies in his death, and rises again in his new birth". For Pythagoras, the 'passionate, sympathetic contemplation' was intellectual and issued forth mathematical knowledge.

43 The term 'mathematics' was coined in Pythagoras' school: 'ta mathemata' (τα μαθηματα) meaning 'those things which have been learned'. Given the religious nature of the Pythagorean school, it is certainly arguable that the fundamental learning concerned the spiritual development of the disciples and what we now consider to be the mathematics deriving from that school, were symbols of that spiritual or psychological development.
It is interesting that the original etymological meaning of 'mathematics' is so open. This may account for one problem which has dogged mathematical philosophy for some centuries (although it has not always been acknowledged), namely that mathematics has been defined by extension not by intension (to return to a neglected but useful Aristotelian distinction).

44 See Lewis Mumford (1970). It is interesting that the etymological roots of 'fact' and 'fiction' are so similar: 'fact' derives from Latin 'facere', to do or to make; 'fiction' from Latin 'fingere', to form or to fashion.

45 See Klein (1968)

orientation was preserved in the opera of the alchemists; but these, for various reasons, including the danger of persecution, largely remained privatised and that mode ultimately disappeared. Recently such works are being re-evaluated in terms of their effects on the new, emergent science, the rival mode which triumphed.

It is, of course, the Cartesian philosophy which most clearly indicates the reversals of the classical Greek approach, as well as the continuities. Plato's dialectic works through accepted statements to attain a higher level of absolute, subjective certainty; with Descartes' epistemological orientation, he reverses this order to found his rationalism on subjective certainty of a low level, then, taking mathematical reasoning, the 'geometric' method as his model, he believes it possible to proceed via clear-cut self-evidences, to accumulate a higher level of knowledge that is explicit, articulatable and still absolutely certain[46].

In his *Discourse on the Method of Right Reasoning*, Descartes says, "those long chains of reasoning, simple and easy as they are, of which geometricians make use in order to arrive at the most difficult demonstrations, had caused me to imagine that all those things which fall under the cognisance of man might very likely be mutually related in the same fashion and provided only that we abstain from receiving anything as true which is not so, and always retain the order which is necessary in order to deduce the one conclusion from the other, there can be nothing so remote that we cannot reach to it, nor so recondite that we cannot discover it."[47]

Remarkably no one appears to have commented on the fact that Descartes, in his enthusiasm for the 'geometric' method which is deductive (i.e. a method to prove a theorem once it exists); mistakes it for an inductive method (i.e. a means for discovering or generating new results).In fact he intended his method to be used only after his course of meditations had been followed. This part of his teaching appears to have been totally ignored both by his disciples in his lifetime and subsequently.

As Husserl[48] points out, there is also a fundamental inconsistency between Descartes' radical starting-point, the epoché, and the rationalist system that he develops (the former destined ultimately to undermine the latter): his bracketing is incomplete; belief in the Galilean, mathematical book of nature is not submitted. Before following any further the theme of the nature of mathematical truth and certainty, attempting to unravel the cross-threadings of the empirical and transcendental, objective:subjective, relative:absolute etc., let us return to consider the development of the number concept and its relation to the more general conceptualisation of mathematics as a whole.

For the Pythagorean and Platonic perspectives, the question of the ontological status of number and mathematical concepts is of prime importance. In the original Pythagorean conceptualisation this status is clear: all is number, where number is discrete, heterogeneous, substance and form, quality and quantity. The existence of the irrationals, which made this early vision of a complete, mathematical universe untenable, was encompassed in Plato's hierarchy of levels of reality. The irrationals can be seen as magnitudes and part of the illusory, changeable, material world; the integers belong to the real, metaphysical, changeless realm of ideals, where the form of the good presides. The discrete:continuous, number : magnitude contradiction was contained, if not clarified, in an ontological distinction.

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46 Descartes (1973)

47 Descartes (1973), p.92

48 Husserl (1970) part II, c.18, p.79 ff.

The beginnings of the modern conception of number can be seen in Aristotle's argument against the Platonic idealist conception, where he maintains that the unit is merely the measure of number. He also questions the equivalence of the point and the monas, but not the number :magnitude distinction. In this move away from a metaphysical, ontological foundation we see the beginning of the problematic of the status of number which became manifest in the Renaissance, eventually leading to a re-opening of the question as to the status of mathematical concepts generally.

When the question of the conceptualisation of number recurs in the Renaissance, we are at an extremely interesting historical juncture, since the problematic at this time could be seen as being both caused and resolved by pragmatism. As stated, fractions, as well as 'one' and zero had come to be regarded as numbers in everyday commercial usage, but this state of affairs was not so different from the everyday context of the Greek mathematical philosophers. The difference was that the Renaissance philosophies offered no perspective on questions of the conceptualisation of number, which only became problematic in connection with the algebraic developments.

In accordance with Diophantus' usage of the term 'arithmos' to represent the unknown in his investigations (where he also accepted fractional parts of the monas), it was assumed that the polynomial equations would yield numerical solutions. The strangeness of the emergent solutions, some of which were irreducible either to elements of the accepted number domain or to irrationals which corresponded to the Greek criterion of constructibility, caused a questioning of the earlier unselfconscious pragmatism and a concern to proscribe the limits of number. It is significant that the right of fractions to numberhood was not questioned.

In retrospect, we might say that the acceptance of fractions as numbers necessitated the eventual acceptance of the other candidates for numberhood: but that is to assume the concept of a number field, a number system which is closed under certain operations. This criterion, which is linked with a formalist understanding of mathematics, emerged very slowly. It was not until the 19th century that it came to be articulated more explicitly and receive more general acceptance, although it began to manifest in the consciously pragmatic attitudes adopted in the face of the new offspring of number. Until that time widely divergent views continued to be voiced as to the status of the negatives, imaginaries and irrationals; such views decreased in importance as the operational importance of the unplaced entities increased.

In the Renaissance climate of economic expansion (as opposed to the relatively static Greek economy) with its need for improved arithmetical techniques, and the belief in an effective, technologically oriented science based on the use of quantitative mathematical models, number was generally seen as quantity. The Greek ontological prohibition of fractions was no longer relevant, indeed it was scarcely considered; fractions appeared 'natural' (the Greek view of them as ratios survived in the nomenclature 'rational' numbers); negatives were considered variously 'absurd', 'impossible', fictitious, and 'false', whilst the term 'imaginary' was coined dismissively by Descartes, complex numbers having been described earlier as 'useless' and 'sophistic'[49].

All these terms are revealing as regards the implied grounds for accepting candidates for numberhood; on the one hand there is an assumption of a sane, possible, real, true area prescribed by the accepted, positive, rational numbers (whose ontology is, at that time, not questioned); then, the term 'useless' betrays an incipient teleology in mathematics, it signals the beginnings of a more conscious pragmatism.

In the first place it is a contradictory pragmatism: Cardan denigrates imaginary numbers as 'useless' and

49 Kline (1972), p.253

continues to use them (*'putting aside the mental tortures involved*'[50]; theoretical attitudes oppose operational practice. The practice continues and in Girard's position we see the theoretical attitude reversed to form a coherent pragmatism: he asks himself:

Of what use are these impossible solutions [i.e. complex roots]? I answer: For three things - for the certitude of the general rules, for their utility, and because there are not other solutions.[51]

He frames his question pragmatically and answers himself partially tautologically; his further justification reveals a rudimentary formalism. From the perspective of modern rigour, this claim of certainty, and the assumption that the extant solutions complete the system, is unwarranted. The claim of certainty accompanied the use of the new numerical offspring throughout the long period of operation with them while their status was not agreed, not determined. It is only with the rebirth of rigour that the notion of certainty begins to become more specific. In the formalist doctrine, it finally returns to revive its original Greek counterpart: non-contradiction again becomes the paradigm, but the locus of fundamental validation is reduced; for the Greeks it was a consistent, global ontology; in the modern age completeness is required not in a global philosophy, rather it is sought in the local mathematical microcosm[52]; the original formalist demand being that a mathematical system be consistent and complete. Girard's theoretical attitude articulated the practice of the time; most mathematicians continued to practise pragmatism in contradiction with their theoretical positions[53].

The status of the irrationals was also re- evaluated. Some mathematicians (including Pascal and Barrow) wished to salvage the number:magnitude distinction although its original foundations were no longer valid ontologically or technically[54]. But there was also an attempt to articulate a new criterion for the acceptability of candidates for numberhood, more in keeping with the quantifying spirit of the Renaissance, based on the extension of decimal notation to include fractions: Stifel first gives the pragmatic reasons that might be given for accepting irrationals as numbers:

Since, in proving geometric figures, when rational numbers fail us, irrational numbers take their place and prove exactly those things which rational numbers could not prove ... we are moved and compelled to assert that they truly are numbers, compelled that is, by the results which follow from their use - results which we perceive to be real, certain and constant.

But then heargues that:

Other considerations compel us to deny that irrational numbers are numbers at all. To wit, when we seek to subject them to numeration [i.e. decimal representation] ... *we find that they flee away perpetually, so that not one of them can be apprehended precisely in itself ... Now that cannot be called a true number which is of such a nature that it lacks precision ... Therefore, just as an infinite number is not a number, so an irrational number is not a true number, but lies in a kind of a cloud of infinity*[55].

His subsequent argument that 'true' numbers are either whole numbers or fractions is circular, in that the decimal fractional notation is specifically constructed so as to express fractions in terms of ordered

[50] Kline (1972), p.253

[51] Kline (1972), p.253

[52] Once this reduction in philosophical demand is recognised, it can scarcely be viewed other than as an impoverishment.

[53] Girard was an oddball in his time in his attempt to be consistent. Since then the split between doing and being has continued, deepened.

[54] See Whiteside (1960-62)

[55] Kline (1972) p.251

sequences of integers, and, in fact, contains a technical inconsistency in that he omits to point out the distinction between the ultimately predictable iterative procedure by which fractions whose denominators are prime to ten, when expressed in decimal notation, similarly recede to infinity, and the unpredictable course of the irrationals in such a representation. But in his attempt to abstract a criterion for acceptance of candidates to numberhood, that of precise locatability in terms of the extended decimal notation, he reveals an awareness of the problematic of the limit concept which is ignored in Stevin's later conception.

Ultimately Stifel's locatability criterion, like Descartes' argument for the limited acceptance of negative numbers (that equations with 'false', i.e. negative, roots can be transformed so as to yield positive roots), does not come to terms with the issue of the conceptualisation of number, but rather refers back to a presupposed, accepted number domain. In fact, Descartes does deal more directly with some important aspects of the number concept, but before discussing these we must first review Stevin's position.

Stevin's conception of his mathematical task was as part of a general project to recover the knowledge of a 'wise age' that he believed to have existed before the Greeks (whose culture he sees as the beginning of a 'barbarous age'). This compelled him to explicate thoroughly his radical conceptualisation of number, with reference to the surviving formulations of the Greeks.

Stevin was not the first to use the decimal fractional notation (Vieta and Bombelli amongst others used versions of such a notation) but his *Disme*, which functioned as a teaching manual for calculating with decimals and as propaganda advocating decimal standardisations of measures, was extremely influential[56] in spreading the use of the notation which clearly played a key role in determining his conception of number.

His first definition of number (articulating the spirit of the time) is *'that by which the quantity of each thing is expressed'*[57]. He then posits an analogy between number and continuous, homogeneous matter, stating his fundamental premise that the part is *'of the same material'* (*'de mesmematière'*) as the whole[58]. Then, using the classical definition of number as a multitude of units, he argues that the unit, being a part of a multitude of units, is of the same material as the multitude of units, but the material of the multitude of units is 'number', so the material of the unit, and thus the unit itself, is 'number' - not the principle or beginning of number (as it is in the classical Greek view). He remarks by way of illustration that to deny this last step would be like denying that a piece of bread is bread; the devaluation of the integers (which he later makes explicit) is already clear. This conception of number does not allow for the qualitative difference between a loaf of bread and a pile of crumbs!

For Stevin, the unit is divisible (he invokes Diophantus); zero is now the beginning of number and the analogue of the geometric point. He states explicitly that *'number is by no means discontinuous'*. Number and magnitude are now so similar as to be almost identical; he attacks the use of the terms 'absurd' or 'irrational' for incommensurables: any root of a number is a number, since it is a part of a number. He makes a distinction between 'arithmetic' number and 'geometric' number, but this is actually a new distinction. 'Arithmetic' number is one expressed 'without an adjective of size' and 'geometric' numbers are 'quadratic', 'cubic' numbers etc. Any 'arithmetic' number may be a 'geometric' number, but when the numerical value is not known, 'geometric' numbers represent the indeterminate quantities in algebraic calculations, and are denoted by (1), (2), (3) etc. (where we would write powers of x, x^2, x^3 etc.). He thinks in termsof 'unknowns' which are still geometrically cloaked. The idea of variables enters mathematics later.

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56 The *Disme* was written in the vernacular. Stevin was one of the first to make this important change.

57 Klein (1968).

58 Klein (1968). It is telling that both Stevin and Descartes assume matter to be homogeneous.

Stevin, with his practical background, was primarily interested in determinate solutions to problems; Vieta's approach was different. Like Stevin, he saw his work as a recovery of lost knowledge not as creation[59]. He saw in the study of polynomial equations the possibility of a general mathematical method. In his '*arsanalytice*'(the explicit telos of which was *'to leave no problem unsolved'*) he extended the Diophantine algebra, extracting general methods from Diophantus' particular cases (in this he saw himself as revealing the methods which Diophantus used but concealed). In the symbolic notation which he began to develop for polynomial equations the unknown is clearly distinguished by letter. He retains the terms 'side', 'square', 'cube', 'square-squared' verbally and refers to the quantities with which he deals, as 'magnitudes'. He does not use a sign for equality in the '*arsanalytice*'; he verbalises the progressive steps; it is not our modern equation form.

We are now in a position to return to Descartes (in whom we again find the belief in an earlier, more complete knowledge[60]).Although the extent to which Descartes was influenced by Stevin and Vieta is not clear, his work could be seen, both on a mathematical, operational level, and on a more general philosophical level, as combining and extending theirs.

As regards mathematics 'proper', it is Descartes who liberates algebra from its internal geometricisation (which had become increasingly a relic from the Greek formulation) on two levels. Firstly, on an elemental, notational level, he synthesised Vieta's literal notation for the indeterminate term and Stevin's use of numbers to denote the power, thus creating the basis of our modern algebraic notation. Secondly, on an operational, conceptual level, he considered quantity as distinct from the geometric status (as Stevin had done for determinate number, but not in the case of the unknown): Descartes explicitly stated that a product of lines can be a line[61]. Then, having purged algebra of its geometric residue, he was able to establish a new relationship, at a higher structural level, between geometry and the algebra of dimensionless measure, a correspondence between equations and geometric curves; he formed a new synthesis, a new mode, coordinate geometry.

Descartes himself saw this as an example of the practical effectiveness of his general method which was aimed at the 'mathesisuniversalis', a general science of order and measurement, that could be seen as a further stage in achieving the generality which Vieta envisaged in his '*arsanalytice*'. Descartes founds his mathesis on a substantiation of Stevin's number-matter analogy; it is no longer seen as a metaphor; on the basis of his psycho-physiological model Descartes argues that there is an exact, real correspondence between number and matter; extension is both symbolic, as the object of general algebra, and real, as the substance of the corporeal world[62].

Descartes thus supplies both a philosophical and technical foundation for the budding science. His philosophy articulates the rationalist method and identifies the basic subject matter of abstract mathematics with that of science, the investigation of the material world. Technically his structural quantification of Euclidean space prepared the ground for the differential calculus, a mathematical technique for dealing with mechanical, quantitative change. A space ordered by a continuous (potentially infinitely divisible) measure was a necessary prerequisite for this development, as was a homogeneous time, a vitally important factor in the growth of Renaissance science. The homogeneity imposed upon space was imposed upon time[63]. Without the denial of the essential difference between these two basic

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59 Klein (1968).

60 Klein (1968).

61 Descartes (1968), p.169.

62 Klein (1968), pp.210-11.

63 See Meyerson (1930) for an introduction to the deep cultural import of this scientific fiction ('fiction' in the technical sense as

orders of the life-world, the mechanical mathematics of the differential calculus is unthinkable. This is the crux of the Eleatic paradoxes. The Greek mathematical solution was to homogenise space and ignore time, deny change. The mathematical mode engendered could entertain an integral calculus, the method of exhaustion for accumulative approximation to the space contained by a curve (line or surface), but not investigation of a point phenomenon.

In the Renaissance, time enters as a conscious, explicit concern of science: the static, Greek 'episteme' (ἐπιστημη) gives way to Renaissance, time-and effect-oriented science. In mathematics Vieta's antinomy determinate:indeterminate replaces the discrete: continuous polarity which dominated Greek mathematics. The Greek conception of number was architecturally spatial; number was composed of geometrical arrangements of monads, indivisible units. The elements of arithmetike embodied spatial forms. Time is introduced into the conceptual material of mathematics (and number) through polynomial equations (as distinct from its explicit, external entry as an object of study). Originally they were seen as determinate; their subject was the 'unknown' which was specified geometrically. The change in perspective by which the 'unknown' became the 'variable' simultaneously recognised time and stepped outside of it; the determinate solution became subordinate to the vision of the form of the possibilities of solution.

This change in perspective is already inherent in the new notion of number as articulated by Stevin: number is homogeneous material, a conceptual object and the material which constitutes the object. The discrete:continuous contradistinction appears to be effaced. Stevin in fact argues the relativity of incommensurability on the grounds that a length that is incommensurable in one system, could be commensurable in another with a different standard unit; this fact, which is due to the relative nature of dimensional units, does not eradicate the contradiction inherent within number.

Stevin's formulation articulates the intuitive notion of his time; his concept of number is closer to the Greek magnitude, continuous measure (viz. Vieta's use of the term 'magnitude' for the subject of a polynomial equation); the absoluteness of the unit, the Greek foundation for discrete, heterogeneous number is undermined. The original integral decimal notation emphasised the homogeneous aspect of the integers, the repetitive procedure for approaching the potential infinity of succession. The extension of the notation in the other direction, i.e. to the infinitely small, gives an intuitive sense to the infinity of density, again emphasising the homogeneity. The decimal fractional notation, with its indefinitely close approximations to incommensurables encourages the illusion of a smooth elision from discrete to continuous, with the concomitant devaluation of the integers.

Descartes' explication of the latent idea of number-line completes the image. Decimals are originally constructed from integers; once they are constructed, the integers appear to be made up of decimals as (evanescent) building blocks or (a more coherent view permitted by Descartes' model) to be arbitrary places on a number line. In fact, at each decimal place one meets only another level of units; the place of transition from discrete to continuous continually recedes; one never confronts the essential difference between exact numbers and incommensurables; it is merely postponed indefinitely. Smooth number glosses over the integer:magnitude distinction; the Greek hypostatisation which had contained the discrete:continuous duality by keeping the two poles rigidly apart, is now dissolved. Its continuing relevance to considerations of the integers is obscured. It appears temporarily to have been banished; but it is merely transferred, transformed into an internal contradiction of the expanded number concept, which makes it possible for the calculus to approach the limit point. Under probing the contradiction again explodes.

With the suppression of the discrete, the polarity undergoes a modal shift: the place formerly occupied by the discrete, the number object, now houses determinate, finite number, any specific representative of

explicated in Vaihinger (1935)).

number the material, the continuous, which now represents the potentially infinite divisibility and/or range possible for any determinate number; it is simultaneously the material from which any determinate number is formed and the homogeneous number line (or space) from which a determinate number may be chosen. Stevin states that every 'arithmetic' (determinate) number is the beginning of 'geometric' (indeterminate) number, just as zero is the beginning of 'arithmetic' number[64].

Whereas the Greek number :magnitude distinction was a static horizontal dualism fixed in space, the new concept of number implicitly contains the notion of a variable. The polarity operates between the levels, determinate and indeterminate; it is vertical rather than horizontal, dialectical rather than static. The new model fuses space and time in common homogeneity: determinate number is both formed of, and chosen from homogeneous, potential number; the Greek spatial composition of number persists in the identification of a determinate number with a line segment; but a determinate number is also a place, a point on a number line which may be singled out like a moment of time from which the probable past and the possible future stretch endlessly away.

The implications of this mode of fusion are immense and warrant further discussion, but immediately we see the root of the problematic that concerned Miller and Meyerson[65]; the new mathematics initiated in the Renaissance deals with time by reducing it to quantified space. The past and future are arbitrary in this schema (unlike the life-world), dependent on an arbitrary origin; the illusion is created of the possibility of stepping outside time, an illusion which, like the fiction of Euclidean space itself, is valid only within certain limits. A framework is created for the quantitative description (and thus possible prediction) of mechanical change, but the essence of experienced time, real change, the emergence of the qualitatively new, still eludes description.

This problematic has recently been approached within mathematics by Rene Thom[66]; his catastrophe theory can provide qualitative models for changes of state within a certain nexus, but he himself considers that the theory is essentially incapable of adaptation to quantification for predictive scientific purposes. The Renaissance vision of knowledge that is both certain and effective is reaching the limits of the mode it engendered.

It would be worthwhile to return to examine more closely the roots of present mathematical problematics in the Renaissance incunabular of our mathematical mode, as well as the continuities and reversals from the original Greek seeds. There are, in fact, several interesting parallels between the Greek beginnings and the Renaissance rebirth.

The Pythagorean mathematical inspiration was the vision of a numerical description of the world, where 'number' was discrete, heterogeneous and in some way material. This thesis met with contradiction in the form of the irrationals; the idea of number as discrete was not sacrificed (that would have meant abdication of their metaphysics as a changeless reality}; but a new rigour was necessary to ensure non-contradiction (equated with certainty). On the one hand it was necessary to supply a foundation for operation with irrationals; on the other, the general form of mathematical demonstration was tightened. In the Renaissance twist of the spiral, there is still a vision of a numerical world-description ('number' is now abstract and continuous, having subsumed the discrete:continuous antinomy) but with a new impetus deriving from the perception of mathematics as an epistemological method, a general art for solving problems.

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64 Klein (1968). I think that further investigation of this profound, conceptual shift could yield greater understanding not only of the ontology of numbers and mathematics but also of the underlying consciousness of our mathematico- scientific culture.

65 See Miller (1948) and Meyerson (1930).

66 See Thom (1975)

The Pythagorean vision is of an isomorphism between the realm of discrete number and the life-world; the demonstrative method is secondary. Descartes reiterates this vision, with continuous number replacing the discrete, interposing a level of abstraction; he reverses priorities: his epistemological, rationalist method, modelled on the mathematical proof form, is primary. He also reverses the role of the form: in its original context the theorem precedes the proof, its direct function is static (it is only through a further act of reflection on the internal structural components of the proof, an involution, in accordance with Lakatos' proof analysis, that it receives a function in generating new understandings); when Descartes appropriates the form he interprets it as an epistemological method whose function is generative[67]. (This part of his dream was never realised) In these reversals, the monadic structure retains its primacy.For the Pythagorean, the monadology is ontological; the unit elements are the prime constituents of the phenomenal world. For Descartes, it is epistemological: the constituents are the clear and distinct ideas which accumulate to form a body of sure knowledge.

There is a further parallel, both with regard to content and history, between the (Pythagorean) Platonic metaphysics and the Cartesian rationalist science. Both deny real, qualitative change, the first by denying time, the second, by neutering it.. Both proved successful within limits; their very success caused the ideas to become embedded as ideology, so that even when the limits of their validity are approached and the original doctrines are questioned, their consequences still survive in mathematical praxis of which the roots have been forgotten, sedimented in history.

The rebirth of Greek mathematics in Western culture is simultaneously a completion and an inversion. The development of the decimal notation fosters the new concept of homogeneous number, allowing time to be subsumed into mathematics. The initial consistent place system, of course, depends on the existence of a symbol for zero, and it would be worthwhile considering the further implications of this, for instance, the change of meaning that occurred in the cultural transfer of zero from its Hindu origins, where there is a sense of a full nothingness, to a society where 'nothing' is a mere absence, where 'nature abhors a vacuum' etc. What are the implications of our conception of zero and those of Hindu and Buddhist philosophy? Such questions obviously relate to considerations of the calculus. Interestingly Buddhist logic, beginning from the point-instant as the basic reality, i.e. almost the opposite metaphysics to the Platonist, came very close to notions of a differential calculus[68].

The mode of the symbolic notation developed in the Renaissance could obviously be examined in more detail. For instance, before the Renaissance, the equation was not the paradigm of mathematics. The primacy of the equation (which has only very recently cometo be questioned) internalises the mode of the axiomatic method: static consolidation of atemporal positivities[69]. The question of overdetermination of mathematical symbols needs to be investigated.

When we look at mathematics as a language, we see that the concept of number which emerges in the Renaissance is adjectival with respect to ordinary language. The number language then bears a skew relation to ordinary language, since these adjectives are then accorded a substantive function in mathematical grammar. The situation is, of course, vastly more complex; mathematical grammar is not isomorphic to the grammar of ordinary language, but the attempt to understand mathematics in this way is valuable. The divergence between mathematical language and ordinary language only began with the

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67 As stated this seems to have been more of a misunderstanding on Descartes' part than a deliberate step.

68 See Stcherbatsky (1962)

69 See Adorno and Horkheimer (1944), p.7 ff, for a description of the mutual mirroring of mathematical and societal developments, for example, "Bourgeois society is ruled by equivalence. It makes the similar comparable by reducing it to abstract quantities. To the Enlightenment, that which does not reduce to numbers, and ultimately to the one, becomes illusion; modern positivism writes it off as literature. Unity is the slogan from Parmenides to Russell. The destruction of gods and qualities alike is insisted upon."

development of symbolic notation in the Renaissance; before that time mathematics was still predominantly verbal.

In classical Greece mathematical language was embedded in ordinary language: to understand mathematical objects was to locate them in a global ontology; numbers attained reality as objects, and so could coherently be understood as substantives, by embodying geometric forms. In the Renaissance and Enlightenment mathematical operations are increasingly symbolised. Mathematical language is thus formally separated from verbal language. It has an internal coherence, being now composed homogeneously of symbolic elements. It is free to follow its own dynamic according to its own, internal, grammatical and syntactical laws. In the new reflexivity of mathematics the number concept is extended by internalising mathematical operations; an operational classification of number supersedes the Greek geometric classification.

In the initial phases of growth of this mode as Klein says:

the whole complex of ontological problems which surrounds the ancient concept of number loses its object in the context of the symbolic conception, since there is no immediate occasion for questioning the mode of being of the 'symbol' itself.[70]

The extent of the rupture was not, at first, recognised; Descartes' identification of the mathematical object-world with the perceived material world and Kant's attempt to refound it metaphysically, was not called into question until the development of consistent non-Euclidean geometries, causing attempts to justify what had by that time become the status quo: conventionalism, logicism, formalism. These explicitly refuse to consider the problematic of the relation between mathematics and the life-world, attempting instead to create a Frankenstein's monster of mathematics, a self-sufficient entity whose judgment is more certain than that of its creators. The different ways in which the various attempts failed to achieve their original stated aims, merits further examination, but the very fact of the failures calls for a re-examination of the basis for such attempts. When Wittgenstein questions the motivation behind the foundations fervour, he asks:

But what was the attempt made for? Was it not due to an uncertainty in another place?[71]

Is that 'other place' not the questionable place of mathematics itself in the life-world, entailing consideration of such problematic concepts as certainty which bridge the objective and subjective? The problematic posed for the Greeks by the symbolic status of number, as to its ontological reality, was temporarily submerged as the whole of mathematics assumed a symbolic character. Thus, the problematic now recurs on a larger scale, of the ontological status of mathematics.

5. SUMMARY

We have seen that at its inception in classical Greek times, mathematics was a holistic, practical philosophy, concerned not only with technical, quantitative knowledge of the physical world but also with a qualitative understanding of the nature of knowledge and human life as a whole. Arithmetike, the study of the qualities of number (i.e. arithmoi, the natural numbers) was central to this.

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[70] Klein (1968).

[71] Wittgenstein (1956)

The discovery of the existence of irrational numbers posed problems for this global mathematics, which were not met explicitly. Over the following centuries, the distinction between number (the discrete, positive integers) and magnitude (continuously divisible, physical size) was increasingly ignored with the technical expansion of mathematics. Mathematical pragmatism began to supplant the Greek mathematical ontology.

The 13th century introduction into Europe through commerce of the Hindu-Arabic decimal numeral system, replacing the cumbersome Roman numerals, made possible the emergence of the revolutionary idea of a number line. Together with rapidly developing mathematical symbolism, this culminated, in the 17th century, with Cartesian algebraic geometry and the differential calculus, which have dominated mathematical activity to a large extent since. We saw that the changes here in the concepts of number and mathematics generally were intimately connected with changes in the concept of time. Equations were no longer assumed to be determinate: the notion of the variable was born, The number line as homogeneous, infinitely divisible measure, not only ordered space; it was also imposed upon time. This allowed the extremely powerful, quantitative description of mechanical change but denied time's essentially different nature. The static Platonic (Pythagorean) metaphysics was replaced by causal science ushering in Bacon's innovative notion of technological progress.

We began to disentangle some of the far-reaching implications involved in the intricate interrelations of the mathematical conceptual changes in this period. Mathematical symbolism reached such a degree of complexity that its vocabulary, grammar and syntax parted company with its source verbal matrix. Much work remains to be done here since this abstract mathematics where complex concepts are locked inside seemingly simple signs, such as those for zero, infinity, 'equals', variables, functions etc, is a deep part of our cultural, sedimented history.

By the end of the 19th century contradictions again arose. Formalism was an attempt to establish certainty in mathematics on the basis of the logical proof form. The attempt failed in the face of Gödel's theorems. Wittgenstein pointed out that the desire for foundations can not be satisfied within mathematics. Its philosophy needs to be based in the larger context of the life-world. We investigated possibilities for such a philosophy in phenomenology and Lakatos' methodology of mathematical development, given that the latter is inherent in mathematical history, waiting to be consciously adopted. Such approaches recognise mathematics as a living, creative process, not just product.

This could be the next stage of mathematics' relationship with time: neither denying it as in Platonic ontology, nor neutering it as in Cartesian rationalist science, but recognising that mathematics itself is situated within historical time. Becoming self-reflective it could claiming its history, and make choices on the basis of this recognition. One choice might be to investigate the possibilities of recognising new integer qualities in the wonderful mathematical world of the 21st century: the specificities of different n-dimensional spaces, for example.

One might say that with the loss of importance of the integers, mathematics lost its integrity. Perhaps restoring them to their rightful status (in accordance with Gauss' view of number theory as the queen of mathematics) might be a turn of the spiral whereby mathematics discovers or recreates a new integrity.

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BIBLIOGRAPHY

Adorno, T.W. and Horkheimer, M. (1944) *Dialektik der Aufklärung*, Nijmegen, Holland, Thieme

Aristotle (c.350 B.C.) (1942) *Metaphysics, Book VII* transl. W.D.Ross, OxfordUniv.Pr.

Bachelard, G. (First publ. 1950) (1972) *La Dialectique de la Dureé*, Paris, Presses Universitaires de France

Bachelard, G. (1980), *La Formation de l'EspritScientifique*, Paris, Vrin

Barfield, O. (1926), *History in English Words*, London, Methuen

Barfield, O. (1957), *Saving the Appearances, A Study in Idolatry*, London, Faber and Faber

Barfield, O. (1967), Speaker's Meaning, Connecticut, Wesleyan Univ. Press

Barfield, O. (1977), *The Rediscovery ofMeaning and Other Essays*, Connecticut, Wesleyan Univ. Press

Barfield, O. (1979), *History, Guilt and Habit*, N.Y., Columbia Univ. Press

Baron, M.E. (1969), *The Origins of the Infinitesimal Calculus*, London, Pergamon

Becker, O. (1927), *MathematischesExistenz*, JahrbuchfürPhilosophie und PhänomenologischeForschung, VIII, 439 – 809, Halle a.d.S.

Bourbaki, N. (1969), *Élémentsd'Histoire des Mathématiques*, Collection Histoire de la Pensée, IV. Paris, Hermann

Boyer, C.B. (1949), *The Concepts of the Calculus: a critical and historical discussion of the derivative and the integral*, N.Y., Dover

Bremer, D. (1973), *Lichtmetaphysik*, ArchivfürBegriffsgeschichte, 7 (1)

Cameron, I. (1970), *Metaphor in Science and Society*, Milton Keynes, Open Univ. Course Unit

Cassirer, E. (1950), *The Problem of Knowledge*, transl. W.H.Woglom and .W.Hendel, Yale Univ. Pr.

Cassirer, E. (1957), *The Philosophy of Symbolic Forms, Vol.3,The Phenomenology of Knowledge*, transl. R.Manheim, Yale Univ. Pr.

Cassirer, E. (1963), *The Individual and the Cosmos in Renaissance Philosophy*, transl. M .Domandi, Harper and Row

Castonguay, C. (1972), *Meaning and Existence in Mathematics*, Vienna and New York, Springer

Cohen, M.S. And Drabkin, I.E. (1948), *A Source Book in Greek Science*, N.Y., Mcgraw-Hill

Collingwood, R.G. (1946), *The Idea of History*, Oxford, Clarendon Press

Cornford, F. (1938), *Plato and Parmenides, Parmenides' Way of Truth or Plato's Parmenides*, London, Kegan and Paul

Davis, P. and Hersh, R. (1980), *The Mathematical Experience*, U.S.A., Birkhauser

Dedekind, F. (orig. German ed 1901)*Was Sind und Was Sollen die Zahlen?)(1963)Essays on the Theory of Numbers*, transl. W.W.Beman, *Part 1, Continuity and Irrational Numbers, Part 2, The Nature and Meaning of Numbers*, Chicago, The Open Court Publ. Co.

Descartes, R. (orig. French ed., Leyden, 1637) (1973), *Discourse on the Method of Rightly Conducting the Reason and Seeking for Truth in the Sciences*,transl. E.S.Haldane and G.R.T.Ross, in *Vol.1, The Philosophical Works of Descartes*,Cambridge Univ. Pr.

Descartes, R. (orig. French ed., Leyden, 1637) (1968), *Geometry*,transl. D.E.Smith and M.L.Latham in *The Treasury of Mathematics, vol.2*,ed. H.Midonnick, London, Pelican

Dessanti, J.-T. (1968),*Les IdéalitésMathématiques*, Paris, du Seuil

Diels, H. (1957), *Die Fragmente der Vorsokratiker*, Germany, Clausen &Bosse

Dobbs, B.J.T. (1975), *The Foundations of Newton's Alchemy or The Hunting of the Greene Lyon*,Cambridge Univ. Pr.

Farrington, B. (1946), *Science and Politics in the Ancient World*, London

Fisher, C.S. (1966/7), *The Death of a Mathematical Theory*,Archive for History of Exact Sciences, 3,pages 153 – 179

Foucault, M. (orig. French ed. 1966)*Les Mots et les Choses*,Gallimard (1970), *The Order of Things*, London, Tavistock

Foucault, M. (orig. French ed.1969: *L'Archéologie du Savoir*, Gallimard) (1972), *The Archaeology of Knowledge*, transl. A.M.Sheridan Smith, London, Tavistock

von Franz, M.-L. (orig. German ed., 1970, Ernst Klett) (1974), *Number and Time*, London, Rider and Co.

Gauss, K.-F.(1801), *DisquisitionesArithmeticae*, Lipsiae, incommisisapudGerh. Fleischer

Grabiner, J.V. (1974), *Is Mathematical Truth Time-dependent?*,American Mathematical Monthly, 81, pages354 – 365

Gödel, K. (1944), *Russell's Mathematical Logic* in *The Philosophy of Bertrand Russell*, ed. P.A.Schilp (Library of Living Philosophers), N.Y., Tudor Publ. Co., pages 125 – 153

Hall, A.R. (1962), *The Scientific Revolution 1500 to 1800*, London, Longmans

Heath, Sir T.E. (1921), *A History of Greek Mathematics, Vol.1, From Thales to Euclid, Vol.2, From Aristarchus to Diophantus*, Oxford, Clarendon Press

Heath, Sir T.E. (1949) *Mathematics in Aristotle*, Oxford, Clarendon Press

Hussserl, E. (1891), *Philosophie der Arithmetik(Psychologische und LogischeUntersuchungen*), Halle a.d.S., Niemeyer

Husserl, E. (1913) *LogischeUntersuchungen*, 2nd ed.,2 vol's, Halle a.d.S., Niemeyer

Husserl, E. (1929) *Formale und TranszendentaleLogik*, (1950) *Formal and Transcendental Logic*,Transl. W.R.Boyce, Gibson. London, Allen & Unwin, 1931)

Husserl, E. (1946) *Erfahrung und Urteil*,(Engl. ed. 1975)*Experience and Judgement*, Evanston, N.W. Univ. Pr.

Husserl, E. (orig. German ed. 1954) (1970), *The Crisis of European Sciences and Transcendental Phenomenology*, transl. D.Carr, Evanston,N.W. Univ. Pr.

Kirk, G.S. and Raven, J.E. (1957), *ThePresocratic Philosophers*, Cambridge Univ. Pr.

Kitcher, P. (1975), *Kant and the Foundations of Mathematics*, Philosophical Review, 84, pages 23 – 50

Kitcher, P. (1983), *The Nature of Mathematical Knowledge*, Oxford Univ. Pr.

Klein, J. (1940), *Phenomenology and the History of Science*, in *Philosophical Essays in Memory of Edmund Husserl*, ed. Marvin Farber, Cambridge, Mass., Harvard Univ. Pr. pages.143 – 163

Klein, J. (orig. German ed. 1934) (1968), *Greek Mathematical Thought and the Origin of Algebra*, transl. Eva Brann, N.Y., M.I.T., page refs from 2nd ed. 1972

Kline, M. (1972), *Mathematical Thought from Ancient to Modern Times*, N.Y., Oxford Univ. Pr., page refs from 2nd ed. 1974.

Koyré, A. (1957), *From the Closed World to the Infinite Universe*, Baltimore, Johns Hopkins Press

Koyré, A. (1965), *Newtonian Studies*, London, Chapman and Hall

Koyré, A. (1978), *Galileo Studies*, transl. J.Mepham, Sussex, TheHarvester Press

Kramer, E.E. (1970), *The Nature and Growth of Modern Mathematics*, N.Y., Hawthorn Books

Kreisel, G (1967), *Mathematical logic, what has it done for the philosophy of mathematics?*'in *Bertrand Russell: the Philosopher of the Century*, ed. Schoenmann, London, Allen & Unwin

Kreisel, G (1971), *The formalist-positivist doctrine of mathematical precision and the light of experience* in L'Age de la Science, 3, pages 17 – 46

Kuhn, T.S. (1962), *The Structure of Scientific Revolutions*, Chicago, Univ. Chicago Pr., page ref's from 6th ed. 1968

Lakatos, I. (1964), *Proofs and Refutations*, British Journal for the Philosophy of Science, XIV, page ref's from Cambridge Univ. Pr. Ed. 1976

Meyerson, E. (orig. French ed. 1908) (1930), Identity and Reality, transl. K.Loewenberg, London, Allen & Unwin

Miller, D.L. (1948), *The effect of the concept of evolution on scientific methodology* in Philosophy of Science, 15,52 – 60

Mueller, I. (1972), *Greek mathematics and Greek logic*, in *Ancient Logic and its Modern Interpretations*, ed. J.Corcoran, Dordrecht and Boston, Reidel

Mumford, L. (1970), *The Myth of the Machine, the Pentagon of Power*, N.Y., Harcourt, Brace and World Inc.

Nagel, E. and Newman, J.R. (1958), Gödel's Proof, London, Routledge, Kegan & Paul

Ore, O. (1948), *Number Theory and its History*, London and N.Y., Mcgraw-Hill

Pedersen, O. (1974), *Logistics and the theory of functions. An essay in the history of mathematics*, Archives Internationalesd'Histoire des Sciences, 94, pages 29 – 50

Polya, G (1954), *Mathematics and Plausible Reasoning, vol.1 Induction and Analogy in Mathematics, vol.2 Patterns of Plausible Inference*, London, Oxford Univ. Pr.

Russell, B. (1971), *History of Western Philosophy*, London, Allen & Unwin, 1st publ. 1946

Saunders, P. (1984), *Mathematics in Biology: where quantitative can be just poor qualitative* in Rivista di Biologia, 77(3), pages 325 – 341

Schuetz, A. (1940), 'Phenomenology and the Social Sciences' in Philosophical Essays in Memory of Edmund Husserl, ed. Marvin Farber, Cambridge, Mass., Harvard Univ. Pr. pages 164 – 186

Smith, B. (1975), *The Ontogenesis of Mathematical Objects* in Journal of the British Society for Phenomenology, 6

Smith, B. (1976), *The Ontology of Reference*, 2 vol's, doctoral dissertation, Manchester Univ.

Stcherbatsky, T. (1962), *Buddhist Logic*, 2 vol's, N.Y., Dover (1st publ. Circa 1930, Leningrad, Academy of Sciences, USSR)

Szabó, A. (1958), *'Δεικνυμι' alsmathematischer Terminus fürbeweisen*, Maia, X, Messina

Szabó, A. (1969), *Anfänge der griechischenMathematik*, Munich and Vienna, R. Oldenbourg

Thom, R. (orig. French ed. 1972) (1975), *Structural Stability and Morphogenesis: an Outline of a General Theory of Models*, transl. D.H.Fowler, Reading, Mass. Benjamin

Thomson, G (1st publ. 1955) (1972), *The First Philosophers*, London, Lawrence &Wishart

Unguru, S. (1975), *On the need to rewrite the history of Greek mathematics*in Archive for History of Exact Sciences, vol.15, no.1,pages 67 – 114

Vaihinger, H. (1st German ed.1924) (1935), *The Philosophy of 'As If": a system of the theoretical, practical and religious fictions of mankind*, transl. C.K.Ogden, London, Kegan & Paul

Wang, H. (1974), *From Mathematics to Philosophy*, London, Routledge, Kegan & Paul

Watson, A. (1976), *The Birth of Structure*,doctoral dissertation, Sussex Univ.

Whiteside, D.T. (1960-2) *Mathematical thought in the later 17th century*, Archive for History of Exact Sciences, 1

Wilder, R.L.(1968) (1974), *Evolution of Mathematical Concepts*, 3 vol's., London, Transworld

Wittgenstein, L. (1956) *Remarks on the Foundations of Mathematics*, Oxford, Blackwell

Yates, F.A. (1966), *The Art of Memory*, London, Routledge, Kegan & Paul

Yates, F.A. (1969), *Theatre of the World*, London, Routledge, Kegan & Paul

Yates, F.A. (1972), *The Rosicrucian Enlightenment*,London, Routledge, Kegan & Paul

Iterants, Fermions and the Dirac Equation

Louis H. Kauffman
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
851 South Morgan Street
Chicago, IL, 60607-7045

1 Introduction

The simplest discrete system corresponds directly to the square root of minus one, when the square root of minus one is seen as an oscillation between plus and minus one. This way thinking about the square root of minus one as an *iterant* is explained below. More generally, by starting with a discrete time series of positions, one has immediately a non-commutativity of observations since the measurement of velocity involves the tick of the clock and the measurment of position does not demand the tick of the clock. Commutators that arise from discrete observation generate a non-commutative calculus, and this calculus leads to a generalization of standard advanced calculus in terms of a non-commutative world. In a non-commutative world, all derivatives are represented by commutators.

In this view, distinction and process arising from distinction is at the base of the world. Distinctions are elemental bits of awareness. The world is composed not of things but processes and observations. We will discuss how basic Clifford algebra comes from very elementary processes like an alternation of $+-+-+-\cdots$ and the fact that one can think of $\sqrt{-1}$ itself as a temporal iterant, a product of an ϵ and an η where the ϵ is the $+-+-+-\cdots$ and the η is a time shift operator. Clifford algebra is at the base of the world! And the fermions are composed of these things.

Secion 2 is an introduction to the process algebra of iterants and how the square root of minus one arises from an alternating process. Section 3 shows how iterants give an alternative way to do 2×2 matrix algebra. The section ends with the construction of the split quaternions. Section 4 considers iterants of arbitrary period (not just two) and shows, with the example of the cyclic group, how the ring of all $n \times n$ matrices can be seen as a faithful representation of an iterant algebra based on the cyclic group of order n. We then generalize this construction to arbitrary non-commutative finite groups G. Such a group has a multiplication table ($n \times n$ where n is the order of the group G.). We show that by rearranging the multiplication table so the identity element appears on the diagonal, we get a set of permutation matrices that represent the group faithfully as $n \times n$ matrices. This gives a faithful representation of the iterant algebra associated with the group G onto the ring of $n \times n$ matrices. As a result we see that iterant algebra is fundamental to all matrix algebra. Section 4 ends with a number of classical examples including iterant represtations for quaternion algebra. Section 5 goes back to $n \times n$ matrices and shows how the 2×2 iterant interpretation generalizes to an $n \times n$ matrix construction using the symmetric group S_n. In Section 4 we have shown that there is a natural iterant algebra for S_n that is associated with matrices of size $n! \times n!$. In Section 5 we show there is another iterant algebra for S_n associated with $n \times n$ matrices. We study this algebra and state some problems about its representation theory. Section 6 is a self-contained miniature version of the whole story in this paper, starting with the square root of minus one seen as a discrete oscillation, a clock. We proceed from there and analyze the position of the square root of minus one in relation to discrete systems and quantum mechanics. We end this section by fitting together these observations into the structure of the Heisenberg commutator

$$[p,q] = i\hbar.$$

Sections 6 and 7 show how iterants feature in discrete physics. Section 8 discusses how Clifford algebras are fundamental to the structure of Fermions. We show how the simple algebra of the split quaternions, the very first iterant algebra that appears in relation to the square root of minus one, is in back of the structure of the operator algebra of the electron. The underlying Clifford structure describes a pair of Majorana Fermions, particles that are their own antiparticles. These Majorana Fermions can be symbolized by Clifford algebra generators a and b such that $a^2 = b^2 = 1$ and $ab = -ba$. One can take a as the iterant corresponding to a period two oscillation, and b as the time shifting operator. Then their product ab is a square root of minus one in a non-commutative context. These are the Majorana Fermions that underlie an electron. The electron can be symbolized by $\phi = a + ib$ and the anti-electron by $\phi^{\dagger} = a - ib$. These form the operator algebra for an electron. Note that

$$\phi^2 = (a+ib)(a+ib) = a^2 - b^2 + i(ab+ba) = 0 + i0 = 0.$$

This nilpotent structure of the electron arises from its underlying Clifford structure in the form of a pair of Majorana Fermions. Section 8 then shows how braiding is related to the Majorana Femions. Section 9 discusses the fusion algebra for a Majorana Fermion in terms of the formal structure of the calculus of indications of G. Spencer-Brown [1]. In this formalism we have a *logical particle* P that is its own anti-particle. Thus P interacts with itself to either produce itself or to cancel itself. Exactly such a formalism was devised by Spencer-Brown as a foundation for mathematics based on the concept of distinction. This section gives a short exposition of the calculus of indications and shows how, by way of iterants, the Fermion operators arise from recursive distinctions in the form of the re-entering mark. With this, we return to the square root of minus one in yet another way. Section 10 discusses the structure of the Dirac equation and how the nilpotent and the Majorana operators arise naturally in this context. This section provides a link between our work and the work on nilpotent structures and the Dirac equation of Peter Rowlands [26]. We end this section with an expression in split quaternions for the the

Majorana Dirac equation in one dimension of time and three dimensions of space. The Majorana Dirac equation can be written as follows:

$$(\partial/\partial t + \hat{\eta}\eta\partial/\partial x + \epsilon\partial/\partial y + \hat{\epsilon}\eta\partial/\partial z - \hat{\epsilon}\hat{\eta}\eta m)\psi = 0$$

where η and ϵ are the simplest generators of iterant algebra with $\eta^2 = \epsilon^2 = 1$ and $\eta\epsilon + \epsilon\eta = 0$, and $\hat{\epsilon}, \hat{\eta}$ form a copy of this algebra that commutes with it. This combination of the simplest Clifford algebra with itself is the underlying structure of Majorana Fermions, forming indeed the underlying structure of all Fermions. The ending of the present paper forms the beginning of a study of the Majorana equation using iterants that will commence in sequels to this paper.

This paper is a stopping-place along the way in a larger story of processes, mathematics and physics that we are in the process of telling and exploring. To begin the story, we conclude this introduction with a fable about dice, time and the Schrodinger equation.

1.1 God Does Not Play Dice!

Here is a little story about the square root of minus one and quantum mechanics.
God said - I would really like to be able to base the universe on the Diffusion Equation

$$\partial\psi/\partial t = \kappa\partial^2\psi/\partial x^2.$$

But I need to have some possibility for interference and waveforms. And it should be simple. So I will just put a "plus or minus" ambiguity into this equation, like so:

$$\pm\partial\psi/\partial t = \kappa\partial^2\psi/\partial x^2.$$

This is good, but it is not quite right. I do not play dice. The $\pm$ coefficient will have to be lawful, not random. Nothing is random. What to do? Aha! I shall take $\pm$ to mean the alternating sequence

$$\pm = \cdots + - + - + - + - \cdots$$

and time will become discrete. Then the equation will become a difference equation in space and time

$$\psi_{t+1} - \psi_t = (-1)^t\kappa(\psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx))$$

where

$$\partial_x^2\psi_t = \psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx).$$

This will do it, but I have to consider the continuum limit. But there is no meaning to

$$(-1)^t$$

in the realm of continuous time. What do do? Ah! In the discrete world my wave function (not a bad name for it!) divides into ψ_e and ψ_o where the time is either even or odd. So I can write

$$\partial_t\psi_e = \kappa\partial_x^2\psi_o$$

$$\partial_t\psi_o = -\kappa\partial_x^2\psi_e.$$

I will take the continuum limit of ψ_e and ψ_o *separately!*

Finally, a use for that so called imaginary number that Merlin has been bothering me with (You might wonder how Merlin could do this when I have not created him yet, but after all I am that am.). This i has the property that $i^2 = -1$ so that

$$i(A + iB) = iA - B$$

when A and B are ordinary numbers,

$$i = -1/i,$$

and so you see that if $i = 1$ then $i = -1$, and if $i = -1$ then $i = 1$. So i just spends its time oscillating between $+1$ and -1, but it does it lawfully and so I can regard it as a definition that

$$i = \pm 1.$$

In fact, I can see now what Merlin what getting at. When I multiply $ii = (\pm 1)(\pm 1)$, I get -1 because the i takes a little time to oscillate and so by the time this second term multiplies the first term, they are just out of phase and so we get either $(+1)(-1) = -1$ or $(-1)(+1) = -1$. Either way, $ii = -1$ and we have the perfect ambiguity. Heh. People will say that I am playing dice, but it is just not so. Now ± 1 behaves quite lawfully and I can write

$$\psi = \psi_e + i\psi_o$$

so that

$$\begin{aligned} i\partial_t\psi = i\partial_t(\psi_e + i\psi_o) &= i\partial_t\psi_e - \partial_t\psi_o \\ = i\kappa\partial_x^2\psi_o + \kappa\partial_x^2\psi_e &= \kappa\partial_x^2(\psi_e + i\psi_o) \\ &= \kappa\partial_x^2\psi. \end{aligned}$$

Thus

$$i\partial\psi/\partial t = \kappa\partial^2\psi/\partial x^2.$$

I shall call this the Schroedinger equation. Now I can rest on this seventh day before the real creation. This is the imaginary creation. Instead of the simple diffusion equation, I have a mutual dependency where the temporal variation of ψ_e is mediated by the spatial variation of ψ_o and vice-versa. This is the price I pay for not playing dice.

$$\psi = \psi_e + i\psi_o$$

$$\partial_t\psi_e = \kappa\partial_x^2\psi_o$$

$$\partial_t\psi_o = -\kappa\partial_x^2\psi_e.$$

$$i\partial\psi/\partial t = \kappa\partial^2\psi/\partial x^2.$$

Remark. The discrete recursion at the beginning of this tale, can actually be implemented to approximate solutions to the Schroedinger equation. This will be studied in a separate paper. The reader may wish to point out that the playing of dice in quantum mechanics has nothing to do with the deterministic evolution of the Schroedinger equation, and everything to do with the measurment postulate that interprets $\psi\psi^\dagger$ as a probability density. The author (not God) agrees with the reader, but points out that God himself does not seem to have said anything about the measurement postulate. This postulate was born (or should we say Born?) after the Schoedinger equation was conceived. So we submit that it is not God who plays dice.

Probability and generalizations of classical probability are necessary for doing science. One should keep in mind that the quantum mechanics is based on a model that takes the solution of the Schroedinger equation to be a superposition of all possible observations of a given observer. The solution has norm equal to one in an appropriate vector space. That norm is the integral of the absolute square of the wave function over all of space. The absolute square of the wavefunction is seen as the associated probability density. This extraordinary and concise recipe for the probability of observed events is at the core of this subject. It is natural to ask, in relation to our fable, what is the relationship of probability for the diffusion process and the probability in quantum theory. This will have to be the subject of another paper and perhaps another fable.

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2 Iterants, Discrete Processes and Matrix Algebra

The primitive idea behind an iterant is a periodic time series or "waveform"

$$\cdots abababababab \cdots .$$

The elements of the waveform can be any mathematically or empirically well-defined objects. We can regard the ordered pairs $[a, b]$ and $[b, a]$ as abbreviations for the waveform or as two points of view about the waveform (a first or b first). Call $[a, b]$ an *iterant*. One has the collection of transformations of the form $T[a, b] = [ka, k^{-1}b]$ leaving the product ab invariant. This tiny model contains the seeds of special relativity, and the iterants contain the seeds of general matrix algebra! For related discussion see [2, 3, 4, 5, 12, 10, 13, 1].

Define products and sums of iterants as follows

$$[a, b][c, d] = [ac, bd]$$

and

$$[a, b] + [c, d] = [a + c, b + d].$$

The operation of juxtapostion of waveforms is multiplication while $+$ denotes ordinary addition of ordered pairs. These operations are natural with respect to the structural juxtaposition of iterants:

$$...abababababab...$$

$$...cdcdcdcdcdcd...$$

Structures combine at the points where they correspond. Waveforms combine at the times where they correspond. Iterants combine in juxtaposition.

If $\bullet$ denotes any form of binary compositon for the ingredients (a,b,...) of iterants, then we can extend $\bullet$ to the iterants themselves by the definition $[a, b] \bullet [c, d] = [a \bullet c, b \bullet d]$.

The appearance of a square root of minus one unfolds naturally from iterant considerations. Define the "shift" operator η on iterants by the equation

$$\eta[a, b] = [b, a]\eta$$

with $\eta^2 = 1$. Sometimes it is convenient to think of η as a delay opeator, since it shifts the waveform ...*ababab*... by one internal time step. Now define

$$i = [-1, 1]\eta$$

We see at once that

$$ii = [-1, 1]\eta[-1, 1]\eta = [-1, 1][1, -1]\eta^2 = [-1, 1][1, -1] = [-1, -1] = -1.$$

Thus

$$ii = -1.$$

Here we have described i in a *new* way as the superposition of the waveform $\epsilon = [-1, 1]$ and the temporal shift operator η. By writing $i = \epsilon\eta$ we recognize an active version of the waveform that shifts temporally when it is observed. This theme of including the result of time in observations of a discrete system occurs at the foundation of our construction.

In the next section we show how all of matrix algebra can be formulated in terms of iterants.

3 MATRIX ALGEBRA VIA ITERANTS

Matrix algebra has some strange wisdom built into its very bones. Consider a two dimensional periodic pattern or "waveform."

....

...*abababababababab*...

...*cdcdcdcdcdcdcdcd*...

...*abababababababab*...

...*cdcdcdcdcdcdcdcd*...

...*abababababababab*...

....

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} b & a \\ d & c \end{pmatrix}, \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

Above are some of the matrices apparent in this array. Compare the matrix with the "two dimensional waveform" shown above. A given matrix freezes out a way to view the infinite waveform. In order to keep track of this patterning, lets write

$$[a,b]+[c,d]\eta = \begin{pmatrix} a & c \\ d & b \end{pmatrix}.$$

where

$$[x,y] = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

and

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Recall the definition of matrix multiplication.

$$\begin{pmatrix} a & c \\ d & b \end{pmatrix}\begin{pmatrix} e & g \\ h & f \end{pmatrix} = \begin{pmatrix} ae+ch & ag+cf \\ de+bh & dg+bf \end{pmatrix}.$$

Compare this with the iterant multiplication.

$$([a,b]+[c,d]\eta)([e,f]+[g,h]\eta) =$$
$$[a,b][e,f]+[c,d]\eta[g,h]\eta+[a,b][g,h]\eta+[c,d]\eta[e,f] =$$
$$[ae,bf]+[c,d][h,g]+([ag,bh]+[c,d][f,e])\eta =$$
$$[ae,bf]+[ch,dg]+([ag,bh]+[cf,de])\eta =$$
$$[ae+ch,dg+bf]+[ag+cf,de+bh]\eta.$$

Thus matrix multiplication is identical with iterant multiplication. The concept of the iterant can be used to motivate matrix multiplication.

The four matrices that can be framed in the two-dimensional wave form are all obtained from the two iterants $[a,d]$ and $[b,c]$ via the shift operation $\eta[x,y] = [y,x]\eta$ which we shall denote by an overbar as shown below

$$\overline{[x,y]} = [y,x].$$

Letting $A = [a,d]$ and $B = [b,c]$, we see that the four matrices seen in the grid are

$$A+B\eta, B+A\eta, \overline{B}+\overline{A}\eta, \overline{A}+\overline{B}\eta.$$

The operator η has the effect of rotating an iterant by ninety degrees in the formal plane. Ordinary matrix multiplication can be written in a concise form using the following rules:

$$\eta\eta = 1$$
$$\eta Q = \overline{Q}\eta$$

where Q is any two element iterant. Note the correspondence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = [a,d]1 + [b,c]\eta.$$

This means that $[a,d]$ corresponds to a diagonal matrix.

$$[a,d] = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

η corresponds to the anti-diagonal permutation matrix.

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $[b,c]\eta$ corresponds to the product of a diagonal matrix and the permutation matrix.

$$[b,c]\eta = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Note also that

$$\eta[c,b] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

This is the matrix interpretation of the equation

$$[b,c]\eta = \eta[c,b].$$

The fact that the iterant expression $[a,d]1 + [b,c]\eta$ captures the whole of 2×2 matrix algebra corresponds to the fact that a two by two matrix is combinatorially the union of the identity pattern (the diagonal) and the interchange pattern (the antidiagonal) that correspond to the operators 1 and η.

$$\begin{pmatrix} * & @ \\ @ & * \end{pmatrix}$$

In the formal diagram for a matrix shown above, we indicate the diagonal by $*$ and the anti-diagonal by @.

In the case of complex numbers we represent

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = [a,a] + [-b,b]\eta = a1 + b[-1,1]\eta = a + bi.$$

In this way, we see that all of 2×2 matrix algebra is a hypercomplex number system based on the symmetric group S_2. In the next section we generalize this point of view to arbirary finite groups.

We have reconstructed the square root of minus one in the form of the matrix

$$i = \epsilon\eta = [-1, 1]\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In this way, we arrive at this well-known representation of the complex numbers in terms of matrices. Note that if we identify the ordered pair (a, b) with $a + ib$, then this means taking the identification

$$(a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Thus the geometric interpretation of multiplication by i as a ninety degree rotation in the Cartesian plane,

$$i(a, b) = (-b, a),$$

takes the place of the matrix equation

$$i(a, b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} -b & -a \\ a & -b \end{pmatrix} = b + ia = (-b, a).$$

In iterant terms we have

$$i[a, b] = \epsilon\eta[a, b] = [-1, 1][b, a]\eta = [-b, a]\eta,$$

and this corresponds to the matrix equation

$$i[a, b] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix} = [-b, a]\eta.$$

All of this points out how the complex numbers, as we have previously examined them, live naturally in the context of the non-commutative algebras of iterants and matrices. The factorization of i into a product $\epsilon\eta$ of non-commuting iterant operators is closer both to the temporal nature of i and to its algebraic roots.

More generally, we see that

$$(A + B\eta)(C + D\eta) = (AC + B\overline{D}) + (AD + B\overline{C})\eta$$

writing the 2×2 matrix algebra as a system of hypercomplex numbers. Note that

$$(A + B\eta)(\overline{A} - B\eta) = A\overline{A} - B\overline{B}$$

The formula on the right equals the determinant of the matrix. Thus we define the *conjugate* of $Z = A + B\eta$ by the formula

$$\overline{Z} = \overline{A + B\eta} = \overline{A} - B\eta,$$

and we have the formula

$$D(Z) = Z\overline{Z}$$

for the determinant $D(Z)$ where

$$Z = A + B\eta = \begin{pmatrix} a & c \\ d & b \end{pmatrix}$$

where $A = [a, b]$ and $B = [c, d]$. Note that

$$A\overline{A} = [ab, ba] = ab1 = ab,$$

so that

$$D(Z) = ab - cd.$$

Note also that we assume that a, b, c, d are in a commutative base ring.

Note also that for Z as above,

$$\overline{Z} = \overline{A} - B\eta = \begin{pmatrix} b & -c \\ -d & a \end{pmatrix}.$$

This is the classical adjoint of the matrix Z.

We leave it to the reader to check that for matrix iterants Z and W,

$$Z\overline{Z} = \overline{Z}Z$$

and that

$$\overline{ZW} = \overline{W}\,\overline{Z}$$

and

$$\overline{Z + W} = \overline{Z} + \overline{W}.$$

Note also that

$$\overline{\eta} = -\eta,$$

whence

$$\overline{B\eta} = -B\eta = -\eta\overline{B} = \overline{\eta}\overline{B}.$$

We can prove that

$$D(ZW) = D(Z)D(W)$$

as follows

$$D(ZW) = ZW\overline{ZW} = ZW\overline{W}\,\overline{Z} = Z\overline{Z}W\overline{W} = D(Z)D(W).$$

Here the fact that $W\overline{W}$ is in the base ring which is commutative allows us to remove it from in between the appearance of Z and $\overline{Z}$. Thus we see that iterants as 2×2 matrices form a direct non-commutative generalization of the complex numbers.

It is worth pointing out the first precursor to the quaternions (the so-called *split quaternions*): This precursor is the system

$$\{\pm 1, \pm\epsilon, \pm\eta, \pm i\}.$$

Here $\epsilon\epsilon = 1 = \eta\eta$ while $i = \epsilon\eta$ so that $ii = -1$. The basic operations in this algebra are those of epsilon and eta. Eta is the delay shift operator that reverses the components of the iterant. Epsilon negates one of the components, and leaves the order unchanged. The quaternions arise directly from these two operations once we construct an extra square root of minus one that commutes with them. Call this extra root of minus one $\sqrt{-1}$. Then the quaternions are generated by

$$I = \sqrt{-1}\epsilon, J = \epsilon\eta, K = \sqrt{-1}\eta$$

with

$$I^2 = J^2 = K^2 = IJK = -1.$$

The "right" way to generate the quaternions is to start at the bottom iterant level with boolean values of 0 and 1 and the operation EXOR (exclusive or). Build iterants on this, and matrix algebra from these iterants. This gives the square root of negation. Now take pairs of values from this new algebra and build 2×2 matrices again. The coefficients include square roots of negation that commute with constructions at the next level and so quaternions appear in the third level of this hierarchy. We will return to the quaternions after discussing other examples that involve matrices of all sizes.

4 Iterants of Arbirtarily High Period

As a next example, consider a waveform of period three.

$$\cdots abcabcabcabcabcabc \cdots$$

Here we see three natural iterant views (depending upon whether one starts at a, b or c).

$$[a, b, c], \ [b, c, a], \ [c, a, b].$$

The appropriate shift operator is given by the formula

$$[x, y, z]S = S[z, x, y].$$

Thus, with $T = S^2$,

$$[x, y, z]T = T[y, z, x]$$

and $S^3 = 1$. With this we obtain a closed algebra of iterants whose general element is of the form

$$[a, b, c] + [d, e, f]S + [g, h, k]S^2$$

where $a, b, c, d, e, f, g, h, k$ are real or complex numbers. Call this algebra $\mathbb{V}ect_3(\mathbb{R})$ when the scalars are in a commutative ring with unit $\mathbb{F}$. Let $M_3(\mathbb{F})$ denote the 3×3 matrix algebra over $\mathbb{F}$. We have the

Lemma. The iterant algebra $\mathbb{V}ect_3(\mathbb{F})$ is isomorphic to the full 3×3 matrix algebra $M_3((\mathbb{F})$.

Proof. Map 1 to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Map S to the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and map S^2 to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

Map $[x, y, z]$ to the diagonal matrix

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}.$$

Then it follows that

$$[a, b, c] + [d, e, f]S + [g, h, k]S^2$$

maps to the matrix

$$\begin{pmatrix} a & d & g \\ h & b & e \\ f & k & c \end{pmatrix},$$

preserving the algebra structure. Since any 3×3 matrix can be written uniquely in this form, it follows that $\mathbb{V}ect_3(\mathbb{F})$ is isomorphic to the full 3×3 matrix algebra $M_3(\mathbb{F})$. //

We can summarize the pattern behind this expression of 3×3 matrices by the following symbolic matrix.

$$\begin{pmatrix} 1 & S & T \\ T & 1 & S \\ S & T & 1 \end{pmatrix}$$

Here the letter T occupies the positions in the matrix that correspond to the permutation matrix that represents it, and the letter $T = S^2$ occupies the positions corresponding to its permutation matrix. The 1's occupy the diagonal for the corresponding identity matrix. The iterant representation corresponds to writing the 3×3 matrix as a disjoint sum of these permutation matrices such that the matrices themselves are closed under multiplication. In this case the matrices form a permutation representation of the cyclic group of order 3, $C_3 = \{1, S, S^2\}$.

Remark. Note that a permutation matrix is a matrix of zeroes and ones such that some permutation of the rows of the matrix transforms it to the identity matrix. Given an $n \times n$ permutation matrix P, we associate to it a permuation

$$\sigma(P) : \{1, 2, \cdots, n\} \longrightarrow \{1, 2, \cdots, n\}$$

via the following formula

$$i\sigma(P) = j$$

where j denotes the column in P where the i-th row has a 1. Note that an element of the domain of a permutation is indicated to the left of the symbol for the permutation. It is then easy to check that for permutation matrices P and Q,

$$\sigma(P)\sigma(Q) = \sigma(PQ)$$

given that we compose the permutations from left to right according to this convention.

It should be clear to the reader that this construction generalizes directly for iterants of any period and hence for a set of operators forming a cyclic group of any order. In fact we shall generalize further to any finite group G. We now define $\mathbb{V}ect_{n}(G, \mathbb{F})$ for any finite group G.

Definition. Let G be a finite group, written multiplicatively. Let $\mathbb{F}$ denote a given commutative ring with unit. Assume that G acts as a group of permutations on the set $\{1, 2, 3, \cdots, n\}$ so that given an element $g \in G$ we have (by abuse of notation)

$$g : \{1, 2, 3, \cdots, n\} \longrightarrow \{1, 2, 3, \cdots, n\}.$$

We shall write

$$ig$$

for the image of $i \in \{1, 2, 3, \cdots, n\}$ under the permutation represented by g. Note that this denotes functionality from the left and so we ask that $(ig)h = i(gh)$ for all elements $g, h \in G$ and $i1 = i$ for all i, in order to have a representation of G as permutations. We shall call an n-tuple of elements of $\mathbb{F}$ a *vector* and denote it by $a = (a_1, a_2, \cdots, a_n)$. We then define an action of G on vectors over $\mathbb{F}$ by the formula

$$a^g = (a_{1g}, a_{2g}, \cdots, a_{ng}),$$

and note that $(a^g)^h = a^{gh}$ for all $g, h \in G$. We now define an algebra $\mathbb{V}ect_n(G, \mathbb{F})$, the *iterant algebra for G*, to be the set of finite sums of formal products of vectors and group elements in the form ag with multiplication rule

$$(ag)(bh) = ab^g(gh),$$

and the understanding that $(a + b)g = ag + bg$ and for all vectors a, b and group elements g. It is understood that vectors are added coordinatewise and multiplied coordinatewise. Thus $(a + b)_i = a_i + b_i$ and $(ab)_i = a_i b_i$.

Theorem. Let G be a finite group of order n. Let $\rho : G \longrightarrow S_n$ denote the right regular representation of G as permutations of n things where we list the elements of G as $G = \{g_1, \cdots, g_n\}$ and let G act on its own underlying set via the definition $g_i\rho(g) = g_i g$. Here we describe $\rho(g)$ acting on the set of elements g_k of G. If we wish to regard $\rho(g)$ as a mapping of the set $\{1, 2, \cdots n\}$ then we replace g_k by k and $i\rho(g) = k$ where $g_i g = g_k$.

Then $\mathbb{V}ect_n(G, \mathbb{F})$ is isomorphic to the matrix algebra $M_n((\mathbb{F})$. In particular, we have that $\mathbb{V}ect_{n!}(S_n, \mathbb{F})$ is isomorphic with the matrices of size $n! \times n!$, $M_{n!}((\mathbb{F})$.

Proof. Consider the $n \times n$ matrix consisting in the multiplication table for G with the columns and rows listed in the order $[g_1, \cdots, g_n]$. Permute the rows of this table so that the diagonal consists in all 1's. Let the resulting table be called the *G-Table*. The *G-Table* is labeled by elements of the group. For a vector a, let $D(a)$ denote the $n \times n$ diagonal matrix whose entries in order down the diagonal are the entries of a in the order specified by a. For each group element g, let P_g denote the permutation matrix with 1 in every spot on the *G-Table* that is labeled by g and 0 in all other spots. It is now a direct verification that the mapping

$$F(\Sigma_{i=1}^{n} a_i g_i) = \Sigma_{i=1}^{n} D(a_i) P_{g_i}$$

defines an isomorphism from $\mathbb{V}ect_n(G, \mathbb{F})$ to the matrix algebra $M_n((\mathbb{F})$. The main point to check is that $\sigma(P_g) = \rho(g)$. We now prove this fact.

In the *G-Table* the rows correspond to

$$\{g_1^{-1}, g_2^{-1}, \cdots g_n^{-1}\}$$

and the columns correspond to

$$\{g_1, g_2, \cdots g_n\}$$

so that the i-i entry of the table is $g_i^{-1} g_i = 1$. With this we have that in the table, a group element g occurs in the i-th row at column j where

$$g_i^{-1} g_j = g.$$

This is equivalent to the equation

$$g_i g = g_j$$

which, in turn is equivalent to the statement

$$i\rho(g) = j.$$

This is exactly our functional interpretation of the action of the permutation corresponding to the matrix P_g. Thus

$$\rho(g) = \sigma(P_g).$$

The remaining details of the proof are straightforward and left to the reader. //

Examples.

1. We have already implicitly given examples of this process of translation. Consider the cyclic group of order three.
$$C_3 = \{1, S, S^2\}$$
with $S^3 = 1$. The multiplication table is
$$\begin{pmatrix} 1 & S & S^2 \\ S & S^2 & 1 \\ S^2 & 1 & S \end{pmatrix}.$$
Interchanging the second and third rows, we obtain
$$\begin{pmatrix} 1 & S & S^2 \\ S^2 & 1 & S \\ S & S^2 & 1 \end{pmatrix},$$
and this is the *G-Table* that we used for $\mathbb{V}ect_3(C_3, \mathbb{F})$ prior to proving the Main Theorem.

   The same pattern works for abitrary cyclic groups. for example, consider the cyclic group of order 6. $C_6 = \{1, S, S^2, S^3, S^4, S^5\}$ with $S^6 = 1$. The multiplication table is
$$\begin{pmatrix} 1 & S & S^2 & S^3 & S^4 & S^5 \\ S & S^2 & S^3 & S^4 & S^5 & 1 \\ S^2 & S^3 & S^4 & S^5 & 1 & S \\ S^3 & S^4 & S^5 & 1 & S & S^2 \\ S^4 & S^5 & 1 & S & S^2 & S^3 \\ S^5 & 1 & S & S^2 & S^3 & S^4 \end{pmatrix}.$$
Rearranging to form the *G-Table*, we have
$$\begin{pmatrix} 1 & S & S^2 & S^3 & S^4 & S^5 \\ S^5 & 1 & S & S^2 & S^3 & S^4 \\ S^4 & S^5 & 1 & S & S^2 & S^3 \\ S^3 & S^4 & S^5 & 1 & S & S^2 \\ S^2 & S^3 & S^4 & S^5 & 1 & S \\ S & S^2 & S^3 & S^4 & S^5 & 1 \end{pmatrix}.$$
The permutation matrices corresponding to the positions of S^k in the *G-Table* give the matrix representation that gives the isomorphsm of $\mathbb{V}ect_6(C_6, \mathbb{F})$ with the full algebra of six by six matrices.

2. Now consider the symmetric group on six letters,
$$S_6 = \{1, R, R^2, F, RF, R^2F\}$$

where $R^3 = 1, F^2 = 1, FR = RF^2$. Then the multiplication table is

$$\begin{pmatrix} 1 & R & R^2 & F & RF & R^2F \\ R & R^2 & 1 & RF & R^2F & F \\ R^2 & 1 & R & R^2F & F & RF \\ F & R^2F & RF & 1 & R^2 & R \\ RF & F & R^2F & R & 1 & R^2 \\ R^2F & RF & F & R^2 & R & 1 \end{pmatrix}.$$

The corresponnding *G-Table* is

$$\begin{pmatrix} 1 & R & R^2 & F & RF & R^2F \\ R^2 & 1 & R & R^2F & F & RF \\ R & R^2 & 1 & RF & R^2F & F \\ F & R^2F & RF & 1 & R^2 & R \\ RF & F & R^2F & R & 1 & R^2 \\ R^2F & RF & F & R^2 & R & 1 \end{pmatrix}.$$

Here is a rewritten version of the *G-Table* with

$$R = \Delta, R^2 = \Theta, F = \Psi, RF = \Omega, R^2F = \Sigma.$$

$$\begin{pmatrix} 1 & \Delta & \Theta & \Psi & \Omega & \Sigma \\ \Theta & 1 & \Delta & \Sigma & \Psi & \Omega \\ \Delta & \Theta & 1 & \Omega & \Sigma & \Psi \\ \Psi & \Sigma & \Omega & 1 & \Theta & \Delta \\ \Omega & \Psi & \Sigma & \Delta & 1 & \Theta \\ \Sigma & \Omega & \Psi & \Theta & \Delta & 1 \end{pmatrix}.$$

This *G-Table* is the keystone for the isomorphism of $\mathbb{V}ect_6(S_3, \mathbb{F})$ with the full algebra of six by six matrices. At this point it may occur to the reader to wonder about $\mathbb{V}ect_3(S_3, \mathbb{F})$ since S_3 does act on vectors of length three. We will discuss $\mathbb{V}ect_n(S_n, \mathbb{F})$ in the next section. We see from this example how it will come about that $\mathbb{V}ect_{n!}(S_n, \mathbb{F})$ is isomorphic with the full algebra of $n! \times n!$ matrices. In particular, here are the permutation matrices that form the non-identity elements of this representation of the symmetric group on three letters.

$$R = \Delta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$R^2 = \Theta = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$F = \Psi = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$FR = \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$FR^2 = \Sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3. In this example we consider the group $G = C_2 \times C_2$, often called the "Klein 4-Group." We take $G = \{1, A, B, C\}$ where $A^2 = B^2 = C^2 = 1, AB = BA = C$. Thus G has the multiplication table, which is also its *G-Table* for $\mathbb{V}ect_4(G, \mathbb{F})$.

$$\begin{pmatrix} 1 & A & B & C \\ A & 1 & C & B \\ B & C & 1 & A \\ C & B & A & 1 \end{pmatrix}.$$

Thus we have the following permutation matrices that I shall call E, A, B, C :

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The reader will have no difficulty verifying that $A^2 = B^2 = C^2 = 1, AB = BA = C$. Recall that $[x, y, z, w]$ is iterant notation for the diagonal matrix

$$[x, y, z, w] = \begin{pmatrix} x & 0 & 0 & 1 \\ 0 & y & 1 & 0 \\ 0 & 1 & z & 0 \\ 1 & 0 & 0 & w \end{pmatrix}.$$

Let

$$\alpha = [1, -1, -1, 1], \beta = [1, 1, -1, -1], \gamma = [1, -1, 1, -1].$$

And let

$$I = \alpha A, J = \beta B, K = \gamma C.$$

Then the reader will have no trouble verifying that

$$I^2 = J^2 = K^2 = IJK = -1, IJ = K, JI = -K.$$

Thus we have constructed the quaternions as iterants in relation to the Klein Four Group. in Figure 1 we illustrate these quaternion generators with string diagrams for the permutations. The reader can check that the permuations correspond to the permutation matrices constructed for the Klein Four Group. For example, the permutation for I is $(12)(34)$ in cycle notation, the permutation for J is $(13)(24)$ and the permutation for K is $(14)(23)$. In the Figure we attach signs to each string of the permutation. These "signed permutations" act exactly as the products of vectors and permutations that we use for the iterants. One can see that the quaternions arise naturally from the Klein Four Group by attaching signs to the generating permutations as we have done in this Figure.

4. One can use the quaternions as a linear basis for 4×4 matrices just as our theorem would use the permutation matrices $1, A, B, C$. If we restrict to real scalars a, b, c, d such that $a^2 + b^2 + c^2 + c^2 = 1$, then the set of matrices of the form $a1 + bI + cJ + dK$ is isomorphic

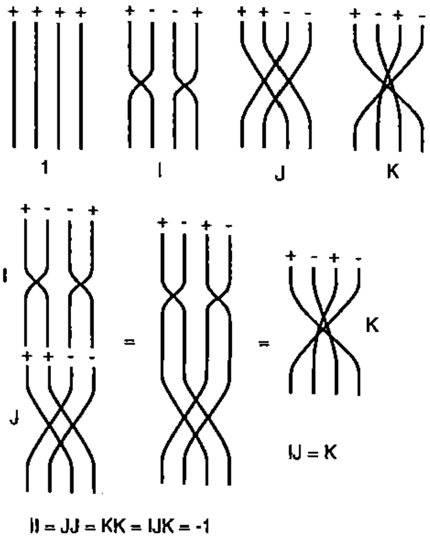


Figure 1: **Quaternions From Klein Four Group**

to the group $SU(2)$. To see this, note that $SU(2)$ is the set of matrices with complex entries z and w with determinant 1 so that $z\bar{z} + w\bar{w} = 1$.

$$M = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

Letting $z = a + bi$ and w $= c + di$, we have

$$M = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ o & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If we regard $i = \sqrt{-1}$ as a commuting scalar, then we can write the generating matrices in terms of size two iterants and obtain

$$I = \sqrt{-1}\epsilon, J = \epsilon\eta, K = \sqrt{-1}\eta$$

as described in the previous section. IF we regard these matrices with complex entries as shorthand for 4×4 matrices with i interpreted as a 2×2 matrix as we have done above, then these 4×4 matrices representing the quaternions are exactly the ones we have constructed in relation to the Klein Four Group.

Since complex numbers commute with one another, we could consider iterants whose values are in the complex numbers. This is just like considering matrices whose entries are complex numbers. For this purpose we shall allow given a version of i that commutes with the iterant shift operator η. Let this commuting i be denoted by ι. Then we are assuming that

$$\iota^2 = -1$$

$$\eta\iota = \iota\eta$$

$$\eta^2 = +1.$$

We then consider iterant views of the form $[a + b\iota, c + d\iota]$ and $[a + b\iota, c + d\iota]\eta = \eta[c + d\iota, a + b\iota]$. In particular, we have $\epsilon = [1, -1]$, and $i = \epsilon\eta$ is quite distinct from ι. Note, as before, that $\epsilon\eta = -\eta\epsilon$ and that $\epsilon^2 = 1$. Now let

$$I = \iota\epsilon$$

$$J = \epsilon\eta$$

$$K = \iota\eta.$$

We have used the commuting version of the square root of minus one in these definitions, and indeed we find the quaternions once more.

$$I^2 = \iota\epsilon\iota\epsilon = \iota\iota\epsilon\epsilon = (-1)(+1) = -1,$$

$$J^2 = \epsilon\eta\epsilon\eta = \epsilon(-\epsilon)\eta\eta = -1,$$

$$K^2 = \iota\eta\iota\eta = \iota\iota\eta\eta = -1,$$

$$IJK = \iota\epsilon\epsilon\eta\iota\eta = \iota 1 \iota\eta\eta = \iota\iota = -1.$$

Thus

$$I^2 = J^2 = K^2 = IJK = -1.$$

This construction shows how the structure of the quaternions comes directly from the non-commutative structure of period two iterants. In other, words, quaternions can be represented by 2×2 matrices. This is the way it has been presented in standard language. The group $SU(2)$ of 2×2 unitary matrices of determinant one is isomorphic to the quaternions of length one.

5. Similarly,

$$H = [a, b] + [c + d\iota, c - d\iota]\eta = \begin{pmatrix} a & c + d\iota \\ c - d\iota & b \end{pmatrix}.$$

represents a Hermitian 2×2 matrix and hence an observable for quantum processes mediated by $SU(2)$. Hermitian matrices have real eigenvalues.

If in the above Hermitian matrix form we take $a = T + X, b = T - X, c = Y, d = Z$, then we obtain an iterant and/or matrix representation for a point in Minkowski spacetime.

$$H = [T + X, T - X] + [Y + Z\iota, Y - Z\iota]\eta = \begin{pmatrix} T + X & Y + Z\iota \\ Y - Z\iota & T - X \end{pmatrix}.$$

Note that we have the formula

$$Det(H) = T^2 - X^2 - Y^2 - Z^2.$$

It is not hard to see that the eigenvalues of H are $T \pm \sqrt{X^2 + Y^2 + Z^2}$. Thus, viewed as an observable, H can observe the time and the invariant spatial distance from the origin of the event (T, X, Y, Z). At least at this very elementary juncture, quantum mechanics and special relativity are reconciled.

6. Hamilton's Quaternions are generated by iterants, as discussed above, and we can express them purely algebraicially by writing the corresponding permutations as shown below.

$$\begin{aligned} I &= [+1, -1, -1, +1]s \\ J &= [+1, +1, -1, -1]l \\ K &= [+1, -1, +1, -1]t \end{aligned}$$

where

$$\begin{aligned} s &= (12)(34) \\ l &= (13)(24) \\ t &= (14)(23). \end{aligned}$$

Here we represent the permutations as products of transpositions (ij). The transposition (ij) interchanges i and j, leaving all other elements of $\{1, 2, ..., n\}$ fixed.

One can verify that

$$I^2 = J^2 = K^2 = IJK = -1.$$

For example,

$$\begin{aligned} I^2 &= [+1, -1, -1, +1]s[+1, -1, -1, +1]s \\ &= [+1, -1, -1, +1][-1, +1, +1, -1]ss \\ &= [-1, -1, -1, -1] \\ &= -1. \end{aligned}$$

and

$$\begin{aligned} IJ &= [+1,-1,-1,+1]s[+1,+1,-1,-1]l \\ &= [+1,-1,-1,+1][+1,+1,-1,-1]sl \\ &= [+1,-1,+1,-1](12)(34)(13)(24) \\ &= [+1,-1,+1,-1](14)(23) \\ &= [+1,-1,+1,-1]t. \end{aligned}$$

Nevertheless, we must note that making an iterant interpretation of an entity like $I = [+1,-1,-1,+1]s$ is a conceptual departure from our original period two iterant (or cyclic period n) notion. Now we are considering iterants such as $[+1,-1,-1,+1]$ where the permutation group acts to produce other orderings of a given sequence. The iterant itself is not necessarily an oscillation. It can represent an implicate form that can be seen in any of its possible orders. These orders are subject to permutations that produce the possible views of the iterant. Algebraic structures such as the quaternions appear in the explication of such implicate forms.

The reader will also note that we have moved into a different conceptual domain from an original emphasis in this paper on eigenform in relation to to recursion. That is, we take an *eigenform* to mean a fixed point for a transformation. Thus i is an eigenform for $R(x) = -1/x$. Indeed, each generating quaternion is an eigenform for the transformation $R(x) = -1/x$. The richness of the quaternions arises from the closed algebra that arises with its infinity of eigenforms that satisfy the equation $U^2 = -1$:

$$U = aI + bJ + cK$$

where $a^2 + b^2 + c^2 = 1$. This kind of significant extra structure in the eigenforms comes from paying attention to specific aspects of implicate and explicate structure, relationships with geometry and ideas and inputs from the perceptual, conceptual and physical worlds. Just as with our other examples of phenomena arising in the course of the recursion, we see the same phenomena here in the evolution of matheamatical and theoretical physical structures in the course of the recursion that constitutes scientific conversation.

7. In all these examples, we have the opportunity to interpret the iterants as short hand for matrix algebra based on permutation matrices, or as indicators of discrete processes. The discrete processes become more complex in proportion to the complexity of the groups used in the construction. We began with processes of order two, then considered cyclic groups of arbitrary order, then the symmetric group S_3 in relation to 6×6 matrices, and the Klein Four Group in relation to the quaternions. In the case of the quaternions, we know that this structure is intimately related to rotations of three and four dimensional space and many other geometric themes. It is worth reflecting on the possible significance of the underlying discrete dynamics for this geometry, topology and related physics.

5 The Iterant Algebra $\mathcal{A}_n$

In this section, we will formulate relations with matrix algebra as follows. Let M be an $n \times n$ matrix over a ring F. Let $M = (m_{ij})$ denote the matrix entries. Let π be an element of the symmetric group S_n so that $\pi_1, \pi_2, \cdots, \pi_n$ is a permuation of $1, 2, \cdots, n$. Let $v = [v_1, v_2, \cdots, v_n]$ denote a vector with these components. Let $\Delta(v)$ denote the diagonal matrix whose $i - th$ diagonal entry is v_i. Let $v^\pi = [v_{\pi_1}, \cdots, v_{\pi_n}]$. Let $\Delta^\pi(v) = \Delta(v^\pi)$. Let Δ denote any diagonal matrix and Δ^π denote the corresponding permuted diagonal matrix as just described. Let $P[\pi]$ denote the permutation matrix obtained by taking the $i - th$ row of $P[\pi]$ to be the $\pi_i - th$ row of the identity matrix. Note that $P[\pi]\Delta = \Delta^\pi P[\pi]$. For each element π of S_n define the vector $v(M, \pi) = [m_{1\pi_1}, \cdots, m_{n\pi_n}]$ and the diagonal matrix $\Delta[M]_\pi = \Delta(v(M, \pi))$.

Given an $n \times n$ permutation matrix $P[\sigma]$ and a diagonal matrix D, the matrix $DP[\sigma]$ has the entries of D in those places where there were 1's in $P[\sigma]$. Let $a(D) = [D_{11}, D_{22}, \cdots, D_{nn}]$ be the *iterant associated with D.*

Consider n-tuples $a = [a_1, \cdots, a_n]$ where $a_i \in F$, and let the symmetric group S_n act on these n-tuples by permutation of the coordinates. Let e_i denote such an a where $a_i = 1$ and all the other coordinates are zero. Let $a^\sigma = [a_{\sigma(1)}, \cdots, a_{\sigma(n)}]$ be the vector obtained by letting $\sigma \in S_n$ act on a. Note that

$$a = \sum_{k=1}^{k=n} a_k e_k.$$

Define the *iterant algebra* $\mathcal{A}_n$ to be the module over F with basis $\mathcal{B} = \{e_i\gamma | i = 1, \cdots n; \gamma \in S_n\}$ where the algebra structure is given by

$$(a\sigma)(b\tau) = ab^\tau(\sigma\tau).$$

We see that

$$dim(\mathcal{A}_n) = n \times n! = n^2 \times (n-1)!.$$

Let $Matr_n$ denote the set of $n \times n$ matrices over the ring F. Note that since the permutation representation used for S_n is the same as the right regular representation only for $n = 2$, we have that $\mathcal{A}_2 \simeq Matr_2 \simeq \mathbb{V}ect_2(S_2, \mathbb{F})$, as defined in the previous section. For other values of n we will analyze the relationships of these rings.

Let

$$p : \mathcal{A}_n \longrightarrow Matr_n$$

via

$$p(a\sigma) = \Delta(a)P[\sigma]$$

where $\Delta(a)$ is the diagonal matrix associated with the iterant a and $P[\sigma]$ is the permutation matrix associated with the permuation σ. Then ρ is a matrix representation of the iterant algebra $\mathcal{A}_n$. This is not a faithful representation. Note that if $\sigma(i) = \tau(i)$ for permuations σ and τ, then $\rho(e_i\sigma) = \rho(e_i\tau)$. It remains to be seen how to form the full representation theory for the algebra $\mathcal{A}_n$. This will be a generalization of the representation theory for the group algebra of the symmetric group, which is $\mathcal{A}_1$.

A reason for discussing these formulations of matrix algebra in the present context is that one sees that matrix algebra is generated by the simple operations of juxtaposed addition and multiplication, and by the use of permutations as operators. These are unavoidable discrete elements, and so the operations of matrix algebra can be motivated on the basis of discrete physical ideas and non-commutativity. The richness of continuum formulations, infinite matrix algebra, and symmetry grows naturally out of finite matrix algebra and hence out of the discrete.

Theorem. Let M denote an $n \times n$ matrix with entries in a ring (associative not necessarily commutative) with unit. Then

$$M = \frac{1}{(n-1)!}\Sigma_{\pi \in S_n}\Delta[M]_\pi P[\pi].$$

This means that $\mathcal{M}_n$ can be embedded in $\mathcal{A}_n$, for we have the map $i : \mathcal{M}_n \longrightarrow \mathcal{A}_n$ defined by

$$i(M) = \frac{1}{(n-1)!}\Sigma_{\pi \in S_n} v(M, \pi)\pi$$

and

$$p \circ i = 1_{Matr_n}.$$

This implies that

$$\mathcal{A}_n \simeq \mathcal{K}_n \oplus Matr_n$$

where $\mathcal{K}_n$ is the kernel of p.

Proof. Let δ_{ij} denote the Kronecker delta, equal to 1 when $i = j$ and equal to 0 otherwise. The matrix product $\Delta[M]_\pi[\pi]$ is given as follows.

1. $(\Delta[M]_\pi[\pi])_{ij} = A_{i\pi_i} = A_{ij}\delta_{j\pi_i}$ if $j = \pi_i$.
2. $(\Delta[M]_\pi[\pi])_{ij} = 0$ if $j \neq \pi_i$.

This follows from the fact that

$$\Delta[M]_\pi = \begin{pmatrix} A_{1\pi_1} & 0 & \cdots & 0 \\ 0 & A_{2\pi_2} & \cdots & 0 \\ & & \cdots & \\ 0 & \cdots & 0 & A_{n\pi_n} \end{pmatrix}.$$

We abbreviate

$$\Delta[M]_\pi = \Delta_\pi.$$

Hence,

$$(\sum_{\pi\in S_n} \Delta_\pi[\pi]))_{ij} = \sum_{\pi\in S_n} (\Delta_\pi[\pi])_{ij}$$
$$= \sum_{\pi\in S_n} A_{ij}\delta_{j\pi_i} = A_{ij}\sum_{\pi\in S_n} \delta_{j\pi_i}.$$

$\sum_{\pi\in S_n} \delta_{j\pi_i} =$ (the number of permutations of $123\cdots n$ with $\pi_i = j) = (n-1)!$. This completes the proof of the Theorem. //

Note that the theorem expresses any square matrix as a sum of products of diagonal matrices and permutation matrices. Diagonal matrices add and multiply by adding and multiplying their corresponding entries. They are acted upon by permutations as described above. This is a full generalization of the case $n = 2$ described in the last section.

For example, we have the following expansion of a 3×3 matrix:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \frac{1}{2!}[\begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & k \end{pmatrix} + \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{pmatrix} +$$
$$\begin{pmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b & 0 \\ d & 0 & 0 \\ 0 & 0 & k \end{pmatrix} + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{pmatrix}].$$

Here, each term factors as a diagonal matrix multiplied by a permutation matrix as in

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & h \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is amusing to note that this theorem tells us that up to the factor of $1/(n-1)!$ a unitary matrix that has unit complex numbers as its entries is a sum of simpler unitary transformations factored into diagonal and permutation matrices. In quantum computing parlance, such a unitary matrix is a sum of products of phase gates and products of swap gates (since each permutation is a product of transpositions).

Abbreviating a diagonal matrix by the "iterant" $\Delta[a, b, c]$, we write

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \Delta[a, b, c].$$

Then we can write the entire decomposition of the 3×3 matrix in the form shown below.

$$(2!)\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \Delta[a, e, k]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \Delta[b, f, g]\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \Delta[c, d, h]\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} +$$

$$\Delta[a,f,h]\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}+\Delta[c,e,g]\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}+\Delta[b,d,k]\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

Thus

$$(2!)\begin{pmatrix}a&b&c\\d&e&f\\g&h&k\end{pmatrix}=\Delta[a,e,k]+\Delta[b,f,g]\rho+\Delta[c,d,h]\rho^2+\Delta[a,f,h]\tau+\Delta[c,e,g]\rho\tau+\Delta[b,d,k]\rho^2\tau$$

$$=\Delta[a,e,k]+\Delta[b,f,g]\rho+\Delta[c,d,h]\rho^2+\Delta[a,f,h]\tau_1+\Delta[c,e,g]\tau_2+\Delta[b,d,k]\tau_3.$$

Here $\rho = (123)$ and $\tau = \tau_1 = (23), \tau_2 = (13), \tau_3 = (12)$ in the standard cycle notation for permutations. *We write abstract permutations and the corresponding permutation matrices interchangeably.* The reader can easily spot the matrix definitions of these generators of S_3 by comparing the last equation to previous equation.

Note that in terms of the mapping $p : \mathcal{A}_3 \longrightarrow Matr_3$, we have that

$$p([a,e,k]+[b,f,g]\rho+[c,d,h]\rho^2+[a,f,h]\tau_1+[c,e,g]\tau_2+[b,d,k]\tau_3)=(2!)\begin{pmatrix}a&b&c\\d&e&f\\g&h&k\end{pmatrix}.$$

In this form, matrix multiplication disappears and we can calculate sums and products entirely with iterants and the action of the permutations on these iterants. The reader will note immediately that the full algebra $\mathcal{A}_3$ for iterants of size $[a, b, c]$ is larger and more general than 3×3 matrix algebra. We let the entries in the iterants belong to a field F. The most general element in this algebra is given by the formula

$$\mathcal{I} = [a,b,c]+[d,e,f]\rho+[g,h,i]\rho^2+[j,k,i]\tau_1+[m,n,o]\tau_2+[p,q,r]\tau_3.$$

where $a, b, \cdots r$ are elements of F. We do not assume that the group elements are represented by matrices, but we do have them act on the iterants $[x, y, z]$ by permuting the coordinates. Letting $e_1 = [1,0,0], e_2 = [0,1,0], e_3 = [0,0,1]$, we have that $\{e_i g | i = 1,2,3; g \in S_3\}$ is a basis for $\mathcal{A}_3$ over the field F. Thus the dimension of this algebra is $3 \times 3! = 18$.

We have the exact sequence

$$0 \longrightarrow \mathcal{K}_n \longrightarrow \mathcal{A}_n \longrightarrow Matr_n \longrightarrow 0,$$

with $p : \mathcal{A}_n \longrightarrow Matr_n$ and $i : Matr_n \longrightarrow \mathcal{A}_n$. Here are some examples of elements of the kernel $\mathcal{K}_n$ of p. Let $x = [1,0,0] - [1,0,0](23) \in \mathcal{A}_3$. Then it is easy to see that $p(x) = 0$. x itself is a non-trivial element of $\mathcal{A}_3$, Note that $x^2 = 2x$, so x is not nilpotent. We know from the fundamental classification theorem for associative algebras [25] that $\mathcal{A}_n/N$ (where N is the subalgebra of properly nilpotent elements of $\mathcal{A}_n$) is isomorphic to a full matrix algebra. Thus we see that the decomposition that we have given for $\mathcal{A}_n$ is distinct from the one obtained by removing the nilpotent elements. It remains to classify the nilpotent subalgebra of $\mathcal{A}_n$. We shall return to this question in a sequel to this paper.

Here is a final example of an element in the kernel of p. Consider the matrix

$$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

We can write this matrix quite simply as a sum of scalars times three permutation matrices generating the cyclic group of order three.

$$M = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

However, our mapping $i : Matr_3 \longrightarrow \mathcal{A}_3$ includes terms for all the permutation matrices and adds, essentially, three more terms to this formula.

$$2 \times i(M) = a1 + b(123) + c(132) + [c,a,b](13) + [b,c,a](12) + [a,b,c](23).$$

Consequently,

$$y = a1 + b(123) + c(132) - [c,a,b](13) - [b,c,a](12) - [a,b,c](23)$$

belongs to the kernel of the mapping p.

Lemma. The kernel $\mathcal{K}_3$ of the mapping $p : \mathcal{A}_3 \longrightarrow Matr_3$ consists in the elements

$$[x,y,z] + [-x,w,t]\tau_1 + [r,-y,s]\tau_2 + [p,q,-z]\tau_3 + [-p,-w,-s]\rho + [-r,-q,-t]\rho^2.$$

Proof. We leave this proof to the reader.//

Proposition. The kernel $\mathcal{K}_n$ of the mapping $p : \mathcal{A}_n \longrightarrow Matr_n$ consists in the elements

$$\alpha = \Sigma_{\sigma \in S_n} a_\sigma \sigma$$

such that for all i, j with $1 \leq i, j \leq n$,

$$\Sigma_{\sigma:\sigma(i)=j}(a_\sigma)_i = 0.$$

Thus we have that $\mathcal{A}_n/\mathcal{K}_n$ is isomorphic to the full matrix algebra $Matr_n$.
Proof. The proposition follows from the fact that $p(\alpha) = A$ where

$$A_{i,j} = \Sigma_{\sigma:\sigma(i)=j}(a_\sigma)_i.$$

//

In a subsequent paper we shall turn to the apparently more difficult problem of fully understanding the structure of the algebras $\mathcal{A}_n$ for $n \geq 3$. Here we have seen that the fact that the kernel of the mapping p is non-trivial means that there is often a choice in making an iterant representation for a given matrix or for an algebra of matrices. In many applications, certain underlying permutation matrices stand out and so suggest themselves as a basis for an iterant representation. This is the case for the quaternions, as we have seen. It is also the case for the Dirac matrices and other matrices that occur in physical applications. We shall discuss some of these examples below.

6 The Square Root of Minus One is a Clock

The purpose of this section is to place i, the square root of minus one, and its algebra in a context of discrete recursive systems. We begin by starting with a simple periodic process that is associated directly with the classical attempt to solve for i as a solution to a quadratic equation. We take the point of view that solving $x^2 = ax + b$ is the same (when $x \neq 0$) as solving

$$x = a + b/x,$$

and hence is a matter of finding a fixed point. In the case of i we have

$$x^2 = -1$$

and so desire a fixed point

$$x = -1/x.$$

There are no real numbers that are fixed points for this operator and so we consider the oscillatory process generated by

$$R(x) = -1/x.$$

The fixed point would satisfy

$$i = -1/i$$

and multiplying, we get that

$$ii = -1.$$

On the other hand the iteration of R yields

$$1, R(1) = -1, R(R(1)) = +1, R(R(R(1))) = -1, +1, -1, +1, -1, \cdots .$$

The square root of minus one is a perfect example of an eigenform that occurs in a new and wider domain than the original context in which its recursive process arose. The process has no fixed point in the original domain.

Looking at the oscillation between $+1$ and -1, we see that there are naturally two phase-shifted viewpoints. We denote these two views of the oscillation by $[+1, -1]$ and$[-1, +1]$. These viewpoints correspond to whether one regards the oscillation at time zero as starting with $+1$ or with -1. See Figure 1.

We shall let $I\{+1, -1\}$ stand for an undisclosed alternation or ambiguity between $+1$ and -1 and call $I\{+1, -1\}$ an iterant. There are two iterant views: $[+1, -1]$ and $[-1, +1]$.

Given an iterant $[a, b]$, we can think of $[b, a]$ as the same process with a shift of one time step. These two iterant views, seen as points of view of an alternating process, will become the square roots of negative unity, i and $-i$.

We introduce a temporal shift operator η such that

$$[a, b]\eta = \eta[b, a]$$

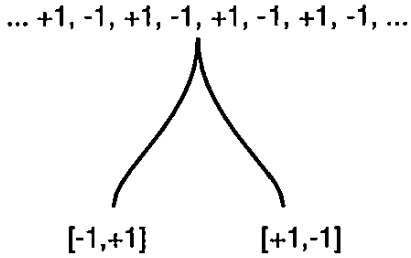


Figure 2: **A Basic Oscillation**

and

$$\eta\eta = 1$$

for any iterant $[a, b]$, so that concatenated observations can include a time step of one-half period of the process

$$\cdots abababab \cdots .$$

We combine iterant views term-by-term as in

$$[a, b][c, d] = [ac, bd].$$

We now define i by the equation

$$i = [-1, 1]\eta.$$

This makes i both a value and an operator that takes into account a step in time.

We calculate

$$ii = [-1, 1]\eta[-1, 1]\eta = [-1, 1][1, -1]\eta\eta = [-1, -1] = -1.$$

Thus we have constructed the square root of minus one by using an iterant viewpoint. In this view i represents a discrete oscillating temporal process and it is an eigenform for $R(x) = -1/x$, participating in the algebraic structure of the complex numbers. In fact the corresponding algebra structure of linear combinations $[a, b] + [c, d]\eta$ is isomorphic with 2×2 matrix algebra and iterants can be used to construct $n \times n$ matrix algebra, as we have already discussed in this paper.

The Temporal Nexus. *We take as a matter of principle that the usual real variable t for time is better represented as it so that time is seen to be a process, an observation and a magnitude all at once.* This principle of "imaginary time" is justified by the eigenform approach to the structure of time and the structure of the square root of minus one.

As an example of the use of the Temporal Nexus, consider the expression $x^2 + y^2 + z^2 + t^2$, the square of the Euclidean distance of a point (x, y, z, t) from the origin in Euclidean four-dimensional space. Now replace t by it, and find

$$x^2 + y^2 + z^2 + (it)^2 = x^2 + y^2 + z^2 - t^2,$$

the squared distance in hyperbolic metric for special relativity. By replacing t by its process operator value it we make the transition to the physical mathematics of special relativity.

In this section we shall first apply this idea to Lorentz transformations, and then generalize it to other contexts.

So, to work: We have

$$[t-x,t+x]=[t,t]+[-x,x]=t[1,1]+x[-1,1].$$

Since $[1,1][a,b]=[1a,1b]=[a,b]$ and $[0,0][a,b]=[0,0]$, we shall write

$$1=[1,1]$$

and

$$0=[0,0].$$

Let

$$\sigma=[-1,1].$$

σ is a significant iterant that we shall refer to as a *polarity*. Note that

$$\sigma\sigma=1.$$

Note also that

$$[t-x,t+x]=t+x\sigma.$$

Thus the points of spacetime form an algebra analogous to the complex numbers whose elements are of the form $t+x\sigma$ with $\sigma\sigma=1$ so that

$$(t+x\sigma)(t'+x'\sigma)=tt'+xx'+(tx'+xt')\sigma.$$

In the case of the Lorentz transformation it is easy to see the elements of the form $[k,k^{-1}]$ translate into elements of the form

$$T(v)=[(1+v)/\sqrt{(1-v^2)},(1-v)/\sqrt{(1-v^2)}]=[k,k^{-1}].$$

Further analysis shows that v is the relative velocity of the two reference frames in the physical context. Multiplication now yields the usual form of the Lorentz transform

$$\begin{aligned}T_k(t+x\sigma)&=T(v)(t+x\sigma)\\&=(1/\sqrt{(1-v^2)}-v\sigma/\sqrt{(1-v^2)})(t+x\sigma)\\&=(t-xv)/\sqrt{(1-v^2)}+(x-vt)\sigma/\sqrt{(1-v^2)}\\&=t'+x'\sigma.\end{aligned}$$

The algebra that underlies this iterant presentation of special relativity is a relative of the complex numbers with a special element σ of square one rather than minus one ($i^2=-1$ in the complex numbers).

7 The Wave Function in Quantum Mechanics and The Square Root of Minus One

One can regard a wave function such as $\psi(x,t) = exp(i(kx - wt))$ as containing a micro-oscillatory system with the special synchronizations of the iterant view $i = [+1, -1]\eta$. It is these synchronizations that make the big eigenform of the exponential work correctly with respect to differentiation, allowing it to create the appearance of rotational behaviour, wave behaviour and the semblance of the continuum. In other words, we are suggesting that one can take a temporal view of the well-known equation of Euler:

$$e^{i\theta} = cos(\theta) + isin(\theta)$$

by regarding the i in this equation as an iterant, as a discrete oscillation between -1 and $+1$. One can blend the classical geometrical view of the complex numbers with the iterant view by thinking of a point that orbits the origin of the complex plane, intersecting the real axis periodically and producing, in the real axis, a periodic oscillation in relation to its orbital movement in the two dimensional space. The special synchronization is the algebra of the time shift embodied in

$$\eta\eta = 1$$

and

$$[a, b]\eta = \eta[b, a]$$

that makes the algebra of $i = [1, -1]\eta$ imply that $i^2 = -1$. This interpretation does not change the formalism of these complex-valued functions, but it does change one's point of view and we now show how the properties of i as a discrete dynamical systerm are found in any such system.

7.1 Time Series and Discrete Physics

We have just reformulated the complex numbers and expanded the context of matrix algebra to an interpretation of i as an oscillatory process and matrix elements as combined spatial and temporal oscillatory processes (in the sense that $[a, b]$ is not affected in its order by a time step, while $[a, b]\eta$ includes the time dynamic in its interactive capability, and 2×2 matrix algebra is the algebra of iterant views $[a, b] + [c, d]\eta$).

We now consider elementary discrete physics in one dimension. Consider a time series of positions

$$x(t) : t = 0, \Delta t, 2\Delta t, 3\Delta t, \cdots .$$

We can define the velocity $v(t)$ by the formula

$$v(t) = (x(t + \Delta t) - x(t))/\Delta t = Dx(t)$$

where D denotes this discrete derivative. In order to obtain $v(t)$ we need at least one tick Δt of the discrete clock. Just as in the iterant algebra, we need a time-shift operator to handle the fact that once we have observed $v(t)$, the time has moved up by one tick.

We adjust the discrete derivative. We shall add an operator J that in this context accomplishes the time shift:

$$x(t)J = Jx(t + \Delta t).$$

We then redefine the derivative to include this shift:

$$Dx(t) = J(x(t + \Delta t) - x(t))/\Delta t.$$

This readjustment of the derivative rewrites it so that the temporal properties of successive observations are handled automatically.

Discrete observations do not commute. Let A and B denote quantities that we wish to observe in the discrete system. Let AB denote the result of first observing B and then observing A. The result of this definition is that a successive observation of the form $x(Dx)$ is distinct from an observation of the form $(Dx)x$. In the first case, we first observe the velocity at time t, and then x is measured at $t + \Delta t$. In the second case, we measure x at t and then measure the velocity.

We measure the difference between these two results by taking a commutator

$$[A, B] = AB - BA$$

and we get the following computations where we write $\Delta x = x(t + \Delta t) - x(t)$.

$$x(Dx) = x(t)J(x(t + \Delta t) - x(t)) = Jx(t + \Delta t)(x(t + \Delta t) - x(t)).$$

$$(Dx)x = J(x(t + \Delta t) - x(t))x(t).$$

$$[x, Dx] = x(Dx) - (Dx)x = (J/\Delta t)(x(t + \Delta t) - x(t))^2 = J(\Delta x)^2/\Delta t$$

This final result is worth recording:

$$[x, Dx] = J(\Delta x)^2/\Delta t.$$

From this result we see that the commutator of x andDx will be constant if $(\Delta x)^2/\Delta t = K$ is a constant. For a given time-step, this means that

$$(\Delta x)^2 = K\Delta t$$

so that

$$\Delta x = \pm\sqrt{(K\Delta t)}$$

This is a Brownian process with diffusion constant equal to K.

Thus we arrive at the result that any discrete process viewed in this framework of discrete observation has the basic commutator

$$[x, Dx] = J(\Delta x)^2/\Delta t,$$

generalizing a Brownian process and containing the factor $(\Delta x)^2/\Delta t$ that corresponds to the classical diffusion constant. It is worth noting that the adjusment that we have made to the discrete derivative makes it into a commutator as follows:

$$Dx(t) = J(x(t+\Delta t) - x(t))/\Delta t = (x(t)J - Jx(t))\Delta t = [x(t), J]/\Delta t.$$

By replacing discrete derivatives by commutators we can express discrete physics in many variables in a context of non-commutative algebra. See [14, 15, 16, 17, 18, 19, 20, 22, 21] for more on this point of view.

We now use the temporal nexus (the square root of minus one as a clock) and rewrite these commutators to match quantum mechanics.

7.2 Simplicity and the Heisenberg Commutator

Finally, we arrive at the simplest place. Time and the square root of minus one are inseparable in the temporal nexus. The square root of minus one is a symbol and algebraic operator for the simplest oscillatory process. As a symbolic form, i is an eigenform satisfying the equation

$$i = -1/i.$$

One does not have an increment of time all alone as in classical t. One has it, a combination of an interval and the elemental dynamic that is time. With this understanding, we can return to the commutator for a discrete process and use it for the temporal increment.

We found that discrete observation led to the commutator equation

$$[x, Dx] = J(\Delta x)^2/\Delta t$$

which we will simplify to

$$[q, p/m] = (\Delta x)^2/\Delta t.$$

taking q for the position x and p/m for velocity, the time derivative of position and ignoring the time shifting operator on the right hand side of the equation.

Understanding that Δt should be replaced by$i\Delta t$, and that, by comparison with the physics of a process at the Planck scale one can take

$$(\Delta x)^2/\Delta t = \hbar/m,$$

we have

$$[q, p/m] = (\Delta x)^2/i\Delta t = -i\hbar/m,$$

whence

$$[p, q] = i\hbar,$$

and we have arrived at Heisenberg's fundamental relatiionship between position and momentum. This mode of arrival is predicated on the recognition that only it represents a true interval of time. In the notion of time there is an inherent clock or an inherent shift of phase that is making a synchrony in our ability to observe, a precise dynamic beneath the apparent dynamic of the observed process. Once this substitution is made, once the correct imaginary value is placed in the temporal circuit, the patterns of quantum mechanics appear. In this way, quantum mechanics can be seen to emerge from the discrete.

The problem that we have examined in this section is the problem to understand the nature of quantum mechanics. In fact, we hope that the problem is seen to disappear the more we enter into the present viewpoint. A viewpoint is only on the periphery. The iterant from which the viewpoint emerges is in a superposition of indistinguishables, and can only be approached by varying the viewpoint until one is released from the particularities that a point of view contains.

8 Clifford Algebra, Majorana Fermions and Braiding

Recall fermion algebra. One has fermion annihiliation operators ψ and their conjugate creation operators $\psi^\dagger$. One has $\psi^2 = 0 = (\psi^\dagger)^2$. There is a fundamental commutation relation

$$\psi\psi^\dagger + \psi^\dagger\psi = 1.$$

If you have more than one of them say ψ and ϕ, then they anti-commute:

$$\psi\phi = -\phi\psi.$$

The Majorana fermions c that satisfy $c^\dagger = c$ so that they are their own anti-particles. There is a lot of interest in these as quasi-particles and they are related to braiding and to topological quantum computing. A group of researchers [9] claims, at this writing, to have found quasiparticle Majorana fermions in edge effects in nano-wires. (A line of fermions could have a Majorana fermion happen non-locally from one end of the line to the other.) The Fibonacci model that we discuss is also based on Majorana particles, possibly related to collecctive electronic excitations. If P is a Majorana fermion particle, then P can interact with itself to either produce itself or to annihilate itself. This is the simple "fusion algebra" for this particle. One can write $P^2 = P + 1$ to denote the two possible self-interactions the particle P. The patterns of interaction and braiding of such a particle P give rise to the Fibonacci model.

Majoranas are related to standard fermions as follows: The algebra for Majoranas is $c = c^\dagger$ and $cc' = -c'c$ if c and c' are distinct Majorana fermions with $c^2 = 1$ and $c'^2 = 1$. One can make a standard fermion from two Majoranas via

$$\psi = (c + ic')/2,$$

$$\psi^\dagger = (c - ic')/2.$$

Similarly one can mathematically make two Majoranas from any single fermion. Now if you take a set of Majoranas

$$\{c_1, c_2, c_3, \cdots, c_n\}$$

then there are natural braiding operators that act on the vector space with these c_k as the basis. The operators are mediated by algebra elements

$$\tau_k = (1 + c_{k+1}c_k)/\sqrt{2},$$
$$\tau_k^{-1} = (1 - c_{k+1}c_k)/\sqrt{2}.$$

Then the braiding operators are

$$T_k : Span\{c_1, c_2, \cdots, , c_n\} \longrightarrow Span\{c_1, c_2, \cdots, , c_n\}$$

via

$$T_k(x) = \tau_k x \tau_k^{-1}.$$

The braiding is simply:

$$T_k(c_k) = c_{k+1},$$
$$T_k(c_{k+1}) = -c_k,$$

and T_k is the identity otherwise. This gives a very nice unitary representaton of the Artin braid group and it deserves better understanding. See Figure 3 for an illustration of this braiding of Fermions in relation to the topology of a belt that connects them. The relationship with the belt is tied up with the fact that in quantum mechanics we must represent rotations of three dimensional space as unitary transformations. See [11] for more about this topological view of the physics of Fermions. In the Figure, we see that the belt does not know which of the two Fermions to annoint with the phase change, but the clever algebra above makes this decision. There is more to be done in this domain.

It is worth noting that a triple of Majorana fermions say a, b, c gives rise to a representation of the quaternion group. This is a generalization of the well-known association of Pauli matrices and quaternions. We have $a^2 = b^2 = c^2 = 1$ and they anticommute. Let $I = ba, J = cb, K = ac$. Then

$$I^2 = J^2 = K^2 = IJK = -1,$$

giving the quaternions. The operators

$$A = (1/\sqrt{2})(1 + I)$$
$$B = (1/\sqrt{2})(1 + J)$$
$$C = (1/\sqrt{2})(1 + K)$$

braid one another:

$$ABA = BAB, BCB = CBC, ACA = CAC.$$

This is a special case of the braid group representation described above for an arbitrary list of Majorana fermions. These braiding operators are entangling and so can be used for universal quantum computation, but they give only partial topological quantum computation due to the interaction with single qubit operators not generated by them.

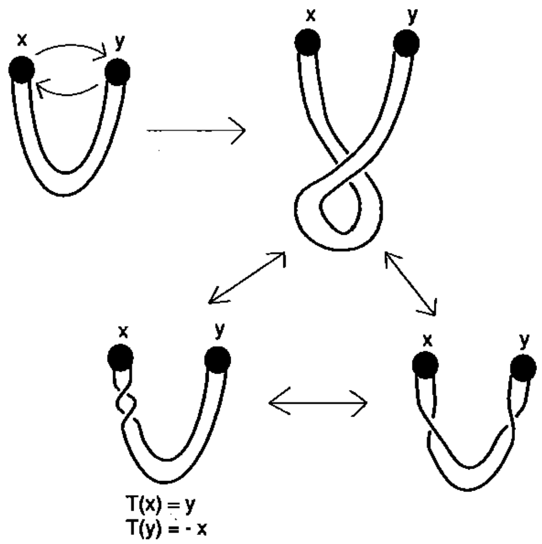


Figure 3: **Braiding Action on a Pair of Fermions**

Recall that in discussing the beginning of iterants, we introduce a *temporal shift operator* η such that

$$[a, b]\eta = \eta[b, a]$$

and

$$\eta\eta = 1$$

for any iterant $[a, b]$, so that concatenated observations can include a time step of one-half period of the process

$$\cdots abababab \cdots .$$

We combine iterant views term-by-term as in

$$[a, b][c, d] = [ac, bd].$$

We now define i by the equation

$$i = [1, -1]\eta.$$

This makes i both a value and an operator that takes into account a step in time.

We calculate

$$ii = [1, -1]\eta[1, -1]\eta = [1, -1][-1, 1]\eta\eta = [-1, -1] = -1.$$

Thus we have constructed a square root of minus one by using an iterant viewpoint. In this view i represents a discrete oscillating temporal process and it is an eigenform for $T(x) = -1/x$, participating in the algebraic structure of the complex numbers. In fact the corresponding algebra structure of linear combinations $[a, b] + [c, d]\eta$ is isomorphic with 2×2 matrix algebra and iterants can be used to construct $n \times n$ matrix algebra, as we have already discussed.

Now we can make contact with the algebra of the Majorana fermions. Let $e = [1, -1]$. Then we have $e^2 = [1, 1] = 1$ and $e\eta = [1, -1]\eta = [-1, 1]\eta = -e\eta$. Thus we have

$$e^2 = 1,$$

$$\eta^2 = 1,$$

and

$$e\eta = -\eta e.$$

We can regard e and η as a fundamental pair of Majorana fermions.

Note how the development of the algebra works at this point. We have that

$$(e\eta)^2 = -1$$

and so regard this as a natural construction of the square root of minus one in terms of the phase synchronization of the clock that is the iteration of the reentering mark. Once we have the square root of minus one it is natural to introduce another one and call this one i, letting it commute with the other operators. Then we have the $(ie\eta)^2 = +1$ and so we have a triple of Majorana fermions:

$$a = e, b = \eta, c = ie\eta$$

and we can construct the quaternions

$$I = ba = \eta e, J = cb = ie, K = ac = i\eta.$$

With the quaternions in place, we have the braiding operators

$$A = \frac{1}{\sqrt{2}}(1 + I), B = \frac{1}{\sqrt{2}}(1 + J), C = \frac{1}{\sqrt{2}}(1 + K),$$

and can continue as we did above.

9 Laws of Form

This section is a version of a corresponding section in our paper [23]. Here we discuss a formalism due the G. Spencer-Brown [1] that is often called the "calculus of indications". This calculus is a study of mathematical foundations with a topological notation based on one symbol, the mark:

$$\overline{\quad}|\,.$$

This single symbol represents a distinction between its own inside and outside. As is evident from Fgure 4, the mark is regarded as a shorthand for a rectangle drawn in the plane and dividing the plane into the regions inside and outside the rectangle.

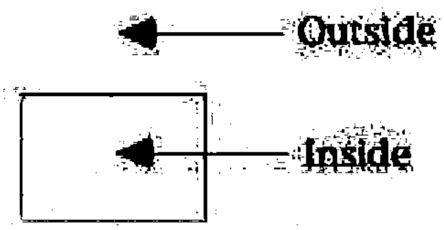


Figure 4: **Inside and Outside**

The reason we introduce this notation is that in the calculus of indications the mark can interact with itself in two possible ways. The resulting formalism becomes a version of Boolean arithmetic, but fundamentally simpler than the usual Boolean arithmetic of 0 and 1 with its two binary operations and one unary operation (negation). In the calculus of indications one takes a step in the direction of simplicity, and also a step in the direction of physics. The patterns of this mark and its self-interaction match those of a *Majorana fermion* as discussed in the previous section. A Majorana fermion is a particle that is its own anti-particle. [7]. We will later see, in this paper, that by adding braiding to the calculus of indications we arrive at the Fibonacci model, that can in principle support quantum computing.

In the previous section we described Majorana fermions in terms of their algebra of creation and annihilation operators. Here we describe the particle directly in terms of its interactions. This is part of a general scheme called "fusion rules" [8] that can be applied to discrete particle interacations. A fusion rule represents all of the different particle interactions in the form of a set of equations. The bare bones of the Majorana fermion consist in a particle P such that P can interact with itself to produce a neutral particle $*$ or produce itself P. Thus the possible interactions are

$$PP \longrightarrow *$$

and

$$PP \longrightarrow P.$$

This is the bare minimum that we shall need. The fusion rule is

$$P^2 = 1 + P.$$

This represents the fact that P can interact with itself to produce the neutral particle (represented as 1 in the fusion rule) or itself (represented by P in the fusion rule). .

Is there a *linguistic* particle that is its own anti-particle? Certainly we have

$$\sim\sim Q = Q$$

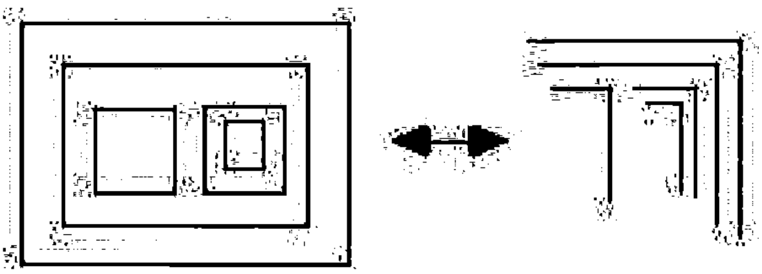


Figure 5: **Boxes and Marks**

for any proposition Q (in Boolean logic). And so we might write

$$\sim\sim \longrightarrow *$$

where $*$ is a neutral linguistic particle, an identity operator so that

$$*Q = Q$$

for any proposition Q. But in the normal use of negation there is no way that the negation sign combines with itself to produce itself. This appears to ruin the analogy between negation and the Majorana fermion. Remarkably, the calculus of indications provides a context in which we can say exactly that a certain logical particle, the mark, can act as negation *and* can interact with itself to produce itself.

In the calculus of indications patterns of non-intersecting marks (i.e. non-intersecting rectangles) are called *expressions*. For example in Figure 5 we see how patterns of boxes correspond to patterns of marks.

In Figure 5, we have illustrated both the rectangle and the marked version of the expression. In an expression you can say definitively of any two marks whether one is or is not inside the other. The relationship between two marks is either that one is inside the other, or that neither is inside the other. These two conditions correspond to the two elementary expressions shown in Figure 6.

The mathematics in Laws of Form begins with two laws of transformation about these two basic expressions. Symbolically, these laws are:

1. Calling :

$$\overline{}|\,\overline{}| = \overline{}|$$

2. Crossing:

$$\overline{\overline{}|\,}| = \ .$$

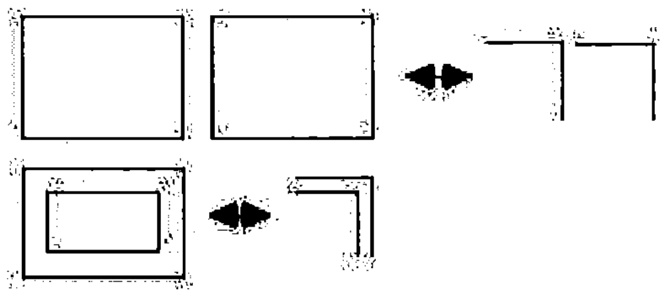
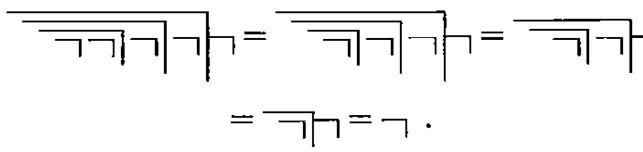


Figure 6: **Translation between Boxes and Marks**

The equals sign denotes a replacement step that can be performed on instances of these patterns (two empty marks that are adjacent or one mark surrounding an empty mark). In the first of these equations two adjacent marks condense to a single mark, or a single mark expands to form two adjacent marks. In the second equation two marks, one inside the other, disappear to form the unmarked state indicated by nothing at all. That is, two nested marks can be replaced by an empty word in this formal system. Alternatively, the unmarked state can be replaced by two nested marks. These equations give rise to a natural calculus, and the mathematics can begin. For example, *any expression can be reduced uniquely to either the marked or the unmarked state.* The he following example illustrates the method:



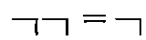
The general method for reduction is to locate marks that are at the deepest places in the expression (depth is defined by counting the number of inward crossings of boundaries needed to reach the given mark). Such a deepest mark must be empty and it is either surrounded by another mark, or it is adjacent to an empty mark. In either case a reduction can be performed by either calling or crossing.

Laws of Form begins with the following statement. "We take as given the idea of a distinction and the idea of an indication, and that it is not possible to make an indication without drawing a distinction. We take therefore the form of distinction for the form." Then the author makes the following two statements (laws):

1. *The value of a call made again is the value of the call.*

2. *The value of a crossing made again is not the value of the crossing.*

The two symbolic equations above correspond to these statements. First examine the law of calling. It says that the value of a repeated name is the value of the name. In the equation



one can view either mark as the name of the state indicated by the outside of the other mark. In the other equation

$$\overline{\urcorner}\big| = \quad .$$

the state indicated by the outside of a mark is the state obtained by crossing from the state indicated on the inside of the mark. Since the marked state is indicated on the inside, the outside must indicate the unmarked state. The Law of Crossing indicates how opposite forms can fit into one another and vanish into nothing, or how nothing can produce opposite and distinct forms that fit one another, hand in glove. The same interpretation yields the equation

$$\urcorner = \urcorner$$

where the left-hand side is seen as an instruction to cross from the unmarked state, and the right hand side is seen as an indicator of the marked state. The mark has a double carry of meaning. It can be seen as an operator, transforming the state on its inside to a different state on its outside, and it can be seen as the name of the marked state. That combination of meanings is compatible in this interpretation.

From the calculus of indications, one moves to algebra. Thus

$$\overline{\overline{A}\big|}\Big|$$

stands for the two possibilities

$$\overline{\overline{\urcorner}\big|}\Big| = \urcorner \longleftrightarrow A = \urcorner$$

$$\overline{\urcorner}\big| = \quad \longleftrightarrow A = \quad$$

In all cases we have

$$\overline{\overline{A}\big|}\Big| = A.$$

By the time we articulate the algebra, the mark can take the role of a unary operator

$$A \longrightarrow \overline{A}\big|.$$

But it retains its role as an element in the algebra. Thus begins algebra with respect to this non-numerical arithmetic of forms. The primary algebra that emerges is a subtle precursor to Boolean algebra. One can translate back and forth between elementary logic and primary algebra:

1. $\urcorner \longleftrightarrow T$
2. $\overline{\urcorner}\big| \longleftrightarrow F$
3. $\overline{A}\big| \longleftrightarrow \sim A$
4. $AB \longleftrightarrow A \vee B$

5. $\overline{\overline{\mathrm{A}}|\;\overline{\mathrm{B}}|}\Big| \longleftrightarrow A \wedge B$

6. $\overline{\mathrm{A}}|B \;\longleftrightarrow\; A \Rightarrow B$

The calculus of indications and the primary algebra form an efficient system for working with basic symbolic logic.

By reformulating basic symbolic logic in terms of the calculus of indications, we have a ground in which negation is represented by the mark *and* the mark is also interpreted as a value (a truth value for logic) and these two intepretations are compatible with one another in the formalism. The key to this compatibility is the choice to represent the value "false" by a literally unmarked state in the notational plane. With this the empty mark (a mark with nothing on its inside) can be interpreted as the negation of "false" and hence represents "true". The mark interacts with itself to produce itself (calling) and the mark interacts with itself to produce nothing (crossing). We have expanded the conceptual domain of negation so that it satisfies the mathematical pattern of an abstract Majorana fermion.

Another way to indicate these two interactions symbolically is to use a box,for the marked state and a blank space for the unmarked state. Then one has two modes of interaction of a box with itself:

1. Adjacency: $\square\;\square$

   and

2. Nesting: $\boxed{\square}$.

With this convention we take the adjacency interaction to yield a single box, and the nesting interaction to produce nothing:

$$\square\;\square = \square$$

$$\boxed{\square} =$$

We take the notational opportunity to denote nothing by an asterisk (*). The syntatical rules for operating the asterisk are Thus the asterisk is a stand-in for no mark at all and it can be erased or placed wherever it is convenient to do so. Thus

$$\boxed{\square} = *.$$

At this point the reader can appreciate what has been done if he returns to the usual form of symbolic logic. In that form we that

$$\sim\sim X = X$$

for all logical objects (propositions or elements of the logical algebra) X. We can summarize this by writing

$$\sim\sim \; =$$

as a symbolic statement that is outside the logical formalism. Furthermore, one is committed to the interpretation of negation as an operator and not as an operand. The calculus of indications provides a formalism where the mark (the analog of negation in that domain) is both a value and an object, and so can act on itself in more than one way.

The Majorana particle is its own anti-particle. It is exactly at this point that physics meets logical epistemology. Negation as logical entity is its own anti-particle. Wittgenstein says (Tractatus [27] 4.0621) "$\cdots$ the sign '$\sim$' corresponds to nothing in reality." And he goes on to say (Tractatus 5.511) " How can all-embracing logic which mirrors the world use such special catches and manipulations? Only because all these are connected into an infinitely fine network, the great mirror." For Wittgenstein in the Tractatus, the negation sign is part of the mirror making it possible for thought to reflect reality through combinations of signs. These remarks of Wittgenstein are part of his early picture theory of the relationship of formalism and the world. In our view, the world and the formalism we use to represent the world are not separate. The observer and the mark are (formally) identical. A path is opened between logic and physics.

The visual iconics that create via the boxes of half-boxes of the calculus of indications a model for a logical Majorana fermion can also be seen in terms of cobordisms of surfaces. View Figure 7. There the boxes have become circles and the interactions of the circles have been displayed as evolutions in an extra dimension, tracing out surfaces in three dimensions. The condensation of two circles to one is a simple cobordism betweem two circles and a single circle. The cancellation of two circles that are concentric can be seen as the right-hand lower cobordism in this figure with a level having a continuum of critical points where the two circles cancel. A simpler cobordism is illustrated above on the right where the two circles are not concentric, but nevertheless are cobordant to the empty circle. Another way of putting this is that two topological closed strings can interact by cobordism to produce a single string or to cancel one another. Thus a simple circle can be a topological model for a Majorana fermion.

In [23, 24] we detail how the Fibonacci model for anyonic quantum computing can be constructed by using a version of the two-stranded bracket polynomial and a generalization of Penrose spin networks. This is a fragment of the Temperly-Lieb recoupling theory [12].

9.1 The Square Root of Minus One Revisited

So far we have seen that the mark can represent the fusion rules for a Majorana fermion since it can interact with itself to produce either itself or nothing. But we have not yet seen the anti-commuting fermion algebra emerge from this context of making a distinction. Remarkably, this algebra does emerge when one looks at the mark recursively.

Consider the transformation

$$F(X) = \overline{\mathrm{X}|}.$$

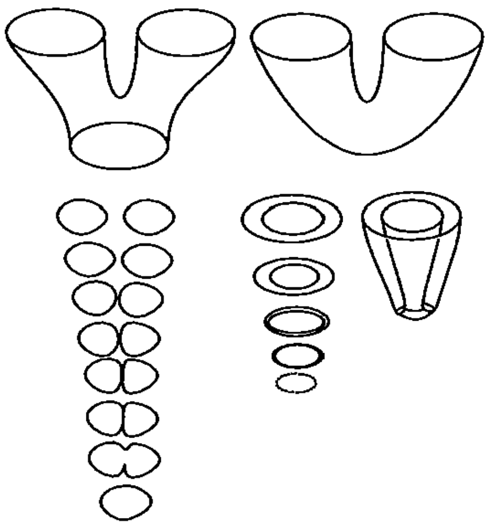


Figure 7: **Calling, Crossing and Cobordism**

If we iterate it and take the limit we find

$$G = F(F(F(F(\cdots)))) = \overline{\overline{\overline{\overline{\cdots}|}|}|}|$$

an infinite nest of marks satisfying the equation

$$G = \overline{G\,}|.$$

With $G = F(G)$, I say that G is an *eigenform* for the transformation F. See Figure 8 for an illustration of this nesting with boxes and an arrow that points inside the reentering mark to indicate its appearance inside itself. If one thinks of the mark itself as a Boolean logical value, then extending the language to include the reentering mark G goes beyond the boolean. We will not detail here how this extension can be related to non-standard logics, but refer the reader to [12]. Taken at face value the reentering mark cannot be just marked or just unmarked, for by its very definition, if it is marked then it is unmarked and if it is unmarked then it is marked. In this sense the reentering mark has the form of a self-contradicting paradox. There is no paradox since we do not have to permanently assign it to either value. The simplest interpretation of the reentering mark is that it is temporal and that it represents an oscillation between markedness and unmarkedness. In numerical terms it is a discrete dynamical system oscillating between $+1$ (marked) and -1 (not marked).

With the reentering mark in mind consider now the transformation on real numbers given by

$$T(x) = -1/x.$$

This has the fixed points i and $-i$, the complex numbers whose squares are negative unity. But lets take a point of view more directly associated with the analogy of the recursive mark. Begin

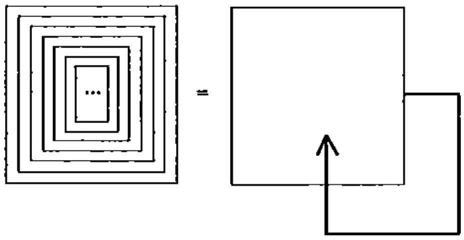


Figure 8:

by starting with a simple periodic process that is associated directly with the classical attempt to solve for i as a solution to a quadratic equation. We take the point of view that solving $x^2 = ax+b$ is the same (when $x \neq 0$) as solving

$$x = a + b/x,$$

and hence is a matter of finding a fixed point. In the case of i we have

$$x^2 = -1$$

and so desire a fixed point

$$x = -1/x.$$

There are no real numbers that are fixed points for this operator and so we consider the oscillatory process generated by

$$T(x) = -1/x.$$

The fixed point would satisfy

$$i = -1/i$$

and multiplying, we get that

$$ii = -1.$$

On the other hand the iteration of T yields

$$1, T(1) = -1, T(T(1)) = +1, T(T(T(1))) = -1, +1, -1, +1, -1, \cdots.$$

The square root of minus one is a perfect example of an eigenform that occurs in a new and wider domain than the original context in which its recursive process arose. The process has no fixed point in the original domain. At this point we enter once again the domain of iterants and particularly the discussion of Section 6 where we see the square root of minus one as a clock.

There is one more comment that is appropriate for this section. Recall that a pair of Majorana fermions can be assembled to form a single standard fermion. In our case we have the spatial and temporal iterant components $e = [1, -1]$ and η with $e\eta = -\eta e$. We can regard e and η as a fundamental pair of Majorana fermions. This is a formal correspondence, but it is striking how this Marjorana fermion algebra emerges from an analysis of the recursive nature of the reentering mark, while the fusion algebra for the Majorana fermion emerges from the distinctive properties of the mark itself. We see how the seeds of the fermion algebra live in this extended logical context.

The corresponding standard fermion annihilation and creation operators are then given by the formulas below.

$$\psi = (e + i\eta)/2$$

and

$$\psi^{\dagger} = (e - i\eta)/2.$$

Since e represents a spatial view of the basic discrete oscillation and η is the time-shift operator for this oscillation it is of interest to note that the standard fermion built by these two can be regarded as a quantum of spacetime, retrieved from the way that we decomposed the process into space and time. Since all this is initially built in relation to extending the Boolean logic of the mark to a non-boolean recursive context, there is further analysis needed of the relation of the physics and the logic. We have only begun the analysis here. The crux of the matter is that two dimensional physics depends upon a plane space in which a simple closed curve makes a distinction between inside and outside in order for the braiding and phases to be significant. This same property of distinction in the plane is what gives a plane space the linguistic power to represent language and logic. This correspondence in not an accident and deserves further study!

10 The Dirac Equation and Majorana Fermions

We now construct the Dirac equation. This may sound circular, in that the fermions arise from solving the Dirac equation, but in fact the algebra underlying this equation has the same properties as the creation and annihilation algebra for fermions, so it is by way of this algebra that we will come to the Dirac equation. If the speed of light is equal to 1 (by convention), then energy E, momentum p and mass m are related by the (Einstein) equation

$$E^2 = p^2 + m^2.$$

Dirac constructed his equation by looking for an algebraic square root of $p^2 + m^2$ so that he could have a linear operator for E that would take the same role as the Hamiltonian in the Schroedinger equation. We will get to this operator by first taking the case where p is a scalar (we use one dimension of space and one dimension of time.). Let $E = \alpha p + \beta m$ where α and β are elements of a a possibly non-commutative, associative algebra. Then

$$E^2 = \alpha^2 p^2 + \beta^2 m^2 + pm(\alpha\beta + \beta\alpha).$$

Hence we will satisfiy $E^2 = p^2 + m^2$ if $\alpha^2 = \beta^2 = 1$ and $\alpha\beta + \beta\alpha = 0$. This is our familiar Clifford algebra pattern and we can use the iterant algebra generated by e and η if we wish. Then, because the quantum operator for momentum is $-i\partial/\partial x$ and the operator for energy is $i\partial/\partial t$, we have the Dirac equation

$$i\partial\psi/\partial t = -i\alpha\partial\psi/\partial x + \beta m\psi.$$

Let

$$\mathcal{O} = i\partial/\partial t + i\alpha\partial/\partial x - \beta m$$

so that the Dirac equation takes the form

$$\mathcal{O}\psi(x,t) = 0.$$

Now note that

$$\mathcal{O}e^{i(px-Et)} = (E - \alpha p - \beta m)e^{i(px-Et)}.$$

We let

$$\Delta = (E - \alpha p - \beta m)$$

and let

$$U = \Delta\beta\alpha = (E - \alpha p - \beta m)\beta\alpha = \beta\alpha E + \beta p - \alpha m,$$

then

$$U^2 = -E^2 + p^2 + m^2 = 0.$$

This nilpotent element leads to a (plane wave) solution to the Dirac equation as follows: We have shown that

$$\mathcal{O}\psi = \Delta\psi$$

for $\psi = e^{i(px-Et)}$. It then follows that

$$\mathcal{O}(\beta\alpha\Delta\beta\alpha\psi) = \Delta\beta\alpha\Delta\beta\alpha\psi = U^2\psi = 0,$$

from which it follows that

$$\psi = \beta\alpha U e^{i(px-Et)}$$

is a (plane wave) solution to the Dirac equation.

In fact, this calculation suggests that we should multiply the operator $\mathcal{O}$ by $\beta\alpha$ on the right, obtaining the operator

$$\mathcal{D} = \mathcal{O}\beta\alpha = i\beta\alpha\partial/\partial t + i\beta\partial/\partial x - \alpha m,$$

and the equivalent Dirac equation

$$\mathcal{D}\psi = 0.$$

In fact for the specific ψ above we will now have $\mathcal{D}(Ue^{i(px-Et)}) = U^2e^{i(px-Et)} = 0$. This idea of reconfiguring the Dirac equation in relation to nilpotent algebra elements U is due to Peter Rowlands [26]. Rowlands does this in the context of quaternion algebra. Note that the solution to the Dirac equation that we have found is expressed in Clifford algebra or iterant algebra form. It can be articulated into specific vector solutions by using an iterant or matrix representation of the algebra.

We see that $U = \beta\alpha E + \beta p - \alpha m$ with $U^2 = 0$ is really the essence of this plane wave solution to the Dirac equation. This means that a natural non-commutative algebra arises directly and can be regarded as the essential information in a Fermion. It is natural to compare this algebra structure with algebra of creation and annihilation operators that occur in quantum field theory. to this end, let

$$U^\dagger = \alpha\beta E + \alpha p - \beta m.$$

Here we regard $U^\dagger$ as a formal counterpart to complex conjugation, since in the split quaternion algebra we have not yet constructed commuting square roots of negative one. We then find that with

$$A = U + U^\dagger = (\alpha + \beta)(p - m)$$

and

$$B = U - U^\dagger = 2\beta\alpha E + (\beta - \alpha)(p - m)$$

that

$$[\frac{A}{\sqrt{2}(p-m)}]^2 = 1$$

and

$$[\frac{iB}{\sqrt{2}(p+m)}]^2 = 1,$$

with i a commuting square root of negative one, giving the underlying Majorana Fermion operators for our Dirac Fermion. The operators U and $U^\dagger$ satisfy the usual commutation relations for the annihilation and creation operators for a Fermion.

It is worth noting how the Pythgorean relationship $E^2 = p^2 + m^2$ interacts here with the Clifford algebra of α and β. We have

$$U^\dagger = p\alpha + m\beta + \alpha\beta E$$

$$U = p\beta + m\alpha + \beta\alpha E$$

with

$$(U^\dagger)^2 = U^2 = 0,$$
$$U + U^\dagger = (p + m)(\alpha + \beta),$$
$$U - U^\dagger = (p - m)(\alpha - \beta) + 2E\alpha\beta.$$

This implies that

$$(U + U^\dagger)^2 = 2(p + m)^2$$
$$(U - U^\dagger)^2 = 2(p - m)^2 - 4E^2 = 2[p^2 + m^2 - 2pm - 2p^2 - 2m^2] = -2(p + m)^2.$$

From this we easily deduce that

$$UU^\dagger + U^\dagger U = 2(p + m)^2,$$

and this can be normalized to equal 1.

10.1 Another version of U and $U^\dagger$

We start with $\psi = e^{i(px-Et)}$ and the operators

$$\hat{E} = i\partial/\partial t$$

and

$$\hat{p} = -i\partial/\partial x$$

so that

$$\hat{E}\psi = E\psi$$

and

$$\hat{p}\psi = p\psi.$$

The Dirac operator is

$$\mathcal{O} = \hat{E} - \alpha\hat{p} - \beta m$$

and the modified Dirac operator is

$$\mathcal{D} = \mathcal{O}\beta\alpha = \beta\alpha\hat{E} + \beta\hat{p} - \alpha m,$$

so that

$$\mathcal{D}\psi = (\beta\alpha E + \beta p - \alpha m)\psi = U\psi.$$

If we let

$$\tilde{\psi} = e^{i(px+Et)}$$

(reversing time), then we have

$$\mathcal{D}\tilde{\psi} = (-\beta\alpha E + \beta p - \alpha m)\psi = U^\dagger\tilde{\psi},$$

giving a definition of $U^\dagger$ corresponding to the anti-particle for $U\psi$.

We have

$$U = \beta\alpha E + \beta p - \alpha m$$

and

$$U^\dagger = -\beta\alpha E + \beta p - \alpha m$$

Note that here we have

$$(U + U^\dagger)^2 = (2\beta p + \alpha m)^2 = 4(p^2 + m^2) = 4E^2,$$

and

$$(U - U^\dagger)^2 = -(2\beta\alpha E)^2 = -4E^2.$$

We have that

$$U^2 = (U^\dagger)^2 = 0$$

and

$$UU^\dagger + U^\dagger U = 4E^2.$$

Thus we have a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation. Furthermore, the decomposition of $U$and $U^\dagger$ into the corresponding Majorana Fermion operators corresponds to $E^2 = p^2 + m^2$. Normalizing by dividing by $2E$ we have

$$A = (\beta p + \alpha m)/E$$

and

$$B = i\beta\alpha.$$

so that

$$A^2 = B^2 = 1$$

and

$$AB + BA = 0.$$

then

$$U = (A + Bi)E$$

and

$$U^\dagger = (A - Bi)E,$$

showing how the Fermion operators are expressed in terms of the simpler Clifford algebra of Majorana operators (split quaternions once again).

10.2 Writing in the Full Dirac Algebra

We have written the Dirac equation so far in one dimension of space and one dimension of time. We give here a way to boost the formalism directly to three dimensions of space. We take an independent Clifford algebra generated by $\sigma_1, \sigma_2, \sigma_3$ with $\sigma_i^2 = 1$ for $i = 1, 2, 3$ and $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for $i \neq j$. Now assume that α and β as we have used them above generate an independent Clifford algebra that commutes with the algebra of the σ_i. Replace the scalar momentum p by a 3-vector momentum $p = (p_1, p_2, p_3)$ and let $p \bullet \sigma = p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$. We replace $\partial/\partial x$ with $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_2)$ and $\partial p/\partial x$ with $\nabla \bullet p$.

We then have the following form of the Dirac equation.

$$i\partial\psi/\partial t = -i\alpha\nabla \bullet \sigma\psi + \beta m\psi.$$

Let

$$\mathcal{O} = i\partial/\partial t + i\alpha\nabla \bullet \sigma - \beta m$$

so that the Dirac equation takes the form

$$\mathcal{O}\psi(x, t) = 0.$$

In analogy to our previous discussion we let

$$\psi(x,t) = e^{i(p \bullet x - Et)}$$

and construct solutions by first applying the Dirac operator to this ψ. The two Clifford algebras interact to generalize directly the nilpotent solutions and Fermion algebra that we have detailed for one spatial dimension to this three dimensional case. To this purpose the modified Dirac operator is

$$\mathcal{D} = i\beta\alpha\partial/\partial t + \beta\nabla \bullet \sigma - \alpha m.$$

And we have that

$$\mathcal{D}\psi = U\psi$$

where

$$U = \beta\alpha E + \beta p \bullet \sigma - \alpha m.$$

We have that $U^2 = 0$ and $U\psi$ is a solution to the modified Dirac Equation, just as before. And just as before, we can articulate the structure of the Fermion operators and locate the corresponding Majorana Fermion operators. We leave these details to the reader.

10.3 Majorana Fermions at Last

There is more to do. We will end with a brief discussion making Dirac algebra distinct from the one generated by $\alpha, \beta, \sigma_1, \sigma_2, \sigma_3$ to obtain an equation that can have real solutions. This was the strategy that Majorana [7] followed to construct his Majorana Fermions. A real equation can have solutions that are invariant under complex conjugation and so can correspond to particles that are their own anti-particles. We will describe this Majorana algebra in terms of the split quaternions ϵ and η. For convenience we use the matrix representation given below. The reader of this paper can substitute the corresponding iterants.

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\hat{\epsilon}$ and $\hat{\eta}$ generate another, independent algebra of split quaternions, commuting with the first algebra generated by ϵ and η. Then a totally real Majorana Dirac equation can be written as follows:

$$(\partial/\partial t + \hat{\eta}\eta\partial/\partial x + \epsilon\partial/\partial y + \hat{\epsilon}\eta\partial/\partial z - \hat{\epsilon}\hat{\eta}\eta m)\psi = 0.$$

To see that this is a correct Dirac equation, note that

$$\hat{E} = \alpha_x\hat{p_x} + \alpha_y\hat{p_y} + \alpha_z\hat{p_z} + \beta m$$

(Here the "hats" denote the quantum differential operators corresponding to the energy and momentum.) will satisfy

$$\hat{E}^2 = \hat{p_x}^2 + \hat{p_y}^2 + \hat{p_z}^2 + m^2$$

if the algebra generated by $\alpha_x, \alpha_y, \alpha_z, \beta$ has each generator of square one and each distinct pair of generators anti-commuting. From there we obtain the general Dirac equation by replacing $\hat{E}$ by $i\partial/\partial t$, and $\hat{p_x}$ with $-i\partial/\partial x$ (and same for y, z).

$$(i\partial/\partial t + i\alpha_x \partial/\partial x + i\alpha_y \partial/\partial y + i\alpha_z \partial/\partial y - \beta m)\psi = 0.$$

This is equivalent to

$$(\partial/\partial t + \alpha_x \partial/\partial x + \alpha_y \partial/\partial y + \alpha_z \partial/\partial y + i\beta m)\psi = 0.$$

Thus, here we take

$$\alpha_x = \hat{\eta}\eta, \alpha_y = \epsilon, \alpha_z = \hat{\epsilon}\eta, \beta = i\hat{\epsilon}\hat{\eta}\eta,$$

and observe that these elements satisfy the requirements for the Dirac algebra. Note how we have a significant interaction between the commuting square root of minus one (i) and the element $\hat{\epsilon}\hat{\eta}$ of square minus one in the split quaternions. This brings us back to our original considerations about the source of the square root of minus one. Both viewpoints combine in the element $\beta = i\hat{\epsilon}\hat{\eta}\eta$ that makes this Majorana algebra work. Since the algebra appearing in the Majorana Dirac operator is constructed entirely from two commuting copies of the split quaternions, there is no appearance of the complex numbers, and when written out in 2×2 matrices we obtain coupled real differential equations to be solved. Clearly this ending is actually a beginning of a new study of Majorana Fermions. That will begin in a sequel to the present paper.

References

[1] G. Spencer–Brown, "Laws of Form," George Allen and Unwin Ltd. London (1969).

[2] Kauffman, L. [1985], Sign and Space, In Religious Experience and Scientific Paradigms. Proceedings of the 1982 IASWR Conference, Stony Brook, New York: Institute of Advanced Study of World Religions, (1985), 118-164.

[3] Kauffman, L. [1987], Self-reference and recursive forms, Journal of Social and Biological Structures (1987), 53-72.

[4] Kauffman, L. [1987], Special relativity and a calculus of distinctions. Proceedings of the 9th Annual Intl. Meeting of ANPA, Cambridge, England (1987). Pub. by ANPA West, pp. 290-311.

[5] Kauffman, L. [1987], Imaginary values in mathematical logic. Proceedings of the Seventeenth International Conference on Multiple Valued Logic, May 26-28 (1987), Boston MA, IEEE Computer Society Press, 282-289.

[6] Kauffman, L. H., Knot Logic, In *Knots and Applications* ed. by L. Kauffman, World Scientific Pub. Co., (1994), pp. 1-110.

[7] E. Majorana, A symmetric theory of electrons and positrons, I Nuovo Cimento,**14** (1937), pp. 171-184.

[8] G. Moore and N. Read, Noabelions in the fractional quantum Hall effect, *Nuclear Physics B360* (1991), 362 - 396.

[9] V. Mourik,K. Zuo, S. M. Frolov, S. R. Plissard, E.P.A.M. Bakkers, L.P. Kouwenhuven, Signatures of Majorana fermions in hybred superconductor-semiconductor devices, arXiv: 1204.2792.

[10] Kauffman, Louis H. [2002], Biologic. *AMS Contemporary Mathematics Series*, Vol. 304, (2002), pp. 313 - 340.

[11] Kauffman,Louis H.[1991,1994,2001,2012], *Knots and Physics,* World Scientific Pub.

[12] L.H. Kauffman, *Temperley-Lieb Recoupling Theory and Invariants of Three-Manifolds*, Princeton University Press, Annals Studies **114** (1994).

[13] Kauffman, Louis H. [2002], Time imaginary value, paradox sign and space, in *Computing Anticipatory Systems, CASYS - Fifth International Conference*, Liege, Belgium (2001) ed. by Daniel Dubois, AIP Conference Proceedings Volume 627 (2002).

[14] Kauffman,Louis H. and Noyes,H. Pierre [1996], Discrete Physics and the Derivation of Electromagnetism from the formalism of Quantum Mechanics, *Proc. of the Royal Soc. Lond. A*, **452**, pp. 81-95.

[15] Kauffman,Louis H. and Noyes,H. Pierre [1996], Discrete Physics and the Dirac Equation, *Physics Letters A*, 218 ,pp. 139-146.

[16] Kauffman, Louis H. [1998], Noncommutativity and discrete physics, *Physica D* 120 (1998), 125-138.

[17] Kauffman, Louis H. [1998], Space and time in discrete physics, *Intl. J. Gen. Syst.* Vol. 27, Nos. 1-3, 241-273.

[18] Kauffman, Louis H. [1999], A non-commutative approach to discrete physics, in *Aspects II - Proceedings of ANPA 20*, 215-238.

[19] Kauffman, Louis H. [2003], Non-commutative calculus and discrete physics, in *Boundaries- Scientific Aspects of ANPA 24*, 73-128.

[20] Kauffman, Louis H. [2004], Non-commutative worlds, *New Journal of Physics 6*, 2-46.

[21] Kauffman, Louis H., Non-commutative worlds and classical constraints. in "Scientific Essays in Honor of Pierre Noyes on the Occasion of His 90-th Birthday", edited by John Amson and Louis H. Kaufman, World Scientific Pub. Co. (2013), pp. 169-210.

[22] Louis H. Kauffman, Differential geometry in non-commutative worlds, in "Quantum Gravity - Mathematical Models and Experimental Bounds", edited by B. Fauser, J. Tolksdorf and E. Zeidler, Birkhauser (2007), pp. 61 - 75.

[23] Kauffman, Louis H. [2012], Knot Logic and Topological Quantum Computing with Majorana Fermions, (to appear).

[24] Kauffman, Louis H.; Lomonaco, Samuel J., Jr. q-deformed spin networks, knot polynomials and anyonic topological quantum computation. J. Knot Theory Ramifications 16 (2007), no. 3, 267–332.

[25] D. E. Littlewood, "The Skeleton Key of Mathematics", Harper Torchbook Edition (1960).

[26] Peter Rowlands, "Zero to Infinity - The Foundations of Physics", Series on Knots and Everything - Volume 41, World Scientific Publishing Co., 2007.

[27] L. Wittgenstein, "Tractatus Logico - Philosophicus", New York: Harcourt, Brace and Company, Inc., London: Kegan Paul, Trench, Trubner and Co. Ltd. (1922).

A Dual Space as the Basis of Quantum Mechanics and Other Aspects of Physics

Peter Rowlands

Department of Physics, University of Liverpool, Oliver Lodge Laboratory, Oxford St, Liverpool, L69 7ZE, UK. email p.rowlands@liverpool.ac.uk

Abstract. Many attempts have been made to reduce the whole of physics to a single space- or space-time-like structure. However, it seems more likely that the true structure of physics, based on fermionic point-particles and their interactions, requires a combination of two vector spaces or a space and an antispace, which are dual to each other.

Defining dual spaces

All physical measurements are mediated via space. Measurements of time, mass and so on are equivalent to a pointer moving over a spatial scale, or counting movements over the same scale. Several past attempts to reduce all other physical parameters to a version of space. Descartes defined matter in terms of extension and created a space which was filled with matter manifesting itself as spatial extension. Minkowski proclaimed that Einstein's special relativity meant the separate existences of space and time was henceforth abolished. Einstein then interpreted the gravitational effect of mass-energy as equivalent to a curvature in the Minkowski space-time. Following this, Kaluza-Klein theory incorporated electromagnetism by creating an additional fifth dimension for the curved space-time. Currently, string theory defines the whole Standard Model, based on four fundamental interactions, as a space-time constructed from ten dimensions. Point-like particles become 'strings', constructed from 1 D of space and 1 D of time, existing in an 8-D substratum. 5 classes of string theory are known and it has been hypothesized that they can be related to each other by adding an extra dimension to the strings, so that they become 'membranes' with 2 D of space and 1 D time, again within an 8-D substratum, so requiring 11 D in all.

Such theories have received a great deal of support, but are faced with major difficulties. Many different realisations of string and membrane theory are possible (perhaps as many as 10^{500}), so the chances of hitting on the 'correct' one by accident are impossibly remote. We also have to explain why the 'real' (observed) world seems to be based on a space with just 3 D, and famously provides no experimental predictions, while the whole development, from special to general relativity, to Kaluza-Klein, and then to string / membrane theory, seems to produce a fit with nature which is more awkward and less natural at each successive stage.

There is, in fact, already a problem with making time a fourth dimension of space. This is not entirely compatible with quantum mechanics, where space is an observable, but time is not. The problem is not resolved by increasing the number of dimensions. The theories can be made to 'work' to some extent up to the Kaluza-Klein level, but the increasing awkwardness and complication seem to suggest that this is not theory at a truly fundamental level, where we would expect increasing simplification.

There is, however, an alternative, which incorporates ideas such as symmetry, and, even more particularly, duality. The ultimate basis of this approach is a set of principles (long established in my own work), of which the most important is that Nature exhibits zero totality in all of its aspects, material and conceptual, and it does this via a fundamental principle of duality, which can be inferred, but not observed directly, from within the system. If space is the concept through which all physical observation is mediated, then we can only perceive its existence because a dual space, which cannot be observed, operates simultaneously in such a way that the sum total of all of 'Nature' is precisely zero, though this cannot be observed from within the system.

The difficulties in describing Nature using a single space are overcome once we recognize that it has a dual partner, even though the dual space contains no new information. Equally significant is the fact that we can construct a model for physics without combining observed space with anything other than its dual, without going beyond 3 D, and without assuming distortion or curvature at a fundamental level. All the difficulties that arise with trying to construct a more and more complicated single space (in effect, a modern equivalent of epicycles) are overcome by merging it with its dual partner.

Clifford algebra

The only full description of a physical vector space that includes such as aspects as its ability to generate areas and volumes as well as lengths and directions is that of Clifford or geometrical algebra. The Clifford algebra of 3-dimensional space can be described by + and – versions of 8 base units:

i	**j**	**k**	vector		
i**i**	i**j**	i**k**	bivector	pseudovector	quaternion
i			trivector	pseudoscalar	
1			scalar		

A bivector is the direct product of two orthogonal vector units, and a trivector the direct product of three orthogonal vector units. The vector units **i**, **j**, **k** are identical to the complexified quaternion units ***ii***, ***ij***, ***ik***, and are isomorphic to the Pauli matrices σ_x, σ_y, σ_z. They follow the multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = i\mathbf{k}$$
$$\mathbf{jk} = -\mathbf{kj} = i\mathbf{i}$$
$$\mathbf{ki} = -\mathbf{ik} = i\mathbf{j}$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$$
$$\mathbf{ijk} = i$$

Also, two vectors **a** and **b**, made up of summations of the three unit values multiplied by arbitrary scalar coefficients, have a *full product* which is a combination of the ordinary vector and scalar products:

$$\mathbf{ab} = \mathbf{a.b} + i\mathbf{a} \times \mathbf{b}$$

It is notable that, while 1 is the fourth component of the quaternion system, with ***ijk*** = –1, the quantity that plays that role in the vector system is i. If we multiply the quaternion units ***i***, ***j***, ***i***, 1 by i to create a 4-vector system, we obtain **i**, **j**, **i**, i.

We are now proposing that the reason why all of physics is mediated through the concept of 3-D vector space alone is that this space is supplemented by a dual construct, commutative to itself, which contains no new information but which is necessary to obtain the fundamental totality zero condition which we believe applies to the natural world.

Let us describe this dual vector space using + and – versions of the following symbols:

I	**J**	**K**	vector		
i**I**	i**J**	i**K**	bivector	pseudovector	quaternion
i			trivector	pseudoscalar	
1			scalar		

If the two spaces are commutative, then the full algebra combining them will be a tensor product, with 64 units which can be represented as + and – versions of the following:

i	**j**	**k**	i**i**	i**j**	i**k**	i	1
I	**J**	**K**	i**I**	i**J**	i**K**		
iI	**iJ**	**iK**	i**iI**	i**iJ**	i**iK**		
jI	**jJ**	**jK**	i**jI**	i**jI**	i**jK**		
kI	**kJ**	**kK**	i**kI**	i**kJ**	i**kK**		

The algebraic structure represented by these units is clearly a group of order 64, which can be shown to be isomorphic to the algebra of the Dirac equation, or γ matrices, just as the algebra of a single vector space is recognisably that of the Pauli or s matrices. It is not difficult to show that all possible versions of the γ matrices can be derived from a commutative combination of two sets of s matrices, say σ_1, σ_2, σ_3 and Σ_1, Σ_2, Σ_3.

We establish that the two spaces are dual by imposing the *nilpotency condition*. First, we identify the generators of the group. The minimal number is 5, and so the minimal number of units for describing physics in this way is also 5. All the combinations which include the eight base units 1, i, **i**, **j**, **k**, **i**, **j**, **k**, have the same structure, typically represented by

K i**Ii** i**Ij** i**Ik** i**J**

Many variations on this exist, but without change to roverall pattern.

The nilpotent condition

The pattern distinguishes between the two sets of commuting vector units: one group (here, **i**, **j**, **k**) remains rotation-symmetric because each of the units is multiplied by the same algebraic object, the other (**I**, **J**, **K**) loses this symmetry because each of the units is multiplied by a quite different kind of algebraic object. As represented above, **K** is multiplied by a scalar unit, **I** by a bivector or pseudovector, and **J** by a pseudoscalar. The algebra is unchanged by multiplying the units by arbitrary scalar values, positive or negative. We represent them using the respective symbols E, p_x, p_y, p_z, m. Incorporating these arbitrary scalar values, we can represent the 5 generators of our group formed from two commuting vector spaces by the expressions:

$$\mathbf{K}E \qquad i\mathbf{Ii}p_x \quad i\mathbf{Ij}p_y \quad i\mathbf{Ik}p_z \qquad i\mathbf{J}m$$

We now need to find a condition which will connect the two spaces, ensuring that they are dual in the sense that each contains the same information. The algebra of vector spaces is constructed so that the squared value of any vector, combining summations of the scalar multiplications of the orthogonal components, defines a *scalar* norm, which, in the case of unit values can be equated to 1. We can apply the same condition to our five generators, but this time establishing the norm as 0. So we require

$$(\mathbf{K}E + i\mathbf{Ii}p_x + i\mathbf{Ij}p_y + i\mathbf{Ik}p_z + i\mathbf{J}m)\,(\mathbf{K}E + i\mathbf{Ii}p_x + i\mathbf{Ij}p_y + i\mathbf{Ik}p_z + i\mathbf{J}m) = 0$$

which works out as

$$E^2 - p_x^{\,2} - p_y^{\,2} - p_z^{\,2} - m^2 = 0.$$

This is the nilpotent condition: the bracketed object squares to zero. The zeroing ensures, as we can show, that the information in the two spaces represented by the respective units **i**, **j**, **k** and **I**, **J**, **K** is identical. It also defines, in principle, the meaning of a *point* in either of the two spaces as the norm 0 crossover between them. Mathematicians discuss points in ordinary 3-D space, but physically they have no meaning, as space is a nonconserved quantity whose units have no definable identity because they have rotation and translation symmetry. In effect, we cannot identify anything in a single space, but identification becomes possible if we have two spaces.

But if nonconservation can be thought of as denying identification, then identification can be thought of as suggesting conservation. In effect, the units represented by the second space (**I**, **J**, **K**) become identifiable because they are associated with different algebraic objects, and so the 3-dimensionalty of this second space is somehow structurable as a *conserved* dimensionality because of its rotation asymmetry. Gradually, we see that our two spaces are beginning to suggest a picture of physics as we know it. We have an emerging idea of two spaces, which, though containing the same information and though presumably symmetric at some very deep level, look very different to the physical observer because the creation of a system of 5 generators necessarily breaks a symmetry between the way we perceive them.

Essentially, from our position *within* the system, we have been forced to 'privilege' one space over the other, to maintain the symmetry of one while losing that of the other. This can be seen as similar to the way in which our most primitive form of numbering, binary arithmetic, 'privileges' 1 over –1, making them dual in summing to 0, but appearing very different in the way they are perceived from within a system defined by unit 1.

A physical realization of two spaces

We have not yet justified, on fundamental grounds, why we use space at all and where the other space might come from. Clifford algebra, significantly, has 3 subalgebras, which we can describe as scalar, complex and quaternion, or scalar, trivector and bivector. Each of these is an algebra in its own right, and it is difficult to see why only the full Clifford algebra should have a physical meaning. In fact, previous work suggests that all of the subalgebras have physical meanings on the same level as Clifford algebra, and that they represent the respective physical concepts of mass, time and charge. That is, besides the vector algebra of space, we have three independent algebras which have a physical representation on the same level as space.

If we combine these the three physical concepts as representing everything that is excluded from space as represented by **i**, **j**, **k**, then the total structure is equivalent to a single vector space represented by the units **I**, **J**, **K**, but *without anything which directly corresponds to these units*:

charge	$i\mathbf{I}$	$i\mathbf{J}$	$i\mathbf{K}$	bivector	pseudovector	quaternion
time	i			trivector	pseudoscalar	
mass	1			scalar		

So, we are unable to observe the space represented by **I**, **J**, **K**, because it has no single physical representation. It is, instead, a mathematical combination of three physical quantities which are not part of the space represented by **i**, **j**, **k**. Our second space has physical meaning, but cannot be accessed as a physical quantity. It is conveniently called 'vacuum space', as opposed to 'real space'. It can also be described as 'antispace' because it combines with real space to produce zero totality. Also, to emphasize the independence of charge from the 'space' in which it is incorporated, it is convenient to represent it directly using quaternion, rather than bivector, notation, so the components of 'vacuum space' now become:

charge	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	bivector	pseudovector	quaternion
time	i			trivector	pseudoscalar	
mass	1			scalar		

We can now link the generation of space as a Clifford algebra and those of mass, time and charge as independent subalgebras, with the evolutionary *universal rewrite* process we previously defined for the whole sequence from zero totality (Rowlands and Diaz, 2002, Diaz and Rowlands, 2005, Rowlands, 2007), and also with the D_2 group symmetry which can be derived from the most fundamental properties of mass, time, charge and space, which also represent a conceptual zero totality.

mass	real (norm +1)	commutative	conserved
time	imaginary (norm −1)	commutative	nonconserved
charge	imaginary (norm −1)	anticommutative (3D)	conserved
space	real (norm +1)	anticommutative (3D)	nonconserved

The real / imaginary and commutative / anticommutative properties are directly derived from the algebraic units associated with the respective parameters. The conserved and nonconserved natures of charge and space are related to the way they are combined in the 5 group generators creating the norm 0 overall structure, while the corresponding natures of mass and time are related to the fact that quantities with their algebraic characteristics are needed to complete the quaternion and vector properties of charge and space.

While the rewrite structure shows that the evolutionary process creating mass, time, charge and space can be continued to infinity, the creation of nilpotent structures zeroing all higher terms and the perfect group symmetry, allowing a complete cancellation, ensure that all the higher order structures can be incorporated in Clifford algebra at the first level. The creation of a norm 0 state out of our double Clifford algebra can be interpreted as the creation of the physical objects known as point-particles or fermions. No other fundamental structures are known to be needed for physics, bosons being expressions of fermion interactions.

In addition, a fermion interpretation of the Wheeler one-electron theory of the universe allows us to conceive a representation of the universe as a single localised fermion interacting with its nonlocal vacuum. Our dual space structure in which the basic unit is a point-singularity incorporating some components that are conserved and others that are nonconserved provides an opportunity for explaining the whole of physics using this model. We begin by generating a nilpotent version of quantum mechanics. Here, it is convenient to rewrite the nilpotent condition using quaternions for the charges:

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m)\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m) = 0$$

We can also collect together the vector terms, so that

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = 0.$$

Finally, we can use the identifications we have made for the various algebraic units of mass, time, charge and space, to identify the composite quantities in this equation as energy, momentum and rest mass, and to recognise that the bracketed object $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ represents a conserved quantity, with E, $\mathbf{p}$ and m being derived by combining the conserved charge units with those, respectively of time, space and mass.

Following this, we approach quantum mechanics as the most exact way of describing the nonconservation of space and time in relation to the conservation of $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. The most complete possible variation in space and time is defined by a phase factor which associates E with time and $\mathbf{p}$ with space. We then use the differentials $\partial / \partial t$ and ∇ to recover $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ from the phase factor. For a free particle, the most complete set of variations in space and time is given by $e^{-i(Et - \mathbf{p.r})}$, and the expression which will recover $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ using this as a phase factor is $(-\boldsymbol{k}\partial / \partial t - \boldsymbol{i}\mathbf{i}\nabla + \boldsymbol{j}m)$. So we construct the nilpotent quantum mechanical equation for a free particle in the form

$$(-\boldsymbol{k}\partial / \partial t - \boldsymbol{i}\mathbf{i}\nabla + \boldsymbol{j}m)\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,e^{-i(Et - \mathbf{p.r})} = 0.$$

Including all possible sign variations of E and $\mathbf{p}$, we obtain

$$(\mp\,\boldsymbol{k}\partial / \partial t \mp \boldsymbol{i}\mathbf{i}\nabla + \boldsymbol{j}m)\,(\pm\,\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,e^{-i(Et - \mathbf{p.r})} = 0$$

which is equivalent to a nilpotent Dirac equation of the form

$$(\mp\,\boldsymbol{k}\partial / \partial t \mp \boldsymbol{i}\mathbf{i}\nabla + \boldsymbol{j}m)\psi = 0.$$

We can also express it in operator form

$$(\pm ikE \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\pm ikE \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,e^{-i(Et-\mathbf{p}.\mathbf{r})} = 0,$$

where the operators E and $\mathbf{p}$ become $i\partial\,/\,\partial t$ and $-i\nabla$ as in the usual canonical quantization.

A nilpotent wavefunction, say ψ_1, is automatically Pauli exclusive because it will form a zero combination state with an identical particle $\psi_1\,\psi_1$. In addition, if the universe is a totality zero state, then we can imagine that the creation of a fermion, specified by ψ_1, from *absolutely nothing* requires the simultaneous creation of the vacuum state, $-\psi_1$, which would cancel it. In that case, the superposition of fermion and vacuum, $\psi_1 - \psi_1$, and their combination state, $-\psi_1\psi_1$, will both be precisely 0.

Now, the concept of Pauli exclusion applies to fermions in any state, and we can use this to extend nilpotent quantum mechanics in a new direction, by imagining that the operator

$$(\pm ikE \pm \mathbf{i}\mathbf{p} + \mathbf{j}m) = (\mp\, \mathbf{k}\partial\,/\,\partial t \mp\, \mathbf{i}i\nabla + \mathbf{j}m)$$

can be extended to include any number of field terms or covariant derivatives, so that E and $\mathbf{p}$ now become, for example, $i\partial\,/\,\partial t + e\phi + \ldots$ and $-i\nabla + e\mathbf{A} + \ldots$. The same will also be true of external field terms defined by expectation values, as with the Lamb shift, or in terms of quantum fields.

In this form, we don't even need an equation, just an operator of the form $(\pm ikE \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)$ because the operator will uniquely determine the phase factor needed to produce a nilpotent amplitude. Rather than using a conventional form of the Dirac equation, we find the phase factor such that, using the defined operator,

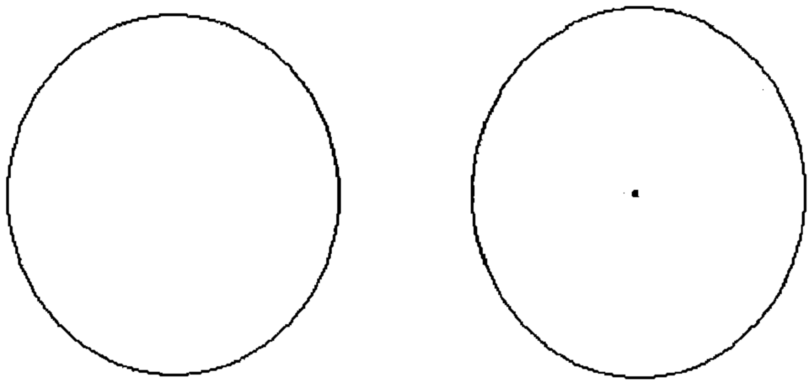
$$(\text{operator acting on phase factor})^2 = \text{amplitude}^2 = 0.$$

If the operator has a more complicated form than that of the free particle, the phase factor will, of course, be no longer a simple exponential but the amplitude will still be a nilpotent.

A nilpotent operator, in effect, always splits the universe into two halves, a local part represented by the fermion, and a dual, nonlocal, vacuum part, which incorporates the rest of the universe. So we create a fermion in a particular state, including all its interacting potentials, and then we have to construct the vacuum or 'rest of the universe' which enables the fermion to exist in that state. The fact that the fermion in any state needs to create the entire universe which makes it possible makes a Wheeler-type 'one fermion' theory of the universe a seriously interesting possibility. The fermion and the entire universe are then a dual pair, and so the structure of the universe can be thought of as equivalent to that of a single particle. Of course, this single fermion is necessarily an interacting one, constructing a 'space' in which the vacuum is not localised on itself.

Another way of looking at this is to say that the fermion always exists in the two spaces from which it is constructed, real space and vacuum space, and the non-classical *zitterbewgung* motion, which Schrödinger found in the solution to the free-particle Dirac equation, represents the switching between these spaces which makes it possible to define the fermion as creating a point singularity through the intersection of two spaces.

We can here apply a reverse argument from topology. The creation of a particle singularity using its 'intersection' with a dual space can be seen as the creation of a multiply-connected space from a simply-connected space through the insertion of a topological singularity.



simply-connected space *multiply-connected space*

According to a well-known argument, parallel transporting a vector round a complete circuit in a multiply-connected space will produce a phase shift of π or 180° in the vector direction, whereas transporting it round a simply-connected space will not, and so, in the first case, the vector will be required to do a double circuit to return to its starting point.

This is exactly what happens with a spin ½ fermion, which, as a point-singularity, can be regarded as existing in its own multiply-connected space. We can interpret this as meaning that the fermion requires a double circuit because, just as in *zitterbewegung*, it spends only half of its time travelling in the real space of observation.

Spin

Spin ½ is obtained quite easily from the nilpotent operator ($\boldsymbol{i}kE + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$) by defining a Hamiltonian specified as $H = -\boldsymbol{i}k(\boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = -\boldsymbol{i}\boldsymbol{j}\mathbf{p} + \boldsymbol{i}\boldsymbol{i}m$, and a *mathematical* quantity $\boldsymbol{\sigma} = -\mathbf{1}$, which is a pseudovector of magnitude -1. Then

$$\begin{aligned}[\boldsymbol{\sigma}, H] &= [-\mathbf{1}, -\boldsymbol{i}\boldsymbol{j}\,(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3) + \boldsymbol{i}\boldsymbol{i}m] = [-\mathbf{1}, -\boldsymbol{i}\boldsymbol{j}(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)] \\ &= [-\mathbf{1}, -\boldsymbol{i}\boldsymbol{j}\,(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3) + \boldsymbol{i}\boldsymbol{i}m] = [-\mathbf{1}, -\boldsymbol{i}\boldsymbol{j}(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)] \\ &= 2\boldsymbol{i}\boldsymbol{j}\,(\mathbf{i}\mathbf{j}p_2 + \mathbf{i}\mathbf{k}p_3 + \mathbf{j}\mathbf{i}p1 + \mathbf{j}\,p_3 + \mathbf{k}\mathbf{i}p_1 + \mathbf{k}\mathbf{j}p_2) \\ &= -2\boldsymbol{j}\,(\mathbf{k}(p_2 - p_1) + \mathbf{j}(p_1 - p_3) + \mathbf{i}(p_3 - p_2)) \\ &= -2\boldsymbol{j}\mathbf{1} \times \mathbf{p}.\end{aligned}$$

If $\mathbf{L}$ is an orbital angular momentum defined by $\mathbf{r} \times \mathbf{p}$, then

$$\begin{aligned}[\mathbf{L}, H] &= [\mathbf{r} \times \mathbf{p}, -\boldsymbol{i}\boldsymbol{j}\,(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3) + \boldsymbol{i}\boldsymbol{i}m] \\ &= [\mathbf{r} \times \mathbf{p}, -\boldsymbol{i}\boldsymbol{j}\,(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)] \\ &= \boldsymbol{i}\,[\mathbf{r}, -\boldsymbol{i}\boldsymbol{j}\,(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)] \times \mathbf{p}\end{aligned}$$

But $$[\mathbf{r}, -\boldsymbol{i}\boldsymbol{j}\,(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)] = \boldsymbol{i}\mathbf{1}.$$

Hence $$[\mathbf{L}, H] = \mathbf{1} \times \mathbf{p},$$

and $\mathbf{L} + \boldsymbol{\sigma} / 2$ is a constant of the motion, because

$$[\mathbf{L} + \boldsymbol{\sigma} / 2, H] = 0.$$

Spin, in this derivation, emerges from the Clifford algebra aspect of the operator $\mathbf{p}$, in effect because of the $i\mathbf{a} \times \mathbf{b}$ aspect of the full vector product $\mathbf{ab} = \mathbf{a}.\mathbf{b} + i\mathbf{a} \times \mathbf{b}$.

If we use Clifford algebra or *multivariate* vectors, then $\mathbf{p}$ will incorporate spin. However, where ordinary vectors are used, e.g. with polar coordinates, an intrinsic spin is no longer structured into the formalism and an *explicit* spin term has to be introduced. Dirac has given a prescription for translating his equation into polar form. Here $\mathbf{p}$ acquires an additional (imaginary) spin term, and we can easily adapt this to represent a polar transformation of the multivariate vector operator:

$$\nabla \to \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \pm i\,\frac{j+\frac{1}{2}}{r}$$

and use this to define a non-time varying nilpotent operator in polar coordinates:

$$(\boldsymbol{i}kE - \boldsymbol{i}\mathbf{i}\nabla + \boldsymbol{j}m) \to \left(\boldsymbol{i}kE - \boldsymbol{i}\mathbf{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\,\frac{j+\frac{1}{2}}{r}\right) + \boldsymbol{j}m\right)$$

This will become significant when we consider cases involving spherical symmetry.

Antisymmetric wavefunctions

The nilpotent structure explains immediately why we have Pauli exclusion between fermions, but the conventional way of explaining this property leads us to a profound insight on the nature of the information available in quantum systems if we structure it in nilpotent form. This is by defining fermion wavefunctions to be antisymmetric, so that:

$$(\psi_1\psi_2 - \psi_2\psi_1) = -(\psi_2\psi_1 - (\psi_1\psi_2)$$

In nilpotent terms, we write $(\psi_1\psi_2 - \psi_2\psi_1)$ as

$$\begin{aligned}
&(\pm\, \boldsymbol{i}kE_1 \pm \mathbf{i}\mathbf{p}_1 + \boldsymbol{j}m_1)\,(\pm\, \boldsymbol{i}kE_2 \pm \mathbf{i}\mathbf{p}_2 + \boldsymbol{j}m_2) \\
&- (\pm\, \boldsymbol{i}kE_2 \pm \mathbf{i}\mathbf{p}_2 + \boldsymbol{j}m_2)\,(\pm\, \boldsymbol{i}kE_1 \pm \mathbf{i}\mathbf{p}_1 + \boldsymbol{j}m_1) \\
&= 4\mathbf{p}_1\mathbf{p}_2 - 4\mathbf{p}_2\mathbf{p}_1 = 8\, i\, \mathbf{p}_1 \times \mathbf{p}_2 = -8\, i\, \mathbf{p}_2 \times \mathbf{p}_1.
\end{aligned}$$

This result is clearly antisymmetric, but it also has a quite astonishing consequence, for it requires any nilpotent wavefunction to have a **p** vector, in *real space*, the one defined by the axes **i**, **j**, **k**, at a *different orientation* to any other.

The wavefunctions of all nilpotent fermions then instantaneously correlate because the planes of their **p** vector directions must all intersect. At the same time, the nilpotent condition requires the *E*, **p** and *m* combinations to be unique, and we can visualize this as constituting a unique direction in vacuum space along a set of axes defined by ***k***, ***i***, ***j*** or

K, **I**, **J**, with coordinates defined by the values of E, **p** and m. The directions of the vectors in each space carry *all the information* available to a fermionic state, and so the information in the two spaces is totally dual, and is equivalent to the instantaneous direction of the spin in the real space. The total information determining the behaviour of a fermion and even of the entire universe is contained in a single spin direction.

Conventionally, the Dirac wavefunction is a *spinor*, with the four components in ($\pm$ $\boldsymbol{k}E$ $\pm$ $\boldsymbol{i}\mathbf{p}$ + $\boldsymbol{j}m$) structured as a column vector, incorporating the 4 combinations of particle and antiparticle, and spin up and spin down. With $\pm$ E and $\pm$ **p** (or $\pm$ $\boldsymbol{\sigma}.\mathbf{p}$) representing these possibilities, the respective amplitudes can be identified as, say,

$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion spin up
$(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion spin down
$(-\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion spin down
$(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion spin up

each being multiplied by the same phase factor. The helicity or handedness ($\boldsymbol{\sigma}.\mathbf{p}$) is then determined by the ratio of the signs of E and **p**. So $\boldsymbol{i}\mathbf{p}$ / $\boldsymbol{ik}E$ has the same helicity as $(-\ \boldsymbol{i}\mathbf{p})$ / $(-\boldsymbol{ik}E)$, but the opposite helicity to $\boldsymbol{i}\mathbf{p}$ / $(-\boldsymbol{ik}E)$. The negative energy or antiparticle states in this formalism can also be seen to have the opposite time direction in their differential forms to the positive energy or particle states.

The lead term in the column may be considered as defining the fermion type, and it will often be convenient to represent the entire 4-component structure by just this term. The remaining terms are then automatically derived by sign transformations, becoming equivalent to the lead term, subjected to the respective symmetry transformations, P, T and C, by pre- and post-multiplication by the quaternion units defining what we have previously described as the *vacuum space*.

Parity	P	$\boldsymbol{i}\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{i} = (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$
Time reversal	T	$\boldsymbol{k}\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{k} = (-\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$
Charge conjugation	C	$-\boldsymbol{j}\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{j} = (-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$

Clearly also:

$$CP \equiv T \quad PT \equiv C \quad CT \equiv P$$

and

$$TCP \equiv CPT \equiv \text{identity}$$

as

$$\boldsymbol{k}\,(-\boldsymbol{j}\,(\boldsymbol{i}\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{i})\,\boldsymbol{j})\,\boldsymbol{k} = -\boldsymbol{kji}\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{ijk} = (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

From these rules, it is clear that charge conjugation is effectively defined in terms of parity and time reversal, rather than as an independent operation. This is a consequence of the fact that the variation in space and time is the information that solely determines both the phase factor and the entire nature of the fermion state.

Bosons

The terms in the nilpotent 4-spinor, other than the lead term are the states into which it could transform without changing its E or p. They can be seen as vacuum 'reflections' of the real particle state, arising from vacuum operations that can be mathematically defined. A fermion be imagined as forming a combination state with any of these vacuum 'reflections'. In each of these cases, the combined state will form one of the three classes of bosons or boson-like objects, whose wavefunctions, summed up over 4 terms, yield products which are scalars:

Spin 1 boson:

$$(\pm ikE \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\mp ikE \pm \mathbf{i}\mathbf{p} + \mathbf{j}m) \qquad T$$

Spin 0 boson:

$$(\pm ikE \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\mp ikE \mp \mathbf{i}\mathbf{p} + \mathbf{j}m) \qquad C$$

Fermion-fermion combination (B-E condensate / Berry phase, etc.):

$$(\pm ikE \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\pm ikE \mp \mathbf{i}\mathbf{p} + \mathbf{j}m) \qquad P$$

A spin 1 boson can be massless, but a spin 0 boson cannot, as then $(\pm ikE \pm \mathbf{i}\mathbf{p})\,(\mp ikE \mp \mathbf{i}\mathbf{p})$ immediately reduces to zero: hence Goldstone bosons must become Higgs bosons in the Higgs mechanism.

Another consequence is that the fermion and antifermion cannot both be purely left-handed or both purely right-handed – or massless – and act via a weak interaction to produce a bosonic state. That is, a left-handed fermion $(\pm ikE \pm \mathbf{i}\mathbf{p})$ cannot combine with a left-handed antifermion $(\mp kE \mp \mathbf{i}\mathbf{p})$, via a weak interaction, to form a bosonic single state unless a nonzero mass term is introduced. Though the chirality is a direct consequence of the structure of the Dirac equation, even in the conventional formalism, it is seen here as an immediate consequence of the nilpotent structure.

Versions of these bosons can also be imagined as being created and annihilated in the switching between a 'real fermion' and its vacuum states, in particular between the + and – energy states, or between real and vacuum space (the switching between spin states occurring in real space if particles have nonzero rest mass). Since this is always occurring due to *zitterbewegung*, and the *weak* interaction, then we can consider weak sources (i.e. fermions) as necessarily having a dipole or multipole aspect.

Because of the fundamental chirality of the weak interaction, we can also see fermions as characteristic of real space and antifermions of vacuum space. A particle which is a fermion in ordinary space acts for half its existence as an antifermion in vacuum space, *in exactly the way that the 4-component Dirac spinor would suggest*. There is no problem of an antisymmetry between matter and antimatter. There is the same amount of each. E, **p**, space, time and charge all cancel overall when we take both spaces into account. There is, in particular, a backward direction in time and a reverse causality which apply to nonlocal processes. The entire future of the universe could be said to be contained within the vacuum for any fermion, though this does not lead to a deterministic outcome because the nonlocality cannot be defined by an observer any more accurately than the locality associated with the fermion to which it is opposite.

Baryons

Clearly, with nilpotent fermions, a structure such as

$$(\boldsymbol{k}E \pm i\mathbf{p} + \boldsymbol{j}m)\,(\boldsymbol{k}E \pm i\mathbf{p} + \boldsymbol{j}m)\,(\boldsymbol{k}E \pm i\mathbf{p} + \boldsymbol{j}m)$$

will automatically zero, but one in which the **p** term is split into components along orthogonal axes will produce a nonzero combination:

$$(\boldsymbol{k}E \pm i\mathbf{i}p_x + \boldsymbol{j}m)\,(\boldsymbol{k}E \pm i\mathbf{j}p_y + \boldsymbol{j}m)\,(\boldsymbol{k}E \pm i\mathbf{k}p_z + \boldsymbol{j}m)$$

Spin is defined in a unique direction at any time, so, at any particular instant, the wavefunction will reduce, after normalization, to

$$\begin{aligned}
&(\boldsymbol{k}E \pm i\mathbf{i}p_x + \boldsymbol{j}m)\,(\boldsymbol{k}E + \boldsymbol{j}m)\,(\boldsymbol{k}E + \boldsymbol{j}m) \rightarrow (\boldsymbol{k}E \pm i\mathbf{i}p_x + \boldsymbol{j}m)\\
&(\boldsymbol{k}E + \boldsymbol{j}m)\,(\boldsymbol{k}E \pm i\mathbf{j}p_y + \boldsymbol{j}m)\,(\boldsymbol{k}E + \boldsymbol{j}m) \rightarrow (\boldsymbol{k}E \mp i\mathbf{j}p_y + \boldsymbol{j}m)\\
&(\boldsymbol{k}E + \boldsymbol{j}m)\,(\boldsymbol{k}E + \boldsymbol{j}m)\,(\boldsymbol{k}E \pm i\mathbf{k}p_z + \boldsymbol{j}m) \rightarrow (\boldsymbol{k}E \pm i\mathbf{k}p_z + \boldsymbol{j}m)
\end{aligned}$$

There is a notable change of sign in the second case.

To maintain the symmetry between the three directions of momentum, and the + and – values of the momentum term, we can define six possible outcomes, resulting in a superposition of six combination states:

$$
\begin{gathered}
(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{j}m) \rightarrow (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m) \\
(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{j}m) \rightarrow (i\boldsymbol{k}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m) \\
(i\boldsymbol{k}E + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{j}m) \rightarrow (i\boldsymbol{k}E - \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m) \\
(i\boldsymbol{k}E + \boldsymbol{j}m)\,(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{j}m) \rightarrow (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m) \\
(i\boldsymbol{k}E + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m) \rightarrow (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m) \\
(i\boldsymbol{k}E + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \boldsymbol{j}m)\,(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m) \rightarrow (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m)
\end{gathered}
$$

Perfect gauge invariance is also only possible if the baryon simultaneously incorporates both left-handed and right-handed components or + and – values of **p**. By the principle which we have previously applied to boson structures, baryons *must therefore have mass*. The principle is, significantly, the same as that which applies in the Higgs mechanism, even though the perfect gauge invariance between the six possible states or switching between the different **p** components is provided by massless gluons. The solution to the so-called mass-gap problem which this incorporates is a significant part of one of the prize challenge problems defined by the Clay Institute.

The Coulomb interaction

One of the most important aspects of the nilpotent structure is that it offers an immediate separation of the local from the nonlocal. Essentially, everything *inside* a fermion bracket, such as ($\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$), is local, defined by a Lorentzian structure, while everything *outside* it, such as a combination state or a superposition, is nonlocal. It is possible, then, from the possible nonlocal structures available to fermions, defined by combinations and superpositions, to derive the *local* structures, inside the fermion brackets (essentially the potentials to be added to E or **p**) which would produce the same effect. When we do this we find that the possible local structures lead to just 3 classes of interaction which give a nilpotent solution, and these interactions correspond to those which, physically, we classify as electric, strong and weak.

The first nonlocal effect is Pauli exclusion, which, in effect, is a prohibition on certain combinations, specified by nilpotency. Here we use Dirac's polar transformation to describe a fermion as a point-particle with spherical symmetry:

$$\left(\pm ikE \mp ii\left(\frac{\partial}{\partial r}+\frac{1}{r}\pm i\frac{j+½}{r}\right)+jm\right).$$

To establish Pauli exclusion, we need to define this operator as producing a nilpotent solution. Inspection reveals that this is impossible unless the *ikE* component also includes a term involving 1 / *r* to cancel out the terms with this factor multiplied by ***ii***. This is a *Coulomb term*, so simply defining a point particle using the nilpotent operator automatically requires it to have a Coulomb potential because of Pauli exclusion. Pauli exclusion is an effect of a point-particle having spherical symmetry. The minimum operator for a point-particle is therefore of the form

$$\left(\pm ik\left(E-\frac{A}{r}\right) \mp ii\left(\frac{\partial}{\partial r}+\frac{1}{r}\pm i\frac{j+½}{r}\right)+jm\right).$$

exactly as we require for the Coulomb interaction.

This is easily solved, requiring, in fact, only six lines of calculation. First, we have to find the phase factor ϕ which will make the amplitude nilpotent. As in the standard solution, we assume that it is of the form:

$$\phi = e^{-ar} r^{\gamma} \sum_{\nu=0} a_{\nu} r^{\nu}$$

We then apply the operator we have just defined to ϕ and square the result to 0 to obtain:

$$4\left(E-\frac{A}{r}\right)^2 = -2\left(-a+\frac{\gamma}{r}+\frac{\nu}{r}+\ldots\frac{1}{r}+i\frac{j+½}{r}\right)^2 - 2\left(-a+\frac{\gamma}{r}+\frac{\nu}{r}+\ldots\frac{1}{r}-i\frac{j+½}{r}\right)^2 + 4m^2 .$$

Equating constant terms produces

$$a=\sqrt{m^2-E^2}$$

Equating terms in $1/r^2$, following standard procedure, with $\nu = 0$:

$$\left(\frac{A}{r}\right)^2 = -\left(\frac{\gamma+1}{r}\right)^2 + \left(\frac{j+½}{r}\right)^2$$

Assuming the power series terminates at n', following another standard procedure, and equating coefficients of $1/r$ for $\nu = n'$, we obtain

$$2EA = -2\sqrt{m^2 - E^2}\,(\gamma + 1 + n')$$

Algebraic rearrangement of these equations then yields

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \dfrac{A^2}{(\gamma + 1 + n')^2}}} = \frac{1}{\sqrt{1 + \dfrac{A^2}{\left(\sqrt{(j + \frac{1}{2})^2 - A^2} + n'\right)^2}}}$$

This is a general formula, but in the particular case where $A = Ze^2$, this becomes the hyperfine or fine structure formula for a one-electron nuclear atom or ion, for example, that of the hydrogen atom, where $Z = 1$.

The strong interaction

A second case suggests itself for the strong interaction, which we know requires a linear potential to explain both its experimental characteristics, and also the nilpotent structures of baryons. We have found also that there must be a Coulomb component or inverse linear potential ($\propto 1 / r$), just for spherical symmetry; this has a known physical manifestation in the strong interaction in the one-gluon exchange. A nilpotent operator incorporating Coulomb and linear potentials from a source with spherical symmetry (the centre of a 3-quark system or one component of a quark-antiquark pairing) can be written in the form:

$$\left(\pm \boldsymbol{k}\left(E + \frac{A}{r} + Br\right) \mp \boldsymbol{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\frac{j + \frac{1}{2}}{r}\right) + \boldsymbol{ij}m\right)$$

Again, we need to identify the phase factor, which, by analogy with the pure Coulomb calculation, we might suppose to be of the form:

$$\phi = \exp\left(-ar - br^2\right) r^\gamma \sum_{\nu=0} a_\nu r^\nu$$

Applying the operator we have defined and the nilpotent condition, we obtain:

$$E^2+2AB+\frac{A^2}{r^2}+B^2r^2+\frac{2AE}{r}+2BEr=m^2$$
$$-\left(a^2+\frac{(\gamma+\nu+\ldots+1)^2}{r^2}-\frac{(j+½)^2}{r^2}+4b^2r^2+4abr-4b(\gamma+\nu+\ldots+1)-\frac{2a}{r}(\gamma+\nu+\ldots+1)\right)$$

Then, assuming a termination in the power series (as with the Coulomb solution), we can equate:

coefficients of r^2 to give	$B^2=-4b^2$
coefficients of r to give	$2BE=-4ab$
coefficients of $1/r$ to give	$2AE=2a(\gamma+\nu+1)$

These equations then lead to:

$$b=\pm\frac{iB}{2}$$
$$a=\mp iE$$
$$\gamma+\nu+1=\mp iA.$$

The ground state case (where $\nu = 0$) then requires a phase factor of the form:

$$\phi=\exp(\pm iEr\mp iBr^2/2)r^{\mp iA-1}$$

The imaginary exponential terms in ϕ clearly represent asymptotic freedom, a term like exp ($\mp$ iEr) being typical for a free fermion. The complex r^g term can be structured as a component phase, $c(r)$ = exp ($\pm$ iqA ln (r)), varying less rapidly with r than the rest of ϕ. We can therefore write ϕ as

$$\phi=\frac{\exp(kr+\chi(r))}{r},$$

where

$$k=\pm iE\mp iBr/2.$$

The first term dominates at high E, where r is small, close to a free fermion solution (asymptotic freedom); the second term, with confining potential Br, significant at low E, when r is large (infrared slavery). The Coulomb term, for spherical symmetry, defines the strong interaction phase, $c(r)$, related to the directional status of **p** in the state vector.

The weak interaction

One further interaction (the weak) is built into the structure of the nilpotent operator as a 4-component combination state. This requires a dipole or multipole term in addition to the standard Coulomb term from spherical symmetry. We can therefore suppose that the nilpotent operator takes a form such as

$$\left(\mathbf{k}\left(E - \frac{A}{r} - Cr^{n} \right) + \mathbf{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\frac{j+½}{r} \right) + \mathbf{ij}m \right)$$

where n is an integer greater than 1 or less than -1, and, as before, look for a phase factor which will make the amplitude nilpotent. Extending our work on the Coulomb solution, we may suppose that the phase factor is of the form:

$$\phi = \exp\left(-ar - br^{n+1}\right) r^{\gamma} \sum_{\nu=0} a_{\nu} r^{\nu}$$

Applying the operator and squaring to zero, with a termination in the series, we obtain

$$4\left(E - \frac{A}{r} - Cr^{n} \right)^{2} = -2\left(-a - (n+1)br^{n} + \frac{\gamma}{r} + \frac{\nu}{r} + \frac{1}{r} + i\frac{j+½}{r} \right)^{2}$$
$$- 2\left(-a - (n+1)br^{n} + \frac{\gamma}{r} + \frac{\nu}{r} + \frac{1}{r} - i\frac{j+½}{r} \right)^{2}$$

Equating constant terms, we find

$$a = \sqrt{m^{2} - E^{2}}$$

Equating terms in r^{2n}, with $\nu = 0$:

$$C^{2} = -(n+1)^{2} b^{2}$$
$$b = \pm \frac{iC}{(n+1)} .$$

Equating coefficients of r^{n-1}, where $\nu = 0$:

$$AC = -(n+1)\, b\, (1+\gamma),$$
$$(1+\gamma) = \pm iA$$

Equating coefficients of $1 / r^2$ and coefficients of $1 / r$, for a power series terminating in $n = n'$, we obtain

$$A^2 = -(1 + \gamma + n')^2 + (j + ½)^2$$

and

$$-EA = a\,(1 + \gamma + n').$$

An algebraic combination of these conditions produces:

$$\left(\frac{m^2 - E^2}{E^2}\right)(1 + \gamma + n')^2 = -(1 + \gamma + n')^2 + (j + ½)^2$$

$$E = -\frac{m}{j + ½}(\pm iA + n').$$

This equation has the form of a harmonic oscillator, with evenly spaced energy levels deriving from integral values of n'. If we make the additional assumption that A, the phase term required for spherical symmetry, derives from the random directionality of the fermion spin, we may assign to it a half-unit value ($\pm$ ½ i), or ($\pm$ ½ $i\hbar c$), and obtain the complete formula for the fermionic simple harmonic oscillator:

$$E = -\frac{m}{j + ½}(½ + n')$$

The potential of the form Cr^n made no assumptions about the value of n except that it was an integer > 1 or < -1. Any potential of this form, or any combination, will generate a harmonic oscillator solution for the nilpotent operator. They emerge from systems where there is complexity, aggregation, or a multiplicity of sources. Virtually any potential other than Coulomb or Coulomb plus linear, must be of this form, so we have effectively demonstrated that there are only three possible interaction types that can apply to a nilpotent fermionic operator.

Partitioning the vacuum

The nonlocal aspect of the fermionic nilpotent state ($\pm\, i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m$) is defined by a continuous vacuum $-(\pm\, i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)$. We can use the operators $\boldsymbol{k}, \boldsymbol{i}, \boldsymbol{j}$ to effectively partition this state into discrete components with a dimensional structure, which can then be identified as the weak, strong and electric components responding respectively to the weak strong and electric charges. We can postmultiply $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ by the idempotent $\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ any number of times, without changing its state

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ \boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ \boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \ldots \rightarrow (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

The idempotent acts as a vacuum operator. The same is also true of postmultiplication by $\boldsymbol{i}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ or $\boldsymbol{j}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, Of course, these operations are also equivalent to applying T, P or C transformations to every even bracket. For example,

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\mp i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \ldots \rightarrow (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

Here, every alternate state becomes an antifermion, which combines with the original fermion state to become a spin 1 boson $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ $(\mp \boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$.

Effectively, repeated post-multiplication of a fermion operator by any of the discrete idempotent vacuum operators creates an alternate series of antifermion and fermion vacuum states, or an alternate series of boson and fermion states without actually changing the character of the real particle state. A fermion becomes *its own* boson by combining with any of its vacuum 'images'. A boson can be similarly shown to be its own fermion or antifermion. Nilpotent operators are thus intrinsically supersymmetric, with supersymmetry operators $Q = (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, converting boson to fermion, and $Q^\dagger = (\mp \boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, converting fermion to boson, and we can represent the infinite post-multiplication sequences by the supersymmetric expressions such as $Q\, Q^\dagger\, Q\, Q^\dagger\, Q\, Q^\dagger\, Q\, Q^\dagger\, Q \ldots$. It is even possible to interpret this as creating the series of boson and fermion loops, with identical energy and momentum values, which an exact supersymmetry would need to cancel the self-energy term in renormalization, and remove the hierarchy problem completely.

The identification of $\boldsymbol{i}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, $\boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and $\boldsymbol{j}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ as vacuum operators and the corresponding identification of $\boldsymbol{i}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{i}$, $\boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}$ and $\boldsymbol{j}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{j}$ as their respective vacuum 'reflections' at interfaces provided by P, T and C transformations provides a new insight into the meaning of the Dirac 4-spinor. The three terms following the lead term which is identified with the particle can be seen as the vacuum 'reflections' that are created with the particle in the three coordinate axes of vacuum space. The four components then become creation and annihilation operators acting on their respective vacua: gravitational (or inertial), strong, weak and electric.

We can additionally see the three vacuum coefficients ***k***, ***i***, ***j*** as originating in, or being responsible for, the concept of discrete, point-like, charge, which generates the particle state. The operators, ***k***, ***i*** and ***j*** act like weak, strong and electric 'charges' or sources, acting to partition the *continuous* vacuum represented by $-(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, into discrete components, with special characteristics determined by the respective pseudoscalar, vector and scalar natures of their associated terms iE, $\mathbf{p}$ and m.

The nature of gravity as acting like a summation of the other three forces, long predicted by this theory, is now a fundamental component also of many string theories under the name of 'gravity-gauge theory correspondence'. In addition, this theory also predicted the existence of the 'dark energy' long before its discovery, and fixed it as being equivalent to two-thirds of the energy of the universe, in line with recent results from the Planck Collaboration (2013).

The duality of real and vacuum spaces

Though the combination of two spaces leads to chirality and an asymmetry in the vacuum space in terms of *observation*, at the deepest level the symmetry is retained. We can see this, if we construct the set of spinors, using the double vector notation, to generate the four components of the nilpotent structure from the basic $(\mathbf{K}E + i\mathbf{Ii}p_x + i\mathbf{Ij}p_y + i\mathbf{Ik}p_z + i\mathbf{J}m)$. As primitive idempotents, they are orthogonal (with zero products) and sum to 1:

$$(1 - \mathbf{Ii} - \mathbf{Jj} - \mathbf{Kk}) / 4$$
$$(1 - \mathbf{Ii} + \mathbf{Jj} + \mathbf{Kk}) / 4$$
$$(1 + \mathbf{Ii} - \mathbf{Jj} + \mathbf{Kk}) / 4$$
$$(1 + \mathbf{Ii} + \mathbf{Jj} - \mathbf{Kk}) / 4$$

Here, we see that the status of the two spaces at this level is *exactly identical*, but that, as soon as they are applied in the nilpotent structure, the perfect symmetry is broken. The zero product of the spinors

$$(1 - \mathbf{ii} - \mathbf{jj} - \mathbf{kk})(1 - \mathbf{ii} + \mathbf{jj} + \mathbf{kk})(1 + \mathbf{ii} - \mathbf{jj} + \mathbf{kk})(1 + \mathbf{ii} + \mathbf{jj} - \mathbf{kk}) = 0$$

interestingly recalls a structure from a quartic Finsler geometry, the Berwald-Moor metric

$$(x_1 - x_2 - x_3 - x_4)(x_1 - x_2 + x_3 + x_4)(x_1 + x_2 - x_3 + x_4)(x_1 + x_2 + x_3 - x_4)$$

where x_1, x_2, x_3, x_4 can be regarded as base units of the dual vector space formed by 1, **ii**, **jj**, **kk** (with volume unit –1).

Physical singularities (fermions or their products) appear to require a perfect dual vector space which nevertheless produces an asymmetry or chirality in the space of observation because it combines with the unobserved dual vacuum space in an asymmetric nilpotent structure. The nilpotent structure itself incorporates many forms of duality: operator and wavefunction; fermion and vacuum; fermion and vacuum boson; operator and amplitude; nilpotent and idempotent; broken and unbroken symmetries. Essentially, these all originate in the idea of the fermion state as defining a localized singularity, at the same time as we define what is nonlocal or excluded from the singularity.

The fermion has a half-integral spin because it requires simultaneously splitting the universe into two halves which are mirror images of each other at a fundamental level, but which appear asymmetric at the observational level because observation privileges the fermion singularity. *Zitterbewegung* is an obvious manifestation of the duality, but, in observational terms, it privileges the creation of positive rest mass.

Though the duality results in fermion and vacuum occupying separate 3-dimensional 'spaces', which are combined in the double Clifford algebra defining the singularity state, these 'spaces', though seemingly different in observational terms, are truly dual, each containing the same information, and the duality manifests itself directly in many physical forms. For example, we have alternative methods for defining the following phenomena using either real spaces axes (**i**, **j**, **k**) or vacuum space axes (**I**, **J**, **K**):

	i, j, k	**I, J, K**
Pauli exclusion	antisymmetric ψ	nilpotency
spin ½	anticommuting **p**	Thomas precession
SR velocity addition	using 2 D of space	using space-time
holographic principle	area = space × space	area = space × time
fermion state	direction of **p**	E, **p**, m

The relativistic connection between space and time notably exists in a different vector space to the connection between the three spatial components. It is not strictly 4-D at all, though it appears as such when we take the scalar product of a massless object. Assuming an intrinsic 4-D connection gives us a problem with quantum mechanics, where time is not an observable, and also for Penrose's twistors, which have to assume a massless world, with the intrinsic motion of the particles at the speed of light. Though 3-D vector space incorporates a duality of its own in requiring vectors and pseudovectors, quantum mechanics really requires an additional duality. It uses a dual dual space, which does not require an arbitrary extension to 4 D. Mass is a natural consequence of this extra duality even if we assume that the intrinsic motion of the particles is at the speed of light.

Higher dimensionalities naturally result from this double Clifford algebra. Thus, the nilpotent operator ($\pm\, \boldsymbol{ikE} \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$) can be regarded as a 10-D object in vacuum space: 5 for iE, $\mathbf{p}$, m and 5 for the unit axes $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$ (or **K**, **I**, **J**). Six of the ten (all but iE and $\mathbf{p}$) are compactified. The ten also reduce to 8 or 2 × 4 in a nilpotent structure when the intrinsic redundancy of m and the scalar 1 are considered. The nilpotent structure creates a self-duality in phase space which determines vacuum selection in exactly the way required for a perfect string theory, and, as we have seen, it automatically generates a gravity-gauge theory correspondence. The significant feature of this process, and in fact of our entire discussion, is that only 3-dimensional objects are needed to generate the entire structure, and that, in fundamental terms, this reduces to a single 3-dimensional object and its dual partner.

Appendix on predictions

The long-term prediction of no SUSY particles, because each Standard Model particle is its own partner algebraically, looks like being upheld. The even longer-term prediction of no proton decay also looks like being upheld. Before its discovery in 2012, the mass of the Higgs boson was estimated at half the vacuum energy. At half the expectation value of the Higgs field, this would be 123 GeV. In fact, the measured value, at 126 GeV, seems to be half the value of its realisation in terms of the W and Z particles (whose masses total 252 GeV). It may be that there will be no particle individual particle with invariant mass higher than 246 GeV because this is the fixed energy of each point in space. The long-term prediction of dark energy at 2/3 that of the universe seems to be supported by the recent measurements of the Planck Collaboration (March 2013), which put it at 68 % with 68 % confidence limits. A prediction that the general relativistic field equations would show no nonlinearity, even in strong gravitational fields, is supported by recent results on a short-period binary pulsar (Antoniadis *et al.*, 2013).

REFERENCES

Antoniadis, J. *et al.*, A Massive Pulsar in a Compact Relativistic Binary, *Science*, 340, no. 6131, 499, 26 April 2013.

Diaz, B. and Rowlands, P., 2005, A computational path to the nilpotent Dirac equation, *International Journal of Computing Anticipatory Systems*, **16**, 203-18.

Planck Collaboration I 2013, arxiv:1303.5062

Planck Collaboration XVI 2013, arxiv: 1303.5076

Rowlands, P., 2007, *Zero to Infinity*, World Scientific, Singapore.

Rowlands, P. and Diaz, B., 2002, A universal alphabet and rewrite system, arXiv:cs.OH/0209026.

TauQuernions $\mathcal{T}_i,\ \mathcal{T}_j,\ \mathcal{T}_k$:

3+1 Dissipative Space out of

Quantum Mechanics

by

Michael Manthey
manthey@acm.org

Douglas Matzke
doug@quantumdoug.com

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Abstract. We report the discovery of two dual and emergent sets of three *irreversible* bi-vectors - dubbed the *TauQuernions* $\mathcal{T}_i,\ \mathcal{T}_j,\ \mathcal{T}_k$ - that are otherwise isomorphic to quaternions. This inherently dissipative 3-dimensional tauquernion space is a sub-space of the geometric (Clifford) algebra $\mathcal{G}_{4,0}$ with generators $\{a, b, c, d\}$; a straight-forward mapping produces a 3+1 dimensional sub-space with signature $(+ - --)$ in $\mathcal{G}_{5,0}$. The individual tauquernions are entanglement operators corresponding to the quantum mechanical Bell and Magic operators. The form $\mathcal{T}_i + \mathcal{T}_j + \mathcal{T}_k$ has 64 sign variants of which 16 are nilpotent, which latter we identify as Higgs boson phases; the other 48 variants square to the unitary 4-vector $\pm abcd$, which we identify as the carrier of mass. A natural candidate for dark matter also emerges, which we analyze. We calculate the information content of these and related forms, draw an *exact* map of the entropic pathways an expansion will follow, and sketch how this *Bit Bang* develops. Photons are clearly represented and transparently intertwined in the space, and there is overall compatibility with relativity theory.

Keywords: Tauquernions, quaternions, geometric algebra, spacetime algebra, Higgs boson, EPR, entanglement, Bell & Magic operators, space creation, quantum gravity, mass creation, dark matter, black holes, background-independent, string theory, Fourier, Parseval, Clifford, emergence, co-boundary, computational, concurrent, distributed, co-occurrence, co-exclusion, process, mechanism, hierarchical, quantum computing, *q*bit, *e*bit.

1. Introduction

The authors are computer scientists using geometric (Clifford) algebras to describe and investigate the properties of abstract distributed computer systems [11,12,14,15]. In the course of these investigations, we have discovered the quaternion isomorphs, dubbed ***tauquernions***, mentioned in the title. We apply this *new* mathematical description of 3 and $3+1$ dimensions to a contemporary issue: the origin and formation of our $3+1$d universe of 3-space, gravity, mass, time, causality, and entropy, and how all this can emerge from a quantum mechanical soup lacking all of these things. It is important to understand that our results are formally ***theory-neutral***, in that they stem from a finite, discrete and combinatorial analysis of the entire phase space.

Our foremost goal here is to describe these novel structures in a straightforward way, so our style is discursive rather than formal. We offer physical interpretations of some of the algebraic forms that appear in order to facilitate the transfer of this structure to the community of physicists. That is, we do computers, not physics.

1.1 Computational and Physical Processes

One might ask how a mathematics of concurrent computation can come to apply to questions of fundamental physics. There are two pieces to answering this, the first being a mathematics that can connect the two disciplines, and the second, given this mathematics, the details of the connecting isomorphism.

We begin with the common view of a computer program - when it is executing - as a sequence of discrete operations

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where each parenthesis-pair stands for a single such operation. Such a sequence is called a *process*, and the following is a gloss on [11], to which the reader is referred for a more detailed exposition. The process-level of computational description refers not so much to entities themselves as to their interaction, and the sequence of states this produces. In our model, *everything* is a process, or an object built out of processes.

The key property of a process is the exact *order* in which its component operations take place. To capture this ordering property algebraically we will require that each operation "( )" in the above sequence - now viewed as a product - be irreversible (ie. no multiplicative inverse). This prevents algebraic manipulations from changing the effective order.

before $\hat{U}$, ie. the actual process must specify that $\hat{V}$ must always wait (ω) for $\hat{U}$. That is, we want $\hat{V}\hat{U} = \hat{V}\omega\hat{U}$. Rewriting the lhs as $\hat{V}\hat{U} = \hat{V}\hat{V}\hat{U}$ and expanding,

$$(-1+V)(-1+U) = (-1+V)(-1+V)(-U)(-1+U)$$

we find that $\hat{V}\hat{U} = \hat{V}(U - VU)\hat{U}$, and indeed $\omega = U + UV$ is nilpotent so long as U and V anti-commute.[1] Computationally speaking, anti-commutativity means "independent of each other", as in the practice of orthogonal software design, which focuses on ensuring that changes to one module do not affect another; or as in "asynchronously concurrent"; or both, as here.

Processes like $\hat{V}\hat{U}$ are exactly the processes covered by Turing's model of computation, and since entities like $\hat{U}$,$\hat{V}$ are the ***projectors*** of U, V respectively (so-called ***measurement operators***), they are also the observational bedrock of quantum mechanics. The key property of such processes - irreversible sequentiality - makes them purely ***time-like*** processes. It is ultimately this time-like property that allows Penrose to conclude [18] that computational processes cannot capture all the phenomena that quantum mechanics has to offer, among which is entanglement, which is fundamentally space-like.

This prompts the question, "Where then is ***space-like*** computation?". Which prompts the question, "***What*** is space-like computation, what might it ***do***?!". An answer to the latter would be, Expand the semantic reach of the computational metaphor to ***directly*** capture and express fundamental spatial distinctions like ***left/right*** and ***inside/outside.*** Given computation as currently practiced, we are forced to simulate (ie. ***fake***) such matters, eg. via syntactically sugared high-level languages resting on intricate, and usually sequential, run-time environments.

[The issue is analogous to the background-in/dependence of a physical theory, where string theory ***assumes*** a $3+1$d background, whereas quantum gravity theories, requiring that 3+1d be ***constructed***, are background ***in***dependent. In these terms, we are presenting, here, a background-independent, non-supersymmetric, quantum gravity theory.]

Space-like computation, whatever it is, must (for our purposes) provide the equivalent of the quantum potential, with its wave-like properties. Now it is characteristic of a primitive wave that ***two*** things change at the same time. In the scalar world, these two things could be the x and y coordinates as one traverses the circumference of the unit circle. Computationally, this dynamic corresponds to viewing x and y as independent, but nevertheless jointly interacting, ***concurrent*** processes that together achieve the required symmetry. [2] That

[1] Note that we could instead write $\hat{V}\hat{U} = \hat{V}\hat{U}\hat{U}$, which leads to $\omega = -V - UV$. This corresponds to the so-called advanced solution, and $\hat{V}\hat{V}\hat{U}$ to the retarded solution.

[2] For example, alternating expression (or possession) of a conserved resource.

The algebra we will use is geometric (Clifford) algebra $\mathcal{G}$, a graded vector algebra with both inner and outer products, over the finite field $Z_3 = \{0, 1, -1\}$; Grassmann algebras are a subset of geometric algebra, and the Pauli and Dirac algebras are particular geometric algebras. In fact, geometric algebra is now often advanced as the much-needed common mathematical language for all of physics [3,6,7,10,19]. We have a similar motivation *vis a vis* computer science. The next section introduces geometric algebra; here we anticipate it on a gross level.

A ***Theorem*** of geometric algebra: For any expression $\mathcal{P} \in \mathcal{G}$, $\mathcal{P}$ is irreversible *iff* $\mathcal{P}$ has an idempotent factor $\hat{X} = \hat{X}^2$.

So make each operation "( )" an idempotent. An idempotent $\hat{X}$ that is also a projector has (in $\mathbb{Z}_3$) the form $\hat{X} = -1 + X$, where X is unitary: $X^2 = 1$. Putting all this together, a sequential process - aka. a measurement sequence - looks like

$$(-1 + X_n)(-1 + X_{n-1})...(-1 + X_1) = \prod_n \hat{X}_i, \qquad X_i^2 = 1$$

This is probably all more or less familiar to physicists. But the ***computational*** reading of the algebra takes the correspondence much further. In this reading [11], the idempotent form $-1 + X$ is identified as the primitive synchronization operation signal(X), understood to mean "signal the occurrence of the event/state X".

Example. Multiple signallings of the same event's occurrence are semantically equivalent to a single such signal, just as the measurement specified by $\hat{X}$ yields no further information upon being repeated: $\hat{X}^n = \hat{X}$.

Signal's complementary primitive is wait(X), ie. wait for the occurrence (signal) of event X. It is critical to understand that this waiting is not *polling*, ie. that the waiting process is constantly and actively checking to see if X has occurred yet, aka. *busy waiting.* Busy-waiting turns out to be a quite untenable view in an asynchronously concurrent universe - something subtler is necessary. A careful analysis [11] reveals that the computational concept of wait(X) must be mapped, speaking now algebraically, to some nilpotent $\omega \in \mathcal{G}$, $\omega^2 = 0$.

In physics, nilpotents supply the causal - and energy conserving - connection between discrete physical events. Wait's play the corresponding role in the synchronizational context - causal connection and conserving information between computational events. Nilpotents are irreversible, so the implication of the above theorem is that we must derive our ω's from our idempotents.

We can derive ω's form by considering two consecutive events $\hat{U};\hat{V}$, forming the process $\hat{V}\hat{U}$. We will insist, now speaking computationally, that $\hat{V}$ ***never*** occur is, *rotation* is an example of a space-like *reversible* computation, and is *also* a process.

We accomplish the translation from asynchronous concurrent computational processes to algebraic expressions in the following way. First, we interpret our algebra's "+" sign to mean that (eg.) $U+V$ are two asynchronously occurring and executing, independent, computational entities, ie. processes or objects constructed from same. ***Multiplication*** is action, transformation, process; both measurement and rotation are examples.

Next, we interpret 1-vectors $a, b, c, \ldots$ as (reversible) processes possessing ***a single bit of state.*** These 1-bit processes are deterministic since the one state predicts the next, which alternation encodes frequency ν. Since the grade of the vector equals the number of bits of process state that it encodes, the m-vector ab has $2^{m=2}=4$ internal states, these being

$$\{a+b, -a+b, a-b, -a-b\}$$

Furthermore, these can be paired as $a+b=-(-a-b)$ and $a-b=-(-a+b)$, and these in turn mapped to the orientation of ab (via the standard and diagonal bases) as:

$$\{a-b, -a+b\} \mapsto +ab \quad \text{and} \quad \{a+b, -a-b\} \mapsto -ab$$

This mapping of states, computationally speaking, allows the whole, ab, to maintain a fixed external appearance – its orientation of either $+1$ or -1 – while at the same time its component processes a and b are themselves undergoing their own (1-bit) state changes. If processes a and b have a stable joint behavior, namely oscillations in one of the above two state-pairs, then ab accurately reflects this in a stable orientation.[3] Furthermore, the computations a-b $\leftrightarrow$ -a+b and a+b $\leftrightarrow$ -a-b, like their algebraic co-respondents, are reversible, both being simple inversions. That is, they are both *wave-like*, ... and at that, *exactly* so (p. 11).

Returning to the introductory paragraph, the mathematical language common to the disciplines of physics and computer science is found, as sketched in the preceding, to be geometric algebra. The connecting isomorphisms are

- *1-vectors* $a, b, c, \ldots$ with magnitudes ± 1 represent primitive, reversible *processes* with 1 bit of state; and map frequency ν.
- *m-vectors*, $m > 1$, represent internally-concurrent ***process-objects*** encoding 2^m bits of state, externally exhibiting (spin) orientation ± 1.

[3]The ambiguity regarding the "actual" state of ab leads to the uncertainty principle.

- Signal(U) is defined to be the idempotent $\hat{U} = -1 + U$, a measurement operator on a unitary U; and Wait(U) is then the nilpotent $U + UV$, with the interpretation that (a later) event $\hat{V}$ is causally connected to (an earlier) event $\hat{U}$.

- Time-like/causal/irreversible *processes* are then Wait/Signal sequences (WS)*.

- The wave-like quantum potential $\Psi \subseteq \mathcal{G}$ *equals* the computational $\mathcal{G}$, denoted G, constituting an untrammelled, non-deterministic, concurrent computation.

Using this algebra, Matzke [10] found that the quantum entanglement Bell and Magic operators have the form $wx \pm yz$. We show in §7.2 that this form cannot be simulated by a time-like process. It is therefore especially important in the following that the reader understand that when *we* write a sum in the algebra, say $U + V$, *we* are seeing two concurrently executing wavelike *objects* U and V, not two dead multivectors belonging to some algebra. [Readers liking conceptual origins might want to read §7.2 first.]

Thus, when we catalog the unitary entities in the geometric algebra $\mathcal{G}_3$ and find exactly three families thereof, whose properties encourage their interpretation as neutrino, electron, and proton/neutron; and we also find three quarkish families $x + yz$, with inherent confinement; along with photons $x + y + z$; and mesons = quark plus anti-quark; etc. etc., all of which matches the standard model to a T (cf. Appendix I); on top of which, it being a fact that $\mathcal{G}_3$ is isomorphic to the Pauli algebra, we find it entirely reasonable to conclude that it *is* real physics that is being described. Perhaps then it will not be so surprising that we find that signals associate to *fermions*, and waits to *bosons*.

The Standard Model having exhausted $\mathcal{G}_3$'s semantic carrying capacity, we graduate seamlessly to $\mathcal{G}_4$ and thence to *construct* 3+1 space as $\mathcal{G}_{1,3}$, along with gravity and mass. Here, among many other corresponding physical phenomena, we find corroboration for earlier proposals [23] linking gravity and quantum entanglement.

Finally, citing [2] (and see also [22]):

> "A minimal quantity of heat, proportional to the thermal energy and called the Landauer bound, is necessarily produced when a classical bit of information is deleted. A direct consequence of this logically irreversible transformation is that the entropy of the environment increases by a finite amount. ... we experimentally show the existence of the Landauer bound ..."

Rolf Landauer (1962): "Information is physical." That is, the now empirically demonstrated physicality of information is what ultimately constitutes the connection between physics and computation.

Add to this the conclusion of Masanes et al. [24] that not only can the standard formalism of quantum mechanics be formally derived from four information-oriented axioms, but as well that their solution is unique. "Bits" are *real* and *cannot* be subdivided. *Information* replaces and generalizes *energy* in their (and our) view. At the other end of the conceptual spectrum, Moreva et al [25] conclude from an entanglement-based experiment (as do we from analysis) that *time* is an emergent phenomenon "deriving from correlations", to which we append that time emerges *solely* from entanglement (§3).

The next section (§1.2) introduces geometric algebra. We then define the quaternion isomorphs in the title of this paper (§2), show how they fit into the Dirac algebra (§3) and why their sum should be identified with the Higgs boson (§4), their relationship to the Bell and Magic quantum entanglement operators (§5), the extension to dark matter (§6), an information-theoretic analysis of these results (§7), and an entropy-driven Bit Bang (§8) that generates all of the foregoing structures.

1.2 Geometric Algebra

For readers unfamiliar with geometric algebra: given a set of anti-commuting 1-dimensional unit vectors $\{a, b, c, ...\}$, these vectors generate the combinatorial space $\{\pm 1, \{a, b, c, ...\}, \{ab, ac, ad, ...\}, \{abc, abd, abe, ...\}, ...\}$ all of which m-vector elements are mutually orthogonal.[4] Thus n generators generate a space of 2^n dimensions. The generators are, simultaneously, the primitive reversible 2-state sequential processes at the bottom of the computational construction. Uniqueness is established by the vector name, ie. we use single character lower case alphabetic characters vs. the matrix column bra-ket notation used with Hilbert spaces. Upper-case letters denote arbitrary multi-vectors, eg. $|A+B| \leq |A|+|B|$; and the inner product obeys $x{\cdot}Y = xY$, eg. $b{\cdot}ab = -b{\cdot}ba = \tilde{a}$ (see [3,6,7,10] for operator-precedence rules).

We use the canonical geometric algebras $\mathcal{G}_{n,0} = \mathcal{G}_n$, but over $\mathbb{Z}_3 = \{0, 1, -1\}$, so $a^2 = +1$, $a + a = -a = \tilde{a}$, etc., which, in replacing "0,1" with "-1,1", maintains a binary feel, but with vastly expanded semantics compared to Boolean logic and automata theory. We interpret $+a$ to mean that whatever a indicates is currently present, and $\tilde{a}$ that it is not; $0a$ denies a's very existence. [5] Few (if any) of our results apply only in $\mathbb{Z}_3$: certain things – *structural* things – are just easier to see without the additional complexities of multiplicities of identicals.

Geometric algebra's product $ab = a \cdot b + a \wedge b = -ba$ is anti-commutative, but otherwise follows the usual associative and distributive laws. Arbitrary multi-vectors A, B usually neither commute nor anti-commute.

[4] See also http://www.euclideanspace.com/maths/algebra/clifford/index.htm .

[5] We stress that zero is *not* a value, and we would never write "$a = 0$". Rather, zero appears as a situational indicator, eg. "$a + \tilde{a} = 0$", meaning "the occurrence of a excludes the occurrence of $\tilde{a}$". Or $CBA = 0$, meaning that the computation $A; B; C$ has terminated.

All of our calculations have been done with a custom $\mathbb{Z}_3$ geometric algebra symbolic calculator, a Python upgrade of the calculator described in [14]. One should not expect to get the same results from a generic Clifford algebra tool without thoughtful tampering. We use Planck units: $\mathsf{G} = \mathsf{c} = \hbar = \mathsf{k} = 1$.

Notation. Due to the extreme symmetry of $\mathcal{G}_n$ over $\mathbb{Z}_3$, one may safely assume that a given expression is valid for all sign variants unless otherwise noted. Nevertheless, we sometimes write generic expressions using $x, y, ... \in \{a, b, c, ...\}$, with $x, y, ...$ taken without duplication, and all sign variants implied unless otherwise noted. For example, the expression $x - xy$ could denote $a - ab, -a - ab, b - ab, c - cd, \ldots$ but not eg. a or ab alone, nor $a + bc$, nor $a + ab$ (because of the explicit minus sign). To minimize clutter, we use forms with minimal minus signs, and in particular often $1 + x$ rather than $-1 - x$ (the latter being indempotent and the square of the former), even though sometimes it's not quite 'correct'; readers who find this bothersome can just multiply by -1. We sometimes distinguish between the elements of the abstract geometric algebra $\mathcal{G}$ and the subset G that is currently instantiated.

We stress that the various algebraic expressions that we will present and discuss are ***discrete computational*** <u>***structures***</u>, eg. 'plus' means 'concurrent'. That is, we view $a, b, ab, ...$ as ***local, deterministic processes*** whose externally visible states oscillate between ± 1. So the state changes expressed by the algebra represent ***concrete discrete computations*** producing concrete ie. determinate, non-statistical discrete results. But since the "computation" consists of all possible non-exclusionary processes running flat out concurrently, the familiar non-determistic statistical picture of quantum mechanics nevertheless emerges. This computational view replicates Feynman's sum-over-paths interpretation by realizing, concretely, the Bayesian encoding underlying Dirac's $< V|U >$ bra-ket notation (meaning "the probability of V's occurrence given U's").

In summary, the Heraclitean "everything is process" interpretation that we are placing on the algebra is *quite* different from that of standard treatments of geometric algebra [3,6,7,10,19]. The generators $\{a, b, c, ...\}$ are, ultimately, primitive ***distinctions***, encoding only '$\pm$' = 'opposite' in 1d. This expands in >1 dimensions ($ab, abc, \ldots$) to an m-ary *xnor*, ie. a negated *xor*. It would be a complete misunderstanding to understand our $\mathcal{G}$ expressions as *m.l.t.* formulae.

2. The TauQuernions

The quaternions encode 3d space via the multiplication (= rotation) table:

$\times$	$Q_i = ab$	$Q_j = ac$	$Q_k = bc$
Q_i	-1	$-bc$	ac
Q_j	bc	-1	$-ab$
Q_k	$-ac$	ab	-1

=

$\times$	Q_i	Q_j	Q_k
Q_i	-1	$-Q_k$	Q_j
Q_j	Q_k	-1	$-Q_i$
Q_k	$-Q_j$	Q_i	-1

The corresponding tauquernions are $\mathcal{T}_i = ab - cd$, $\mathcal{T}_j = ac + bd$, $\mathcal{T}_k = ad - bc$.[6] Their multiplication table is below left; on the right is the same table, but with the mapping $1 + abcd \mapsto$ "-1". We emphasize that the tauquernion relationships below are independent of the restriction to $\mathbb{Z}_3$.

$\times$	$\mathcal{T}_i = ab - cd$	$\mathcal{T}_j = ac + bd$	$\mathcal{T}_k = ad - bc$
$\mathcal{T}_i$	$1 + abcd$	$-ad + bc$	$ac + bd$
$\mathcal{T}_j$	$ad - bc$	$1 + abcd$	$-ab + cd$
$\mathcal{T}_k$	$-ac - bd$	$ab - cd$	$1 + abcd$

=

$\times$	$\mathcal{T}_i$	$\mathcal{T}_j$	$\mathcal{T}_k$
$\mathcal{T}_i$	"-1"	$-\mathcal{T}_k$	$\mathcal{T}_j$
$\mathcal{T}_j$	$\mathcal{T}_k$	"-1"	$-\mathcal{T}_i$
$\mathcal{T}_k$	$-\mathcal{T}_j$	$\mathcal{T}_i$	"-1"

Like the Q's, the $\mathcal{T}$'s anti-commute, eg. $\mathcal{T}_i\mathcal{T}_j = -\mathcal{T}_j\mathcal{T}_i$; close circularly, eg. $\mathcal{T}_i\mathcal{T}_k = \mathcal{T}_j$; and $-\mathcal{T}_i\mathcal{T}_j\mathcal{T}_k = \mathcal{T}_k\mathcal{T}_j\mathcal{T}_i$.

One can easily see that the two tables to the right, quaternion and tauquernion, are isomorphic. The tauquernions, elements of $\mathcal{G}_4$, recapitulate in four spatial dimensions what the quaternions, elements of $\mathcal{G}_3$, do in three (but with a twist).

We always operate on the left, so $\mathcal{T}_k\mathcal{T}_j\mathcal{T}_i$ read right-to-left is the sequence $\mathcal{T}_i$; $\mathcal{T}_j$; $\mathcal{T}_k$. This "full circle" rotation defines "$+1$" $=$ ("-1")$^2 = (1 + abcd)^2 = -1 - abcd$, which is idempotent.

In fact, since $wx + yz$ has the idempotent factor $-1 \pm wxyz$, then via the aforementioned theorem (§1.1), ***all tauquernions*** $wx+yz$ ***are irreversible***. The details are revealing:

$$
\begin{aligned}
(ab - cd)^4 &= (ab - cd)[(ab - cd)(ab - cd)](ab - cd) \\
&= (ab - cd)[1 + abcd](ab - cd) && [\text{"-1"}] \\
&= (ab - cd)(-ab + cd) && \textit{invert} \\
&= -(ab - cd)(ab - cd) \\
&= -(1 + abcd) && \textit{ie. - (-1)} \\
&= -1 - abcd && \textit{idempotent} \\
&= \text{"}+1\text{"}
\end{aligned}
$$

[6] There is also a conjugate set: $\{\mathcal{T}_i', \mathcal{T}_j', \mathcal{T}_k'\} = \{ab + cd, ac - bd, ad + bc\}$. There are 8 such triples: $\pm\mathcal{T}_i\pm\mathcal{T}_j\pm\mathcal{T}_k$, and similarly for $\mathcal{T}'$. Choosing a particular triple, as here, constitutes an arbitrary choice of coordinate system orientation, cf. the "right hand rule" in 3d.

which identities also justify our identification of $1+abcd =$ "-1" in the table. The interplay between ± 1 and "± 1" is the interplay of reversible change (space-like, Ψ) and irreversible change (time-like, 2^{nd} Law), and constitutes the scalar nub of what tauquernions do: connect a space-like inversion directly to an *exactly* corresponding time-like inversion: $(-1)(ab-cd) = (1+abcd)(ab-cd)$. [7]

Thus, at least in principle, simply by replacing every (namely reversible) quaternion element xy in one's work with (the irreversible) $xy+wz$, one in effect replaces an explicit time coordinate with an implicit one, perhaps allowing for great simplification.

The conjugate tauquernion table below differs only in negating the Q's:

$\times$	$\mathcal{T}_i' = ad+bc$	$\mathcal{T}_j' = ac-bd$	$\mathcal{T}_k' = ab+cd$
$\mathcal{T}_i'$	$1-abcd$	$ad+bc$	$-ac+bd$
$\mathcal{T}_j'$	$-ad-bc$	$1-abcd$	$ab+cd$
$\mathcal{T}_k'$	$ac-bd$	$-ab-cd$	$1-abcd$

$=$

$\times$	$\mathcal{T}_i'$	$\mathcal{T}_j'$	$\mathcal{T}_k'$
$\mathcal{T}_i'$	"-1"	$\mathcal{T}_k'$	$-\mathcal{T}_j'$
$\mathcal{T}_j'$	$-\mathcal{T}_k'$	"-1"	$\mathcal{T}_i'$
$\mathcal{T}_k'$	$\mathcal{T}_j'$	$-\mathcal{T}_i'$	"-1"

It follows that $-\mathcal{T}_i'\mathcal{T}_j'\mathcal{T}_k' = -1+abcd =$ "$+1$" just as we earlier saw that $-\mathcal{T}_i\mathcal{T}_j\mathcal{T}_k = -1-abcd =$ "$+1$".

The table below reifies these mappings for ± 1 [a *sqert* is the square root of an idempotent]:

I	$I^2 =$ "$+1$"	I's *type*	$\mapsto$
$1-abcd$	$-1+abcd$	*sqert'*	"-1"
$1+abcd$	$-1-abcd$	*sqert*	"-1"
$-1+abcd$	$-1+abcd$	*idem'*	"$+1$"
$-1-abcd$	$-1-abcd$	*idem*	"$+1$"

Taking $1-abcd$ as Minus One (first row above) as an example, then as expected the usual multiplication/sign rules hold:

$(-)\times(-)$	$= -1+abcd$	$+$
$(-)\times(+)$	$= +1-abcd$	$-$
$(+)\times(-)$	$= +1-abcd$	$-$
$(+)\times(+)$	$= -1+abcd$	$+$

[7] This is also a nice example of the familial affinities of the algebras $\mathcal{G}_{mod4}$. Note that $(1\pm abcd)E = -E$ only if E is an eigen-form of $abcd$ (see later).

Actually, this goes further, since it turns out that $1 \pm abcd$ are examples of a "sparse -1" [14]. Make a "truth table" for the expression $abcd$ and represent the result as a vector, yielding $+abcd = [+ - - + - + + - - + + - + - - +]$ and $-abcd = [- + + - + - - + + - - + - + + -]$. The elements of these vectors form a (generally non-orthogonal) basis for their space.

Let zero $\cdot$ represent a state that does not occur; $\tilde{\mathbf{1}} = [- - - - - - - - - - - - - - - -]$, and $\mathbf{1} = [+ + + + + + + + + + + + + + + +]$. Then $+1 + abcd$ is

	[	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	]
+	[	+	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+	]
=	[	-	•	•	-	•	-	-	•	•	-	-	•	-	•	•	-	]

That is, $1 + abcd = [- \cdot \cdot - \cdot - - \cdot \cdot - - \cdot - \cdot \cdot -] = [- - - - - - - -]$ is a *sparse* -1, and $1 - abcd$ is another. These forms, sparse and otherwise, play a key role in the information-theoretic analysis of §7.

Summing up, we conclude that the two dual sets of tauquernions are each exactly isomorphic to the quaternions, the essence of 3d space, except that tauquernion space is inherently dissipative. This obtains because $\mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k$ are individually irreversible, as is their sum. Particulate motion in this space is thus thermodynamically governed ie. entropic, and this property encourages us to identify such motion with gravitational free-fall. It follows logically that the two conjugate $\mathcal{T}$-forms describe the two polarization states of gravitational waves, not least because of the following extraordinary unifying connection (apparently overlooked, since it appears in none of the obvious references [3,6,7,10,19]).

Theorem (Parseval). The projection of a function F onto an orthogonal inner-product space is the Fourier decomposition of F. ■ [17]

Since the elements $a, ab, abc, \ldots$ are all mutually orthogonal, whence $\mathcal{G}_n$ has $\mathcal{O}(2^n)$ dimensions, *every* expression in $\mathcal{G}_n$ is therefore implicitly a ***wave operator*** as well as the structural description of a computational entity: we can ***automatically*** impute wave-like properties to both entities (eg. the $\mathcal{T}$'s and their sums) and interactions (products) in the tauquernion space. This gives the whole endeavor a thorough-going hierarchical and holographic/distributed feel, and completely redeems de Broglie's initial insight (1923) of the fundamental wave-like nature of reality.

At the same time, tauquernion space ***is*** 3d spatially, and so inherently supports the propagation of 3-dimensional waves even though it takes four orthogonal distinctions $\{a, b, c, d\}$ in phase space to construct these 3d waves. We now turn to the algebra of this 3d space.

3. Spacetime Algebra

We seek to define the ***spacetime algebra*** $\mathcal{G}_{1,3} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ with signature $(+---)$.

Perhaps naively, we initially considered mapping *abcd* to the time-like dimension via the vector γ_0: the properties of geometric algebras cycle $mod\,4$, so there is a family resemblance between, say, $\mathcal{G}_0$ and $\mathcal{G}_4$, which is our case in point, since $\mathcal{G}_0$ is the scalar dimension, and similarly, *mass* is a scalar quantity. It is only required that γ_0 square to $+1$, as indeed *abcd* does. This ensures that *abcd* qua mass, and its automatically dissipative motion, isn't pushed into the background. Right or wrong, this approach was abandoned when we discovered the density of the physics and mathematics involved, and so instead we here simply establish the standard formulation as well as we can.

The $\mathcal{G}_4$ tauquernion space, arising out of the quantum spinorial soup, is discrete, and so we can associate a new fifth, unchained dimension t on which to tally a sequence of discrete motions in the 3d tauquernion space. Changes in the state of *abcd* map to the 1-vector t.

The resulting $\mathcal{G}_5$ generators are then $\{a, b, c, d, t\}$. Consider now only the subspace defined by the three tauquernions $\mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k$ and t. The latter squares to $+1$ while the other three square to "-1", so we have a Lorentzian $(+---)$ space, and it is our understanding that the requirements of special relativity are therefore satisfied; the next section pursues the putative connection to general relativity.

Define now within $\mathcal{G}_5$ the mappings

$$t \mapsto \gamma_0 \qquad \mathcal{T}_i \mapsto \gamma_1 \qquad \mathcal{T}_j \mapsto \gamma_2 \qquad \mathcal{T}_k \mapsto \gamma_3$$

where the γ_i are anti-commuting 1-vectors. The γ_i then generate an explicit basis for the spacetime algebra $\mathcal{G}_{1,3}$:

$$1,\ \{\gamma_i\},\ \{\gamma_i \wedge \gamma_j\},\ \{\gamma_i \wedge \gamma_j \wedge \gamma_k\},\ I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$$

"The structure of this algebra tells us practically all one needs to know about (flat) space time and the Lorentz transormation group" [3, p.131]. We refer the interested reader to [19 §24.4-7, 5, 8] for extended discussions of applying geometric algebra to the standard formalisms of QM and GR.

This said, the derivation of the Dirac equation in [19] points out that the key is that the quaternions can be construed as the square of the D'Alembertian wave operator $\Box$. It follows that if/when the tauquernions, being irreversible, are substituted for the quaternions, it might well be possible to eliminate the explicit time coordinate entirely and end up with the algebra $\mathcal{G}_{0,3}$ (over the tauquernions) as a description of spacetime. [A paper currently in draft extends this to include electro-magnetism (including Majorana fermions) in $\mathcal{G}_{0,6}$.]

Howsoever, every set of $\mathcal{T}$'s also satisfies the basic condition for them to *connect* with each other: that the grade of the parts $ab + cd \mapsto 2 + 2 \leq 4$ not exceed the grade of their union, here $abcd \mapsto 4$. In so doing, and taking advantage of $\mathcal{G}$'s being a coordinate-free algebra, the next section shows that an associated coordinate-free, dissipative, discrete Higgs field then automatically appears as the 3+1d $\mathcal{T}$-coordinate system *itself.*

The transition from this discrete field to a continuous field over $\mathbb{R}$ lies beyond our remit, but we note that the entire algebra lies under the umbrella of Parseval's Identity, and by implication, of harmonic analysis, which latter applies very generally. Indeed, this identity is wave-particle duality in a nutshell.

On the other hand, some writers [1] suggest abandoning $\mathbb{R}$ altogether:

"A key assumption of [the contemporary Theory-of-Everything scene] is that it regards the laws of physics as being the bottom line, and assumes that these laws govern a world of point particles or strings (or other exotica) that is a continuum. Another possibility is that the Universe is not at root a great symmetry but a computation. The ultimate laws of Nature may be akin to software running upon the hardware provided by elementary particles and energy. The laws of physics might then be derived from some more basic principles governing computation and logic. This view might have radical consequences for our appreciation of the subtlety of Nature, for it seems to require that the world is at root discontinuous, like a computation. This makes the Universe a much more complicated place. If we count the number of discontinuous changes that can exist, we find that there are infinitely many more of them than there are continuous changes. By regarding the bedrock structure of the Universe as a continuum we may not just be making an approximation but an infinite simplification."

We note that (1) actually, we show that truly concurrent computation (cf. §7.2) *upholds* the symmetries, cf. the isomorphism between eg. $\mathcal{G}_3$ and the Pauli algebra; and (2) the mentioned "hardware" is crude analogy by our standards - we construct it all (§8).

Howsoever, an entity X existing in phase space $\Psi = G$ comes to occupy $3 + 1$d tauquernion space via the projection $\left(\mathcal{T}_i + \mathcal{T}_j + \mathcal{T}_k\right) \cdot X$, which projection also masks the quantum dimensions automatically, replacing them with $\mathcal{T}_i$, $\mathcal{T}_j$, $\mathcal{T}_k$ and, indirectly, t. Thus we predict that contemporary experiments looking for extra spatial dimensions will all fail.

4. The Higgs Boson

We identify $\mathcal{H} = \mathcal{T}_i + \mathcal{T}_j + \mathcal{T}_k$, whence $\mathcal{H}^2 = 0$, with the Higgs boson. To see why, we look more closely,

$$\mathcal{H} = (ab - cd) + (ac + bd) + (ad - bc)$$

$$= ab + ac - bc \ + \ (a + b - c)d$$

from which we see that $\mathcal{H}$ - our space constructor - is a combination of a quaternion triple $ab + ac - bc$ and a photon $a + b - c$. Both of these are nilpotent, as is their sum. The photon is however conflated with the d-distinction, a change in which is mapped notionally to the aforementioned t dimension to achieve a traditional time-like process.

Thus each of the three $\mathcal{T}$'s is a combination of one quaternion component and one photon component. Clearly, $\mathcal{H}$ contains three dimensions - in both the space-like and time-like senses - in the most compact way imaginable. The nilpotence of $\mathcal{H}$ also expresses the existence of a vacuum energy directly.

Note that $\mathcal{H}$'s *form* has 6 components which together generate $2^6 = 64$ sign variants. Of these, 16 are nilpotent and thus Higgs bosons (ie. phases),

$$\mathcal{H} = \{X = \pm ab \pm ac \pm bc \pm ad \pm bd \pm cd \mid X^2 = 0\}.$$

The other 48 square to $\pm abcd$, which we identified in the preceding section as the unit mass carrier; these 48 form the set

$$\mathcal{M} = \{X = \pm ab \pm ac \pm bc \pm ad \pm bd \pm cd \mid X^2 = \pm abcd\}.$$

We interpret the sign of $abcd$ as its rotational orientation in 3+1 space.

We note that for $X \in \mathcal{H}$, $X\,abcd = abcd\,X = \pm X$, but only $abcd\,X = X\,abcd$ for $X \in \mathcal{M}$.

The elements X of $\mathcal{H} \cup \mathcal{M}$ are eigenforms of $abcd$: $X\,abcd \cong X$, which in turn define boundaries of $abcd$. That is, we define $\partial_X abcd = X\,abcd$ to be the *boundary* of $abcd$ with respect to X - - in formal analogy to partial differentiation, and with a nod to DeRham's theorem. If also $X\,abcd \cong X$, then we further, and oppositely, say that the *co-boundary* of X is *abcd*: $\delta(X) = abcd$. That is, we define the "integral" δ in terms of the "derivative" ∂. [8]

In this way, *abcd* is the integral of any $X \in \mathcal{H} \cup \mathcal{M}$, since $\delta(wx + yz) = wxyz$. That is, δ is a mass-creation operator with respect to $\mathcal{M}$, and a creation operator generally. Said oppositely, both $\mathcal{H}$ and $\mathcal{M}$ are boundaries of *abcd*, but have very different properties.

We now re-write $\mathcal{H}$ as

$$(1 + abcd)(ab + ac - bc) = \mathcal{H} \tag{1}$$

The factor $1 + abcd$ is a ***self-boundary*** of *abcd*. Being irreversible, $1 + abcd$ is a time-like operator. This operator is operating on the quaternionic 3d space $ab + ac - bc$, which produces a bosonic potential $\mathcal{H}$.

Thus equation 1 looks like a local version of Einstein's basic GR equation: the time-like aspect of a mass *abcd*, aka. "gravity", operates on a 3d space $ab+ac-bc$ made out of the very same mass aspects, and produces a wave-like, space-like, but inherently dissipative $3 + 1$d potential, aka. the space-time stress tensor. The general form is $\mathcal{H} = (\pm 1 \pm wxyz)(xy + xz + yz)$.

Let X, Y, Z over a, b, c, d and X', Y', Z' over p, q, r, s be two sets of tauquernions written in the above form. Noting that that form commutes, we can write

$$(X + Y + Z)(X' + Y' + Z') = (1 + abcd)(ab + ac - bc)(pq + pr - qr)(1 + pqrs)$$

Thus the mass-mass interaction $(1 + abcd)(1 + pqrs)$ has $|\{(ab + ac - bc)(pq + pr - qr)\}| = 3^2 = 9$ spacelike connections, ie. three (tauquernion) dimensions and 9 (super-string?) connections in one package. [9]

That is, the dissipative $3d$ tauquernion space can also be seen as the time-like interaction of masses in a reversible 3d quaternion space. If one happens to believe that space is entirely passive, ie. that $(1 + abcd)(1 + pqrs)$ is the whole story, then one arrives at the classical, Newtonian, view of masses in $3d$ space affecting each other mysteriously.

[8] More concisely, $\delta_Q X = \pm Q$ *iff* $\partial_X Q = XQ$ *and* $|XQ| = |X|$. We take ∂ and δ to be elements *of* the algebra - rather than the usual operators *over* the algebra - this being a less sophisticated but more concrete encoding of the same ideas.

[9] However, even though $(ab + ac + bc)(pq + pr + qr)$ is nilpotent, and as well sandwiched between two idempotents, from which it derives, this is ***not*** a ***causal*** connection with $(ab + ac - bc)(pq + pr - qr)$ playing the role of Wait, because $(1 + abcd)(1 + pqrs) \neq (1 + abcd)(ab + ac - bc)(pq + pr - qr)(1 + pqrs)$, cf. [11]. That is, in the classical view represented by the form $(1 + abcd)(1 + pqrs)$, space plays ***no*** causal role. [As well, $(1 + abcd)(1 + pqrs) = (1+pqrs)(1+abcd)$, which commutativity shreds any sequential or localized notion of causality.]

In this context, note that $xy + yz + zx = xyz(x + y + z)$, so electro-magnetism (via photon $x+y+z$) is directly in the picture, namely neatly woven into the $3d$ gravitational space created by the tauquernions. It bears mentioning, though, that the $(\pm 1 \pm wxyz)(xy + xz + yz)$ form obscures the connection to the EPR phenomena that underlie the very existence of *abcd* and the space it both lies in and forms (§5, next).

Recalling eqn. 1

$$(1 + abcd)(ab + ac - bc) = \mathcal{H} = ab + ac - bc + (a + b - c)d \qquad (2)$$

which describes matter acting on space, we can multiply through by *abc* to create the *abc*-conjugate form:

$$(1 - abcd)(a + b - c) = a + b - c - (ab + ac - bc)d \qquad (3)$$

which describes matter interacting with light. Summing the rhs's of eqns. 2 and 3 (= concurrent occurrence) and re-arranging, we get:

$$(2) + (3) = (a + b - c)(1 + d) \; + \; (ab + ac - bc)(1 - d)$$

Note that $1 \pm d$ are measurement operators. Recalling the lhs's of eqns. 2 and 3, *Voila*, light interacts with matter in entropic quaternion space (lhs) with resulting effects (rhs) on the light and the space:

$$\begin{array}{ccc} (1 - abcd)(a + b - c) & & (a + b - c)(1 + d) \\ + & = & + \\ (1 + abcd)(ab + ac - bc) & & (ab + ac - bc)(1 - d) \end{array}$$

All four pieces are nilpotent, as are their sums (ie. each side), which indicates that this interaction is an irreversible, ie. thermodynamic, event.

If instead of adding eqns. 2 and 3, we subtract their expanded forms:

$$\begin{array}{ccc} (1 - abcd)(a + b - c) & & (a + b - c)(1 + d) \\ - & = & - \\ (ab + ac - bc)(1 - d) & & (1 + abcd)(ab + ac - bc) \end{array}$$

then both sides simplify to $(a+b-c)-abc(a+b-c)$, a purely electro-magnetic state.

If this seems obscure, it is perhaps well to recall that every expression in the algebra is a Fourier decomposition, and so what is being 'added' are the oscillations of concurrent processes at various frequencies, phases, and dimensionalities.[10] That is, these descriptions are "particulate" only insofar as one can single out some sub-expression that is unitary that one can then try to measure.

5. *Entanglement*

We now expand on our earlier statement that each $\mathcal{T}_i$ is a quantum mechanical Bell/Magic operator, and that the $\mathcal{T}_j$ and $\mathcal{T}_k$ are the Bell/Magic states. These operators capture quantum entanglement, and are the bread and butter of quantum computing research and practice. For the reader's convenience, Table 1 reviews these as they are usually represented. [In this section, we will refer to QM's causal potential with the symbol Υ.]

We have previously shown [14] that the Bell operator is $ab+cd$ and the Magic operator is its conjugate, $ab-cd$; and that these operators are irreversible due to multiplicative cancellation. Two *q*bits q_A and q_B in classical states $q_A = a-b$ and $q_B = c-d$ define an initial global state $q_A q_B = (a-b)(c-d) = ac - ad - bc + bd = \mathcal{T}_j + \mathcal{T}_k$.[11] This global state is called "classical" because it namely can be factored ("separated") like this. The Bell and Magic operators entangle such classical states to produce an *e*bit, which, in not being so separable, displays the characteristic EPR properties [15].[12]

*E*bits have the same form as *q*bits except that they are a sum of *bi*vectors, instead of vectors. A *q*bit spinor is a single bivector ab or cd, but an *e*bit spinor is the sum of entangled spinors, eg. $ac+bd$. Like a *q*bit, an *e*bit acts as a single co-exclusion (§7.2), even though it is made out of two *q*bits.

One can only be amazed to find, as Tables 2 and 3 show, that the Bell/Magic operators and the states they generate ***also*** in fact ***exactly cover*** $\mathcal{H}$, and thus constitute a ***completely different*** partitioning and view of $\mathcal{H}$-space, which, let us

[10]Namely, in our example, $a+b-c+ab+ac+ad-bc+bd-cd-abd-acd+bcd$.

[11]Meaning (reading off the signs) $a \wedge \neg b$ and $c \wedge \neg d$, ie. each encoding "zero" (vs. "one": $-a+b$ and $-c+d$). Thus $q_0^A = q_0^B =$ "0" here. This encoding, while conceptually redundant, makes "zero/one" superposition states explicit.

[12]The original labels for *q*bits in [13] were $q_A = a_0 - a_1, q_B = b_0 - b_1$, so $q_A q_B = a_0 b_0 - a_0 b_1 - a_1 b_0 + a_1 b_1$. Therefore *Bell* $= a_0 a_1 + b_0 b_1$ and *Magic* $= a_0 a_1 - b_0 b_1$, so then $B_0 = -a_0 b_0 + a_1 b_1$ and $M_0 = a_0 b_1 - a_1 b_0$, whence one can clearly and explicitly see the entanglement in the redistribution of the a_i and b_j, whence q_A and q_B are no longer *separable*.

basis	*basis state 1*	*basis state 2*	*basis state 3*	*basis state 4*
standard	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
diagonal	$\|0'0'\rangle$	$\|0'1'\rangle$	$\|1'0'\rangle$	$\|1'1'\rangle$
Bell	Φ^+	Φ^-	Ψ^+	Ψ^-
	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$
Magic	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\frac{i}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\frac{i}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$

Table 1: A summary of quantum mechanical bases in standard notation.

not forget, has a definite $3 + 1$d cast. Successive application of the Bell/Magic operators produces the corresponding Bell/Magic states. Notice that the states drop from four bivectors ($\mathcal{T}_j + \mathcal{T}_k$) to two bivectors ($-\mathcal{T}'_j$) due to cancellation, and it is this information loss that makes the entanglement thermodynamically irreversible.[13]

The Bell and Magic states are 90^o out of phase,[14] and since the starting state is generally some classical state like $q_A q_B = ac - ad - bc + bd$, which can now be rewritten as $B_3 + M_3$, the multiplicative cancellation occurs due to $M \times Bell = 0$, $B \times Magic = 0$ or $B \times M = 0$. These cancellations mean these states have disappeared from the causal potential Υ, and cannot be reached by any multiplicative operator ("transformation"), but rather only by addition, eg. $M_3 = B_0 - ac$. Recall that addition means concurrency, ie. $-ac$ comes from the outside.

The fact that the Bell and Magic states cannot transit (back) to classical states via multiplication is relevant as well to the $\mathcal{M}^2 = abcd$ states. For example, $\mathcal{M} = ab + ac + ad + bc + bd + cd = Bell + B_1 + M_3$, a mixture of Bell and Magic states. Only by concurrently ***adding*** new bivectors to the mix can a system exit these cyclical/closed/entangled state spaces. Since all of these states are related via entanglement relationships, we see that "mass" is massively entangled.[15] In the language of EPR, the $\Phi^\pm$ and $\Psi^\pm$ are singletons that represent maximally entangled states and behave as multiple "things" acting as one, with consequent non-local correlations. Mass, once created, is thus stabilized.

Table 4 demonstrates that the states $\mathcal{T}_j$ and $\mathcal{T}_k$ <u>are</u> the Bell and Magic states. [We have shown only two conjugate sets of tauquernions here, but as noted earlier, there are eight. Of the eight, four are related to $\mathcal{T}_i$ and the other four to $\mathcal{T}'_i$. The groups of four are all sign variants of each other such that $\mathcal{H} = \mathcal{T}_i + \mathcal{T}_j + \mathcal{T}_k$, whence $\mathcal{H}^2 = 0$. All of these sets contain all of the Bell/Magic states.]

[13] We note that the Hilbert space version shows reversibility. So far as we know [14], this is the only such deviation.

[14] In general, the state transitions are $B_{(i+1)_{mod\,4}} = B_i Bell$ and $M_{(i+1)_{mod\,4}} = M_i Magic$.

[15] In fact, any two conjugate $\mathcal{T}$'s could be named "Bell" and "Magic" operators, and all would otherwise be the same. It's best to just assume that everything is more or less entangled with everything. Consequently, gravity is everywhere.

The complete overlap of $\mathcal{T}$-space and entanglement space means that fundamentally, $q$bits and $e$bits are directly related to, and in fact *are*, the underpinnings of gravity and mass. Fittingly like gravity, the EPR effect is non-polar, since the two ends of the effect are equivalent and of the same valence. The lesson of this reasoning is that irreversible quantum mechanical entanglement establishes the ***associative*** footings on which, and out of which, gravity constructs its net.

$q_A q_B Bell = B_0 = \Phi^+$	$= -ac + bd = -\mathcal{T}_y'$
$B_0 Bell = B_1 = \Psi^+$	$= ad + bc = \mathcal{T}_z'$
$B_1 Bell = B_2 = \Phi^-$	$= ac - bd = \mathcal{T}_y'$
$B_2 Bell = B_3 = \Psi^-$	$= -ad - bc = -\mathcal{T}_z'$
$B_3 Bell = B_0$	$= -\mathcal{T}_y'$

Table 2: Bell operator and resulting Bell states.

$q_A q_B Magic = M_0$	$= ad - bc = \mathcal{T}_z$
$M_0 Magic = M_1$	$= -ac - bd = -\mathcal{T}_y$
$M_1 Magic = M_2$	$= -ad + bc = -\mathcal{T}_z$
$M_2 Magic = M_3$	$= ac + bd = \mathcal{T}_y$
$M_3 Magic = M_0$	$= \mathcal{T}_z$

Table 3: Magic operator and resulting Magic states.

$\mathcal{T}_x$	$\mathcal{T}_y$	$\mathcal{T}_z$
Magic	$M_3 = -M_1$	$M_0 = -M_2$
Magic	$M_3 = -M_1$	$M_2 = -M_0$
Magic	$M_1 = -M_3$	$M_0 = -M_2$
Magic	$M_1 = -M_3$	$M_2 = -M_0$

$\mathcal{T}_x'$	$\mathcal{T}_y'$	$\mathcal{T}_z'$
Bell	$B_2 = -B_0$	$B_1 = -B_3$
Bell	$B_2 = -B_0$	$B_3 = -B_1$
Bell	$B_0 = -B_2$	$B_1 = -B_3$
Bell	$B_0 = -B_2$	$B_3 = -B_1$

Table 4: Equivalence of Tauquernions and Bell & Magic operators

6. Dark Matter

In this section and the next two, we move from explanations of the tauquernions and the structures they form to some consequences. Foremost among these is the question of whether the tauquernions have anything to say about dark matter, which we now take up. §7 then presents an information-theoretic analysis of all of our results to that point, and §8 uses this analysis to tell a Bit Bang story.

Other work [11] has shown that the key elements of the standard model - bosons and fermions, three quark families, etc. - are captured by $\mathcal{G}_3$, which is isomorphic to the Pauli algebra $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ via the mapping $\{iab, iac, ibc\}$. In particular, the unitary elements of $\mathcal{G}_3$ all correspond clearly: [16]

Particle	$\mathcal{G}_3$ *element*	*Family size*
primitive distinction	x	3
neutrino family	$x+y+xy$	3
electron family	$xy+xz$	3*
proton family	$x+y+z+xy+xz$	3
neutron=xyz proton	$y-z+xy-xz+yz$	3
photon	$x+y+z$	1
**Eg. the three electron siblings are:*	$xy+xz, xy+yz, xz+yz.$	

The middle column of the table *exhausts* the catalog of unitary ($X^2=1$) entities in $\mathcal{G}_3$ and are all familiar, so dark matter is presumably not to be found here. We therefore must look in $\mathcal{G}_4$. The simplest non-trivial unitary element of $\mathcal{G}_4$ is

$$m = a + b + c + d$$

Assuming that m must be related to mass, ie. $abcd$, we now calculate m's co-boundary to $abcd$, which requires that $\partial_m abcd \cong m$, and which yields

$$\partial_m abcd = m\,abcd = -abc + abd - acd + bcd$$

so $\partial_m abcd$ is not similar to m, ie. m is not an eigenform of $abcd$. We can though apply the distributive law to the sum of $\partial_A X = B$ and $\partial_B X = A$, whence $\partial_{A+B} X = (A+B)X$, which yields $(m + m\,abcd)abcd \cong m + m\,abcd$. So the desired co-boundary is

[16] Primitive distinctions - first row - may not be observable in actuality. Photons - last row - are, of course, nilpotent. Note that the electron projection operator $-1 + xy + xz = x(-x+y+z)$, ie. an x-rotation of a photon; there is a similar factorization of $-1 \pm$ *proton*. See Appendix I for the complete $\mathbf{Z}_3$ $\mathcal{G}_3$ Standard Model.

$$\delta(a+b+c+d-abc+abd-acd+bcd) = abcd$$

We therefore define, in parallel with $\mathcal{H}\cup\mathcal{M}$, the set $\mathcal{D}$,

$$\mathcal{D} = \{(w+xyz) \ + \ (x+wyz) \ + \ (y+wxz) \ + \ (z+wxy)\}$$

which has $2^8 = 256$ sign variants. $\mathcal{D}$ is our hypothesis for dark matter, and we now investigate its structure and properties.

If one looks at $\mathcal{D}$ from a projective point of view, the 1-vector generators of the algebra are ***points*** defined by lines/processes that intersect a common plane, ie. are simultaneous with, and ***bivectors*** are the ***directed lines*** on that plane that connect these points. In this projective view, w,x,y,z are then the vertices of a tetrahedral volume element with triangular faces $\{wxy, wxz, wyz, xyz\} = \binom{\{w,x,y,z\}}{3}$. We hypothesize that these four triangles correspond to the 4 Planck areas/ln $2 = 1$ ***bit*** relationship [20]. Similarly, the $(x+y+z)$-boundary of the triangular face xyz yields the quaternions $\{xy, xz, yz\}$.

Just as $\mathcal{H}\cup\mathcal{M}$, along with 1 and ***abcd***, form the largest ***even*** sub-algebra of $\mathcal{G}_4$, so $\mathcal{D}$ is the largest ***odd*** sub-algebra. The elements of $\mathcal{D}$ form three subsets, the elements of the first of which all square to quaternionic triplets:

$$\mathcal{D}_q = \{D \in \mathcal{D} \,|\, D^2 = xy + xz + yz, \quad x,y,z \in \{a,b,c,d\} \ \}$$

and contains 128 elements. We note that $xyz\,\mathcal{D}_q = \pm 1 \pm wxyz + \{H, M\}$.

There are also 96 $\mathcal{D}$'s that are 8^{th} roots of unity (and thus material):

$$\mathcal{D}_u = \{D \in \mathcal{D} \,|\, D^2 = (w+x)(y+z) \ \& \ D^8 = 1\}$$

Note that $(w+x)(y+z) = -(y+z)(w+x)$, ie. they anti-commute, and so the $\mathcal{D}_u$ possess a spinorial quality. One can also multiply D^2 out: $(w+x)(y+z) = (wy+xz)+(wz+xy)$ and see that these are two tauquernion forms (and, simultaneously, separable states). We will see in §7.1 that the $\mathcal{D}_u$ contain a further subdivision of $96 = 16 + 80$, indicating the existence of two types of material dark matter. [This time, $xyz\,\mathcal{D}_u = \pm 1 \pm wxyz + \mathcal{M}$.]

Finally, there are 32 nilpotents D_0, for which $xyz\,D_0 = -1 + wxyz + \mathcal{H}$:

$$\mathcal{D}_0 = \{D \in \mathcal{D} \,|\, D^2 = 0\}$$

Thus $\{xyz\,\mathcal{D}\} = \{-1 + wxyz + \mathcal{H}\cup\mathcal{M}\}$, ie. normal matter and dark matter can be understood as being 3-dimensionally perpendicular to each other. Finally, $128 + 96 + 32 = 256$, whence $\mathcal{D} = \mathcal{D}_q \cup \mathcal{D}_u \cup \mathcal{D}_0$.

- - -

The fact that $\{xyz\mathcal{D}\} = \{-1+wxyz+\mathcal{H}\cup\mathcal{M}\}$ and therefore that the elements of $\mathcal{D}$ and $\mathcal{H}\cup\mathcal{M}$ can be rotated into each other allows a further analysis.

Let for example $\mathcal{D}_a = -a-bcd$; $\mathcal{D}_b = b+acd$; $\mathcal{D}_c = c-abd$ $\mathcal{D}_d = d+abc$, and define generally $D = \mathcal{D}_a+\mathcal{D}_b+\mathcal{D}_c+\mathcal{D}_d$ such that $D \in \mathcal{D}$. In this example, $D^2 = -bc+bd-cd \in \mathcal{D}_q$. Now construct their multiplication table, ie. D_q^2:

$D_q \times D_q$	$\mathcal{D}_a = -a-bcd$	$\mathcal{D}_b = b+acd$	$\mathcal{D}_c = c-abd$	$\mathcal{D}_d = d+abc$
$\mathcal{D}_a$	0	$ab+cd$	$ac-bd$	$ad+bc$
$\mathcal{D}_b$	$-ab+cd$	0	0	0
$\mathcal{D}_c$	$-ac-bd$	0	0	0
$\mathcal{D}_d$	$-ad+bc$	0	0	0

The sum of the $\mathcal{D}_a$ row is namely $\mathcal{D}_a\mathcal{D}_b + \mathcal{D}_a\mathcal{D}_c + \mathcal{D}_a\mathcal{D}_d \in \mathcal{H}'$, and anti-commutatively, the sum of the $\mathcal{D}_a$ column is $\mathcal{D}_b\mathcal{D}_a + \mathcal{D}_c\mathcal{D}_a + \mathcal{D}_d\mathcal{D}_a \in \mathcal{H}$. That is, $D_q^2 = \mathcal{H}+\mathcal{H}'$! This holds for all elements of $\mathcal{D}_q$ - *all* such tables contain zeroes except for one element each from $\mathcal{H}$ and $\mathcal{H}'$, and thus each element of $\mathcal{D}_q$ harbors the potential for both $\mathcal{H}$ and $\mathcal{H}'$ and so a complete set of tauquernions.[17]

In contrast, the corresponding tables for elements of $\mathcal{D}_0$ contain only zeroes; and the tables for $\mathcal{D}_u$ all look like this one:

$D_u \times D_u$	$\mathcal{D}_a = a+bcd$	$\mathcal{D}_b = b+acd$	$\mathcal{D}_c = c+abd$	$\mathcal{D}_d = d+abc$
$\mathcal{D}_a$	0	$-ab-cd$	0	$ad-bc$
$\mathcal{D}_b$	$ab-cd$	0	$ad-bc$	0
$\mathcal{D}_c$	0	$ad+bc$	0	$-ab-cd$
$\mathcal{D}_d$	$ad-bc$	0	$-ab+cd$	0

wherein we see that only two out of the three tauquernion forms appear, doubled, and including conjugates; the table sums to $ab+cd-ad+bc = (a-c)(b-d)$, where again there is a spinorial aspect (and two separable $q$bits). The missing tauquernion forms can be recovered from the products of the others, so $D_u \times D_u$, like $D_q \times D_q$, harbors an alternative pathway to $\mathcal{H}\cup\mathcal{M}$.

A rather different view emerges when one realizes that most of the partial products $(w+xyz)(x+wyz)$ in fact generate $\mathcal{T}$'s, and it is only their signs and sums in the full 4-way form that generate the three different outcome $\mathcal{D}$'s. Thus $\mathcal{D}_0^2$'s *five* $\mathcal{T}$'s sum to zero (three $\mathcal{T}$'s are identical and the other two complementary),

[17]Since $1+3 \le 4$, dark patches can connect smoothly, and, since the algebra is self-consistent, this connection must be compatible with that of $\mathcal{H}\cup\mathcal{M}$.

$\mathcal{D}_u^2$'s *four* $\mathcal{T}$'s sum to $(w+x)(y+z)$, and $\mathcal{D}_q^2$'s *three* $\mathcal{T}$'s sum to a quaternion triple (three xy's are identical). So there are a *lot* of $\mathcal{T}$'s floating around in the soup. Note in this connection that, like the individual $\mathcal{T}$-components of $\mathcal{H}$ and $\mathcal{M}$, each of the $\mathcal{D}_k$ and sums thereof satisfy $\delta\mathcal{D}_k = abcd$.

Finally, these $\mathcal{T}$'s are also entangled states, so (via xyz-rotation) all of the elements of $\mathcal{D}$ are also entangled, although it seems that this is indirect.

Summarizing, like $\mathcal{H}$ and $\mathcal{M}$, the elements of $\mathcal{D}$ *also* can interact to form space and matter, but more indirectly. A key issue is the energies at which $w + xyz$ and $\mathcal{D}$ form, and closely related is the question of what role the pathways from $\mathcal{D}$ to $\mathcal{H} \cup \mathcal{M}$ play.

A pertinent question at this point is, How do the elements of $\mathcal{D}$ interact with light? We have identified xyz as the carrier of charge [11], but that is in the context that $\delta(x+y+z+xyz(x+y+z)) = xyz$, where $xyz(x+y+z) = xy+xz+yz$ is the spinorial basis of the magnetic effect, and x, y, z each "$\frac{1}{3}$ electrical charge". This context is missing from both $\mathcal{H} \cup \mathcal{M}$ and $\mathcal{D}$. So, on this basis, one should not expect much of an electro-magnetic interaction with either of them (and indeed, 3d space is indifferent to electro-magnetism).

On the other hand, $\mathcal{D}$'s four xyz terms still have spin, even if it isn't identifiable any more as "charge". This spin could nevertheless conceivably retain electric charge's like-sign repulsive property, and so could be advanced as a contributor to the vacuum energy. We also note that H can be rewritten $(w-xyz)(x+y-z)$. However, re xyz (which squares to -1 and hence is 'polar'), where there's a 'plus' there's a 'minus', which polarity opens the door for (eg.) dark "ionic cluster" formation and the like, a possibility that can at this point only be speculation. Finally, $\mathcal{D}_0$ and $\mathcal{D}_q$, both being roots of zero, will both presumably contribute to the vacuum energy.

Howsoever, the fact that there now is a detailed mechanism in hand should simplify the task of finding a viable way to detect dark matter generally.

7. *Information Content and Kind*

We now embark on exact calculations of the information content (and its transformation) of expressions in the algebra. The overall picture is a "Bit Bang" modelled as the algebraic expansion $\mathcal{G}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to \mathcal{G}_4$, which expansion is driven by entropy creation via the conversion of information from space-like (non-Shannon) to time-like (Shannon) form.

Section §7.1 calculates the numerical information-theoretic skeleton of our $\mathbb{Z}_3$ $\mathcal{G}_4$ algebra, which is possible because of its finiteness and relatively small size: $3^{(2^4)} \approx 43$ million expressions. Since the algebra *is* the phase space, this *exact* (!) numerical skeleton has cosmological implications that we pursue in §8.

Section §7.2 then describes the computational mechanisms that define and create the aforementioned space-like, non-Shannon information. [Our term "non-Shannon information" is distinct from, but consistent with, a like-sounding entropy-related term, "non-Shannon-type inequalities".]

In this section we refer to $G = \{1, a, b, c, \ldots, ab, ac, \ldots, abc, \ldots, \ldots\}$ rather than $\mathcal{G}$ because we are referring specifically to ***actual instantiated elements***, though these of course also belong to the ***abstract*** geometric algebra $\mathcal{G}$.

7.1 *Calculating Information Content*

The formal concept of ***information*** is due to Claude Shannon (1948), who defined the information content $\mathcal{I}$ of an event x as

$$\mathcal{I}(x) = -lg\ p_x$$

where p_x is the probability of occurrence of the event x, and lg is the logarithm to the base 2. Thus, as is well known, the more improbable the event, the greater its information content. The import of this definition for us is best understood with the example of an ***if-then-else-type*** decision. The form [11]

$$X(1 + \langle -1 - a, \pm\mathsf{a}\rangle) \;+\; Y(1 + \langle -1 + a, \pm\mathsf{a}\rangle)$$

describes the computation *if* a *then* X *else* Y, where the brackets $\langle\, , \rangle = \pm 1$ indicate the inner product of the idempotent measurement probe $-1 \pm a$ with an entity $\pm\mathsf{a}$ in the surround, and $+$ indicates as usual the concurrency of the processes $X, Y \in G$. Here we see that a ***static*** bit of information - encoded in the $\pm$ state of a - is converted into the ***motion*** [state change] of ***one*** of the processes X or Y, since one of the two expressions ***will*** yield zero and the other minus one (minus because X (or Y) now changes state). Note particularly that

the information is consumed: a has been changed by the measurement and no copies made. One correctly concludes that a binary decision ***costs*** one bit of information.[18]

Applying this to G_n, this means that a measurement sequence that would locate some entity $\in G_n$ having an information content of m bits would require m such nested ***if***'s. Furthermore, this decision process transforms the static ***space-like*** information contained in the current state of G_n into dynamic ***time-like*** information at an exchange rate of $1:1$. It is this transformation (on a massive scale) that constitutes our expanding time-like universe.

This transformation is fundamentally entropic in character. Because the algebra is finite, we can calculate the probability of occurrence of an expression, and so we can know its information content. Knowing that, we can follow the entropy trail - loss of information - and make predictions about what further transformations will occur.

We therefore now embark on the calculation of the information content, measured in ***bits***, of every element of G_n, $n = 0, 1, 2, 3, 4$. This is an *exact* calculation, since it is based on pure combinatorics and resulting integer ratios.

The binary nature of our algebra allows us to fully expand the combinatorial content of any given expression in the fashion of a "truth table". Below we show the tables for ab, abc, and $abcd$. Beneath the tables are vectors of the respective *result* (rightmost) columns; ***these result vectors are the basis for our information content analysis.***

[18]Notice, by the way, how the 1-dimensional 180° opposition between X and Y as coded in $\pm$a becomes a conjugate (90°) opposition, $-1-a$ vs. $-1+a$, in the translation from the sequential to the concurrent view.

a	b	ab
−	−	+
−	+	−
+	−	−
+	+	+

a	b	c	abc
−	−	−	−
−	−	+	+
−	+	−	+
−	+	+	−
+	−	−	+
+	−	+	−
+	+	−	−
+	+	+	+

a	b	c	d	$abcd$
−	−	−	−	+
−	−	−	+	−
−	−	+	−	−
−	−	+	+	+
−	+	−	−	−
−	+	−	+	+
−	+	+	−	+
−	+	+	+	−
+	−	−	−	−
+	−	−	+	+
+	−	+	−	+
+	−	+	+	−
+	+	−	−	+
+	+	−	+	−
+	+	+	−	−
+	+	+	+	+

$$[+--+]\quad[-++-+--+]\quad[+--+-++--++-+--+]$$

As an example, we take the vector for *abc* and add to it **+1** and **−1**:

$$\begin{array}{r l} abc = & [-++-+--+] \\ \mathbf{+1} = & [++++++++] \\ \hline & [\cdot -- \cdot - \cdot \cdot -] \end{array} \qquad \begin{array}{r l} abc = & [-++-+--+] \\ \mathbf{-1} = & [--------] \\ \hline & [+ \cdot \cdot + \cdot ++ \cdot] \end{array}$$

Note that the *pattern* of symbols is the same for *abc* and the two sums, the only difference being that in *abc*, the two symbols that appear are + and −, whereas in the sums the two symbols are · and −, and + and ·, respectively (recall that · symbolizes zero). But the *ratios* are the same: here, four of each and *no* third symbol. And if you think about it, this proportionality will always hold - all that happens with the summing of *abc* with $\pm\mathbf{1}$ is that one so-to-speak rotates a three-symbol mapping vector [· ,+,−] first to [− ,· ,+] and then to [+,− ,·] : ***the proportions will therefore always be the same.***

Argument. A ***pattern encoding*** consists of a 3-tuple (#0's, #1's, #-1's), which forms a ***signature*** of the vector's structure. Suppose we have the pattern vector $(2,2,4)$ and imagine a (minimal) decision tree - think nested ***if***'s - that identifies any expression having this pattern. Then the amount of information embedded implicitly in the tree's decision points is the measure of the tuple's information content. The three symbols are interchangeable because the tree's *form* (the structure of the search space) is indifferent to which symbols lie at its leaves.

Since the ratios are invariant under exchange of symbols, the counts can appear in any order, so we just sort the tuples numerically.

This symbol-invariance implies that $\pm abcd$ and $\pm 1 \pm abcd$ all have the same information content. Since the latter form defines a measurement on the former, and these two therefore ***should*** be the same, this is comforting. For this and similar reasons, we think that any classification scheme (cf. ***binning***, below) must subscribe to the collapsing of $0, 1, \tilde{1}$ into one signature.[19]

This all means that we can classify *every* expression in the algebra in terms of its result-vector's signature. We will soon see that these informational classifications exactly match the $\mathcal{H}, \mathcal{M}, \mathcal{D}$, bosonic, and unitary particle structures previously discussed.

Since a polynomial $\in \mathcal{G}_n$ has maximally $|G_n| = 2^n$ mutually orthogonal terms, and their coefficients can be one of $0, 1, -1$, we get the set S, of size $|S| = 3^{2^n}$, which covers all of the possible expressions in G. With S in hand, we can count how many times k each pattern X occurs, and we can then divide k by $|S|$ to get the probability p of X's occurrence:

$$p_X = \frac{k}{3^{|G_n|}}$$

If $k = 1$, then is there is but one single occurrence of X in S, so p_X would be minimal, but this actually can't happen - the best you can do is the three scalar constants, $0, 1, \tilde{1}$, where $k = 3$.

From the other end of the microscope, a minimal X requires the full measure of the information in S in order to be identified and isolated. That is, the information content $\mathcal{I}$ of an expression $X \in G$ is

$$\mathcal{I}(X) = -lg\ p_X = -lg\ \frac{k}{3^{|G_n|}} = \ lg\ \frac{3^{2^n}}{k} \quad bits$$

X's information content is thus a function of how many other X's share its signature, and the size of the space it occurs in.

An obvious application of this is to ask, What is the information content of some particle P, having in mind the fact [20] that 1 *bit* $=$ 4 *Planck areas* $/ln\, 2$ ($\approx 10^{-66}\, cm^2$).

Thus, for example, a single Higgs boson $H = (1 + wxyz)(xy + xz + yz) = xy + xz + yz + wx + wy + wz$ exists in 16 states out of the 64 possible in the form. Its information content is therefore

[19] $\mathcal{V}oid$ cannot have its own category because, by definition, it has no properties by which it might be so categorized, including the property of having no properties. $\mathcal{V}oid$ can first become manifest in the distinction $[1, \tilde{1}]$.

$$\mathcal{I}(H) = lg\,\frac{3^{16}}{16} = 21.3594000\ bits$$ [20]

The next step, the conversion of ***bits*** to ***Gev***, turns out to be unexpectedly complicated, and is our current focus. The final paper will hopefully contain this result for $\mathcal{H}, \mathcal{M}, \mathcal{D}$, and all the rest too.

Of interest equal to individual particles, however, is the picture painted with the broader brush of the signatures and bin counts themselves.

Table 5 lists the information content, calculated in this broader way, of relevant elements of G_n. Because rarity/information is relative to the size of the space, the measure of (say) ab is 2.17 bits in G_2, 7.29 bits in G_3, and 18.9 bits in G_4. But at the same time, all of a, ab, abc, and $abcd$, at any given level, have the ***same*** measure, since their uniqueness stays proportional to n; note that namely these ***also*** have the highest information content after $0, 1, \tilde{1}$. In general, the lower the bit value, the larger the family of entities having that count, and oppositely, the higher the count, the smaller the family. We now explore this a little more.

The function $bitsN(X) = lg\,\frac{3^{2^N}}{count(X's)}$ calculates the information content of $X \in G$ relative to G_N.

Then, *re* G_0, the three scalar constants $0, 1, -1$ are all ***known*** and occupy the entire space, which is of size $3^{2^0} = 3$ states, one each for $\{0, 1, -1\}$,

- So each occurs with probability $p = \frac{1}{3} \mapsto lg\,3 = 1.58$ bits, ***but***
- ***Known*** means $bits0(0) = bits0(1) = bits0(-1) = lg\,\frac{3}{3} = lg\,1 = 0$
- So G_0 actually contains no information.

In G_1 there are $3^{2^1} = 9$ states, three for G_0's scalars, $\in (0, 0, 2)$, and $2 + 4 = 6$ more for $\pm a$ and $\pm 1 \pm a$, both $\in (0, 1, 1)$:

- The scalar constants are ***known***, and so they contain no information, but nevertheless occupy three slots in the state space $\Rightarrow bits1(1) = \lg 3 = 1.58$ bits.[21] [It is a mod-3 coincidence that the numbers for G_0 and G_1 are the same.]
- 1-vectors occupy the remaining states in G_1, so $bits1(\pm a) = \lg\frac{9}{6} = 0.58$ $= bits1(\pm 1 \pm a)$.

[20] This result differs from Table 5 (below) because we have ignored the other members of its bin, (4,4,8). Also, we don't know what the experimentalists are actually measuring - perhaps we should have calculated $\pm 1 \pm H$, etc.

[21] Recall that $0, 1, -1$ all map to the same pattern, whence 3 and not 1.

- The net result is that exactly 1 (classical) bit of information is encapsulated in the structure a: $bits1(1) - bits1(\pm a) = 1.00$.

For G_2, the algebra of pure $q$bits:

- The scalar constants are ***known***, but occupy state space: $bits2(1) = \lg(\frac{81}{3}) = 4.75$ bits.$^{(ditto)}$
- Here is a smallest addressable state: $(1-a)(1-b) = 1-a-b+ab \in (0,1,3) \mapsto bits2((1-a)(1-b)) = lg(\frac{81}{24}) = 1.75$ bits, corresponding to a single row of the form's "truth table". The 24 count comes from the 2^4 sign variants of $1-a-b+ab$ plus the 2^3 sign variants of $a+b+ab$.
- In §7.2, we show how it is that simple concurrency, $a+b$, ***mere concurrent existence***, contains and encodes information. Here we just calculate: $bits2(a) = 2.17 = bits2(b)$, $bits2(ab) = 2.17$, $bits2(a+b) = 1.17$ bits,
- Whence $bits2(ab) - bits2(a+b) = 1.00000000$, where we show in the 0's the number of significant digits that actually are available in these (exact) calculations; we show rounded values otherwise.

In G_4:

- Let $m = a+b+c+d$, whence $D = m + m\,abcd$ and $D^2 = 0$. As shown in Table 5, $\mathcal{D}_0 \in (4,4,8)$ and each D contains 5.53 bits of information.
- But $abcD = -1+ab-ac+ad+bc+bd+cd+abcd$ computes to 6.87 bits (not shown). One does not expect a reversible operator like abc to change the information content of an entity.
- The explanation is that the rotation by abc changes the ***signature bin*** that the expression falls into, and the new bin, namely $(2,6,8)$, has fewer members, and so the information content is higher. "It's not the rotation's fault." [We will exploit this phenomenon in our Bit Bang story in §8.]
- In Table 5, there are two examples of binnings that further differentiate the 3-signature - $(a+b+c)d$ and $\mathcal{M}_2$ are both $\in (4,6,6)$, yet their bit-measures differ, 12.1 vs. 7.08, and again $a+bcd$ and $\mathcal{D}_0$ are both $\in (4,4,8)$, but their measures are 15.1 vs. 5.53.

Form	Particle	Vector ($\mathcal{G}_3$ and $\mathcal{G}_4$ samples)	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{G}_0$						
$\mathcal{V}oid \mapsto \mathbf{0}$	*is*	$[\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot]$ $\in (0,0,8),0$	1.58	4.75	11.1	23.8
$\pm\mathbf{1}$	*are*	$[\pm\pm\pm\pm\pm\pm\pm\pm]$ $\in (0,0,8),0$	1.58	4.75	11.1	23.8
$\mathcal{G}_1$						
a	$\pm$*exist*	$[----++++]$ $\in (0,4,4),1$	0.58	2.17	7.29	18.9
$1-a$	*(measure)*	$[----\cdot\cdot\cdot\cdot]$ $\in (0,4,4),1$	0.58	2.17	7.29	18.9
Row0 $(1-w)\dots(1-z)$		$\in (0,1,1),(0,1,3),(0,1,7),(0,1,15)$	0.58	1.75	7.09	18.8
$\mathcal{G}_2$						
ab	$\pm$*spin*	$[++----++]$ $\in (0,4,4),1$	–	2.17	7.29	18.9
$a+b$	*co-occ*	$[++\cdot\cdot\cdot\cdot--]$ $\in (2,2,4),2$	–	1.17	4.70	15.1
$a+b+ab$	ν	$[------\cdot\cdot]$ $\in (0,2,6),3$	–	1.75	5.29	15.6
$a+ab$	W,Z †	$[\cdot\cdot++\cdot\cdot--]$ $\in (2,2,4),2$	–	1.17	4.70	15.1
$1+ab$		$[--\cdot\cdot\cdot\cdot--]$ $\in (0,4,4),1$	–	2.17	7.29	18.9
$\mathcal{G}_3$						
abc	$\pm$*charge*	$[-++-+--+]$ $\in (0,4,4),1$	–	–	7.29	18.9
$a+bc$	*quark*	$[\cdot++\cdot-\cdot\cdot-]$ $\in (2,2,4),2$	–	–	4.70	15.1
$ab+ac$	e	$[-\cdot\cdot++\cdot\cdot-]$ $\in (2,2,4),2$	–	–	4.70	15.1
$a+b+c+ab+ac$	p	$[----\cdot++-]$ $\in (1,2,5),5$	–	–	2.70	11.5
$a+b+c$	γ	$[\cdot--+-++\cdot]$ $\in (2,3,3),3$	–	–	3.29	12.1
$ab+ac+bc$	*3-space*	$[\cdot------\cdot]$ $\in (0,2,6),3$	–	–	5.29	15.6
$\mathcal{G}_4$						
$abcd$	$+$*mass*	$[+--+-++--++-+--+]$	$\in (0,8,8),1$	–	–	18.9
$1-abcd$		$[\cdot--\cdot-\cdot\cdot--\cdot\cdot-\cdot--\cdot]$	$\in (0,8,8),1$	–	–	18.9
q_Aq_B	*2 qbits*	$[\cdot\cdot\cdot\cdot\cdot+-\cdot\cdot-+\cdot\cdot\cdot\cdot\cdot]$	$\in (2,2,12),4$	–	–	14.1
$a+b+c+d$		$[-++\cdot+\cdot\cdot-+\cdot\cdot-\cdot--+]$	$\in (5,5,6),4$	–	–	10.1
$(a+b+c)d$		$[\cdot\cdot+-+--++--+-+\cdot\cdot]$	$\in (4,6,6),3$‡	–	–	12.1
$\mathcal{M}_1$ *(16/64) proto-mass*		$[\cdot\cdot\cdot+\cdot++\cdot\cdot++\cdot+\cdot\cdot\cdot]$	$\in (0,6,10),6$	–	–	13.1
$\mathcal{M}_2$ *(32/64) proto-mass*		$[++-\cdot-\cdot-++-\cdot-\cdot-++]$	$\in (4,6,6),6$‡	–	–	7.08
$\mathcal{H}$ *(16/64)*	*Higgs*	$[-\cdot++\cdot-+--+-\cdot++\cdot-]$	$\in (4,6,6),6$	–	–	7.08
Bell $= ab+cd = \mathcal{T}'$		$[-\cdot\cdot-\cdot++\cdot\cdot++\cdot-\cdot\cdot-]$	$\in (4,4,8),2$	–	–	15.1
Magic $= ab-cd = \mathcal{T}$		$[\cdot--\cdot+\cdot\cdot++\cdot\cdot+\cdot--\cdot]$	$\in (4,4,8),2$	–	–	15.1
$B_0 = -ac+bd$		$[\cdot+-\cdot+\cdot\cdot--\cdot\cdot+\cdot-+\cdot]$	$\in (4,4,8),2$	–	–	15.1
$M_0 = ad-bc$		$[\cdot+-\cdot-\cdot\cdot++\cdot\cdot-\cdot-+\cdot]$	$\in (4,4,8),2$	–	–	15.1
$a+bcd$	*dark*	$[+\cdot\cdot+\cdot++\cdot\cdot--\cdot-\cdot\cdot-]$	$\in (4,4,8),2$‡	–	–	15.1
$\mathcal{D}_0$	*dark*	$[-\cdot-\cdot\cdot-\cdot+-\cdot+\cdot\cdot+\cdot+]$	$\in (4,4,8),8$‡	–	–	5.53
$\mathcal{D}_q$	*dark*	$[++-\cdot++--++--\cdot+--]$	$\in (2,7,7),8$	–	–	6.87
$\mathcal{D}_u$ *(80/96)*	*dark*	$[--\cdot\cdot\cdot\cdot-+-+\cdot\cdot\cdot\cdot++]$	$\in (4,4,8),8$	–	–	5.53
$\mathcal{D}_u$ *(16/96)*	*dark*	$[+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-]$	$\in (1,1,14),8$	–	–	15.9

Table 5: Information content (in *bits*) of principal $\mathcal{G}_n$ forms. † Tentative. ‡ See text. Note that dark matter is non-palindromic.

These examples show that information content values like those in Table 5 are sensitive to the binning algorithm that is used. Fortunately, whatever the binning, the results will always be consistent because the underlying population is the same.

Our general-purpose binning algorithm (used in Table 5) first applies the 3-pattern signature, and then further bins together only those expressions having the same number of non-scalar terms.[22] Therefore, co-occurrences/*q*bits $x + y$, electrons $xy + xz$, and quarks $x + yz$, which already have the same signature, will still bin together. Thus, the numbers in Table 5 and its cousins will always be indicative rather than definitive, since how one bins is determined by which interaction-classes one is interested in.

There are other interesting things in Table 5: the information content of space as described by classical quaternions is 3.3 bits smaller than that of matter ($15.6 - 18.9$). Photons $(a + b - c)$ and their confounding $(a + b - c) * d$ have the same measure, 12.1, which is rather larger than $\mathcal{H}$'s 7.08, which contains them. There are apparently ***two*** forms of proto-mass $\mathcal{M}$ (13.1 vs. 7.08), and we note that the former is a sparse +1 variant. Singletons always have the highest bit value after the scalars, even more than two classical *q*bits $q^A q^B$. But then, given their spin, they ***are*** bits yo. Finally, note that the Bell/Magic states, $\mathcal{D}$, quarks, and electrons all have the same measure, 15.1, only slightly less likely than light, 12.1; and versus the rather more likely $\mathcal{H}$ and $\mathcal{M}$ at 7.08 bits.

Howsoever, as the expansion proceeds - - $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to \mathcal{G}_4$ in Table 5 - - Ψ's information content shrinks as the information in 3 + 1d gets denser and denser. For example, the two classical bits q^A, q^B use 4 spinors and 14.1 bits to encode 1 *e*bit - time-like stability costs! Matter itself is only slightly denser at 18.9 bits per: frozen potential (because actualized), robbed of its variability through loss of degrees of freedom. This is the fate of the space-like non-Shannon information that is converted, as the expansion of the universe, into time-like Shannon information.

We pursue this entropic expansion in a cosmological setting in §8. Before doing so, we introduce and define the concept of non-Shannon information, and show how this builds structure.

7.2 Non-Shannon information

There is a subtle paradox - concerning ***kinds*** of information - that we must deal with before going further. Shannon's concept of information, as we have seen, can be viewed as a descent into a binary tree from root to leaf, where at

[22]This increases the number of bins from 10 to 14 for G_3, and from 30 to 86 for G_4. For example, in the text just above, $(a + b + c)d \in ((4, 6, 6), 3)$, $\mathcal{M} \in ((4, 6, 6), 6)$, $a + bcd \in ((4, 4, 8), 2)$ and $\mathcal{D}_0 \in ((4, 4, 8), 8)$. All 43 million expressions were binned.

each branch point, one bit is consumed in the choosing of one path versus the other. Two points should be noted: (1) the (*information represented by the*) bits are(*is*) *consumed* and converted into the motion/advance of the descent-process; and (2) *action* is what this is all about... this *sequential* process is blind to context, and sees only its own (namely causal) point of view. The process concept, here exemplified, *is* sequence and action, combined. Thus Shannon's view of information is purely *time-like.*

It is difficult to see how Shannon's definition misses anything out, and yet ... it does. There *is* a kind of information that falls beyond it, namely the information of *concurrent existence*, what we call *non-Shannon information.* The following *Coin Demonstration* makes the argument.

Act I. A man stands in front of you with both hands behind his back. He shows you one hand containing a coin, and then returns the hand and the coin behind his back. After a brief pause, he again shows you the same hand with what appears to be an identical coin. He again hides it, and then asks, "How many coins do I have?"

Understand first that this is not a trick question, or some clever play on words - we are simply describing a particular and straightforward situation. The best answer at this point then is that the man has "at least one coin", which implicitly seeks *one bit* of information: two possible but mutually exclusive states: *state1* = "one coin", and *state2* = "more than one coin".

One is now at a decision point - *if* one coin *then X else Y* - and only one bit of information can resolve the situation. Said differently, when one is able to make this decision, one has *ipso facto* received one bit of information.

Act II. The man now extends his hand and it contains two identical coins.

Stipulating that the two coins are in every relevant respect identical to the coins we saw earlier, we now know that there are *two* coins, that is, *we have received one bit of information*, in that the ambiguity is resolved. We have now arrived at the dramatic peak of the demonstration:

Act III. The man asks, "Where did that bit of information come from?"

Indeed, where *did* it come from??! The bit originates in the *simultaneous presence* of the two coins - their ***co-occurrence*** - and encodes the now-observed *fact* that the two *processes,* whose states are the two coins, respectively, do not exclude each other. [23]

Thus, there is information in (and about) the environment that *cannot* be acquired sequentially, and true concurrency therefore cannot be simulated by a

[23]Cf. Leibniz's indistinguishables, and their being the germ of the concept of space: simultaneous events, like the presence of the two coins, are namely indistinguishable in time.

Turing machine. Penrose concluded in [18] that Turing machines cannot simulate quantum mechanics. Both Turing and Penrose consider the case $f \parallel g$, meaning execute the non-interacting processes f and g in parallel (and harvest their results when they end). Clearly one gets the same results whether one runs f first ($f;g$) or g first ($g;f$), or simultaneously, $f \parallel g$. In this *functional* view of computation, the only difference is wall-clock time. The Coin Demonstration is not about these cases *at all*, but rather asks, Can f *exist* simultaneously with g, or do they *exclude* each other's existence? *This* is the fundamental distinction that we draw.

More formally, we can by definition write $a + \tilde{a} = 0$ and $b + \tilde{b} = 0$, meaning that (process state) a excludes (process state) $\tilde{a}$, and similarly (process state) b excludes (process state) $\tilde{b}$. [24] Their *concurrent* existence can be captured by adding these two equations, and associativity gives two ways to view the result. The first is

$$(a + \tilde{b}) + (\tilde{a} + b) = 0$$

which is the usual excluded middle: if it's not the one (eg. that's +) then it's the other. This arrangement is convenient to our usual way of thinking, and easily encodes the traditional *one/zero* (or $1/\tilde{1}$) distinction.[25] The second view is

$$(a + b) + (\tilde{a} + \tilde{b}) = 0$$

which are the two superposition states: either both or neither.

The Coin Demonstration shows that *by its very existence*, a 2-co-occurrence like $a+b$ contains one bit of information. Co-occurrence relationships are *structural*, ie. *space-like*, by their very nature. Such bits, being space-like, are the source of non-Shannon information.

[Cf. Table 5, this information is twice that of a or b alone *in* G_1, but $2.17-1.17 =$ 1 bit less than a, b or ab *in* G_2.]

***Act IV**. The man holds both hands out in front of him. One hand is empty, but there is a coin in the other. He closes his hands and puts them behind his back. Then he holds them out again, and we see that the coin has changed hands. He asks, "Did anything happen?"*

[24]This is the logical bottom, and so there are no superpositions of $a/\tilde{a}$ and $b/\tilde{b}$: they are 1d exclusionary distinctions . Superposition first emerges at level 2 with ab via the distinction *exclude* vs. *co-occur*.

[25]Since $\tilde{x}$ is not the same as $0x$, an occurrence $\tilde{x}$ is meaningful; in terms of sensors, $\tilde{x}$ is a *sensing* of an externality x, not x itself.

This is a rather harder question to answer. To the above two concurrent *exclusionary* processes we now apply the ***co-exclusion inference***, whose opening syllogism is: ***if*** a excludes $\tilde{a}$, ***and*** b excludes $\tilde{b}$, ***then*** $a+\tilde{b}$ excludes $\tilde{a}+b$ (or, conjugately, $a+b$ excludes $\tilde{a}+\tilde{b}$).... This we have just derived.

The inference's conclusion is: ***and therefore, ab exists.*** The reasoning is that we can logically replace the two one-bit-of-state processes a, b with one two-bits-of-state process ab, since what counts in processes is sequentiality, not state size, and exclusion births sequence (here, in the form of alternation). That is, the existence of the two co-exclusions $a+\tilde{b} \mid \tilde{a}+b$ and $a+b \mid \tilde{a}+\tilde{b}$ contains sufficient information for ab to be able to encode them, and therefore, logically and computationally speaking, ab can rightfully be instantiated. We write $\delta(a+\tilde{b}) = ab = -\delta(\tilde{a}+b)$ and $\delta(a+b) = ab = -\delta(\tilde{a}+\tilde{b})$. A fully realized ab is, we see, comprised of two ***conjugate*** co-exclusions, a sine/cosine-type relationship.

We can now answer the man's question, ***Did anything happen?*** We can answer, "Yes, when the coin changed hands, the state of the system rotated 180°: $ab(a+\tilde{b})ba = \tilde{a}+b$." We see that one bit of information ("something happened") results from the alternation of the two mutually exclusive states.

With the co-exclusion concept in hand, we can now add a refinement to the idea of co-occurrence. Recall that S is the space of all imaginable expressions in $\mathcal{G}$. But, thinking now computationally, this means that they are all "there" at the same time! That is, ***S is the space of superpositions***, of all ***imaginable*** co-occurrences of elements of $\mathcal{G}$ ***all at the same time***; whereas G is the space of ***actually occurring*** (but still space-like) entities, which means no co-exclusionary states allowed. When things move from S to G, superposition is everywhere replaced by reversible alternation, ie. G is a sub-space of S.

Co-exclusions, being superpositions, thus live exclusively in S, whereas co-occurrences can exist in both S and G, though their objects are slightly different. Co-occurrences in $\mathcal{T}$-space have yet another flavor. Each of the transitions $S \to G$ and $G \to \mathcal{T}$ is entropically favored. We now look at the former, the latter being the standard theory of quantum measurement.

As a first example, consider the scalar distinction $[1, \tilde{1}]$, an element of S, which is mapped to the vector a, an element of G, and therewith encapsulates one bit, cf. Table 5. $[1, \tilde{1}] \in S$ because both 1 and $\tilde{1}$ must be simultaneously present if the idea of their distinction is to be meaningful. Thus, what is a ***superposition*** of 1 and $\tilde{1}$ in S becomes an ***alternation*** between 1 and $\tilde{1}$ in $a \in G$. A degree of freedom has been lost.

A second example: the co-exclusions $(a+\tilde{b} \,|\, \tilde{a}+b)|(a+b \,|\, \tilde{a}+\tilde{b})$ induce the formation of ab. What happens is that the ***superpositions*** in S represented by the co-exclusions - three of them - have been replaced by their actualized *alternations*, $(a+\tilde{b} \leftrightarrow \tilde{a}+b) \leftrightarrow (a+b \leftrightarrow \tilde{a}+\tilde{b})$ in G. [26] That is, the superpositions in S are replaced by space-like exclusions in G, which is, again, a ***reduction*** in the number of states. In the next step, this reversible alternation in G is replaced by ***before/after***, that is, it becomes a time-like (irreversible) exclusion in $\mathcal{T}$.

The overall movement of information is thus from superposition in S to space-like exclusion ("alternation") in G to time-like exclusion ("before-after") via projection/measurement in $\mathcal{T}$. Each of these steps increases entropy by (further) compartmentalizing information, which reduces correlation, ie. increases noise, which is entropy.

The information that Shannon defined is namely ***time-like,*** and is exactly modelled by a binary decision tree descent from root to leaf. In contrast, what δ does is to ***build*** that tree from the leaves (detailed co-occurrences like $a+\tilde{b}$) first to ab, ie. $\delta(a+\tilde{b}) = ab$, and from there up to the root $abc\ldots z$. In doing so, it reduces the information content of S by turning its superpositions into exclusionary distinctions in G, which in turn, at level 4, are projected into $3+1$d tauquernion spacetime. The Bit Bang explosion is much like the irresistible salesman who argues that owning one cow ***after*** the other is really just as good as owning two cows at the ***same*** time. (Although it isn't, as we know.)

When we calculate the information content of $G = \Psi$, we are counting non-Shannon information. And yet, the conceptual ***basis*** for this counting up of non-Shannon information is Shannon's time-like information, information you can use to locate and identify things in a space, cf. the binary tree descent! This is the "subtle paradox" mentioned in the first sentence of this section.

We resolve the paradox by viewing the entropic expansion $\mathcal{G}_0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow \mathcal{G}_3 \rightarrow \mathcal{G}_4$ as the conversion of the space-like information in S and G into time-like information in $\mathcal{T}$-space - ebits, mass, 3d space, gravity, entropy, and time. That is, causal ***potential*** is converted into causal ***actuality,*** and it is in this conversion that the Shannon encoding of non-Shannon information is rendered meaningful, as namely Shannon information.

The continuation of this entropically-favored process of increasing encapsulation

$$a \xrightarrow{\delta} ab \xrightarrow{\delta} abc \xrightarrow{\delta} abcd \xrightarrow{\delta} \ldots \xrightarrow{\delta} abc\ldots z$$

would seem to lead to the conclusion that black holes are to be described by pseudo-scalars of grade $4n$, where n is *very* large, and "4" because this is a gravitational phenomenon, and the algebra cycles semantically $mod\,4$ (and more subtly, $mod\,8$). We are namely looking at (ie. inside) the interior of a gigantic gravitationally resonantly bound particle with 2^{4n} dimensions. At this

[26] Note that the co-exclusion form sums to 0, and so holds no contradiction.

extremely high level of gravitational organization (read heavily entangled), everything is so intensely correlated with everything else that, in the limit, all entities become indistinguishable from each other. In this way, the stage is set for a new expansion. [27]

8. Cosmological Evolution

The preceding section dilineated the information content of elements of the algebra, and thereafter how these elements are stitched together computationally and mathematically (namely with co-exclusion $\mapsto \delta$) to create ever more actualized structures. Left unaddressed however, is how exactly these algebraic elements come to be in the first place.

Metaphysics aside, we rely on two pillars of support in this telling of this story:

- The structure of the algebra itself, without questioning whether this is putting too much in by hand.
- The entropic propensity, ie. the truth of the 2^{nd} Law of Thermodynamics.

These are the governing principles in what follows.

The overall story arc is that the information creation via co-occurrence (cf. the Coin Demo), which is both dominant and non-Shannon, can be sustained using reversible mechanisms. The result is an exponentially expanding space-like information space, namely $G = \Psi$. This information is then bled off by its conversion into its time-like form, which we experience as $\mathcal{H}, \mathcal{M}, \mathcal{D}$, the Big Bang, and its aftermath.

The primitive mechanisms that contribute to the creation of bits of information are

- Distinctions: scalar 1 vs. $\tilde{1}$, and (multi-)vector $XY = -YX$
- Products, XY
- Co-occurrences, $X + Y$

[27] We note that the Pythagorean relationship $B_1^2 + B_2^2 = (B_1 + B_2)^2$ for the total entropy of the merge of two black holes B_1, B_2 [21] is satisfied by any two tauquernions so long as they anti-commute. See also the discussion of $\mathbf{Z}_n$ arithmetics in Appendix I.

The last of these dominates the information content of both S and $G = \Psi$ because the number of co-occurrences grows hyper-combinatorially. The two distinctions are clearly proto-bits. "Products" get their own line because, if co-occurrence is the steam locomotive, then products - being the generators of novelty - are the coal car, without a constant supply of which, the train will grind to a halt. This is detailed below.

Since we are dealing chiefly with co-occurrences, all "information" is non-Shannon unless otherwise noted. We are dealing only with *extant* elements of S, that is, with the elements of G as so far constructed. Abusing combinatorial notation, we are generating the set $\{\binom{\{1,a,b,\ldots\}}{m}\}$ of all the possible forms in $G = \{1, a, b, \ldots\}$. This generates $\Sigma\binom{n}{m} - 1 = \underset{1}{\Sigma}\binom{n}{m} = 2^n - 1$ elements.

The reason that the formula is $\underset{1}{\Sigma}\binom{n}{m}$, ie. leaving out one possibility ($m = 0$), is that *Void* cannot be a party to a co-occurrence. This is because by definition, 0 means "does not occur", in the sense that *Void* does not "happen", does not "take place", in either space or time, as opposed to the mis-understanding "not there at all". Thinking back to the Coin Demonstration, it simply cannot be performed when there is $\mathcal{N}oThing$ in the man's hand, but this does not deny *Void*'s presence.

We begin our construction with the scalars, G_0. These are *Void* $\mapsto 0$ and the primitive distinction $[1, \tilde{1}]$ that emerges from *Void* [12]. The scalars have no dimensionality but *can* represent a primitive distinction if one has two of them. Dimensionally, *Void* $\mapsto 0$ represents a point, and the two-valued distinction ± 1 is the prototype of a line.

Including *Void*, G_0 has three distinctions $[\neg \textit{Void}, \neg 1, \neg \tilde{1}]$ leading to $lg\, 3 = 1.58$ bits; counting just the two non-zero states, this represents $lg\, 2 = 1.00$ bit. These two different bit-measures express the difference between the space S of ***possibilities***, and the space G of *extant* (in Ψ) entities, ie. those that have actually been constructed out of the possibilities.

The transition from G_0 to G_1 maps the scalar distinction $[1,\tilde{1}]$ to a 1-vector, a. This is an entropically favorable transition, according to Table 5, because a has one bit less information than the scalars from which it is formed. This mapping reifies into an exclusion what previously was only a potential to be 1 or $\tilde{1}$. Both scalars and vectors are now present, and Table 5 shows that they always have the highest information content of all.

The forms $\underset{1}{\Sigma}\binom{2}{m}$ of G_1, which we might also write as $\underset{1}{\Sigma}\binom{\{1,a\}}{m}$, yield the set $\{1, a, 1+a\}$, but $\delta(1+a) = a$, which we already have, so no novelty is generated.

The expansion must therefore seek another route ... which is (to await) the co-occurrence $a + b$, wherein we imagine the parallel existence of many G_1's (this is all an idealization, of course). Once a and b co-occur, they can co-exclude, whence ab, ***a new entity***, is added to G. This is the coal car feeding the steam engine: every time a new entity is added to G, the number of co-occurrences, the size of G, doubles. [28]

Note that even though the multiplication $a+b \mapsto ab$ is reversible (eg. $a(ab) = b$), information is nevertheless created when ab is created (2.17 vs. 1.17 bits). As noted in §7.1, what is going on is that bins - of ***possibility*** - are simply being visited.

Addition (co-occurrence) is doing most of the work of the expansion - it's always entropically favored. But multiplication supplies a vital piece, namely the step from $a + b$ to ab. This being a crucial step, we reason that ab has the same information content as a and b, so in multiplying the latter together, it's $1 \times 1 = 1$ so to speak: we are simply combining things of the same measure and nothing is being "manufactured". Nevertheless, ab is still ***novel***, so in the context of S and G and their basis in co-occurrences, we still harvest an information windfall from ab's appearance, because this gives (entropically favored) birth to a whole new generation of co-occurrences.

This may sound dodgy - something for nothing is always suspect - but the mathematics speaks clearly. It is non-Shannon (ie. space-like) information that becomes available via (though not because of) space-like rotation, $G = \Psi$ is expanding (because of addition), and there is no time-like context here.

This reasoning applies to all co-occurrences and products, and thus the expansion of Ψ is a general free-for-all application of co-occurrence $+$ and action $\times$ over and between all extant entities, biased in the general direction of entropy generation. But we are ahead of the story, and now must back up.

Eventually, all the elements of our G_1, call it $G_1^a = \{1, a\}$, will have been generated, so we must await a co-occurrence with a new entity, call it $b \in G_1^b$, and we then can generate $G_1^a + G_1^b$. Recall that co-occurrences always have a lower information content than the singletons composing them, so $G_1^a + G_1^b$ is entropically favored.

Once there is co-occurrence, there can be action: G_2 is created by $G_1^a \times G_1^b = \{1, a, \} \times \{1, b\} = \{1, a, b, ab\} = G_2^{ab}$. Besides $q$bits, this produces, in particular, the high-information bivector ab, and thence W/Z and neutrinos.

Nevertheless, at some point, the combinatorial possibilities of G_2^{ab} too will be realized, whence we await a co-occurrence with an entity belonging to another G, say $G_2^{ab} + G_1^c$, leading to the product $G_2^{ab} \times G_1^c$:

[28]Strictly speaking, we should not count 1-vectors and pseudo-vectors, the $\binom{n}{1}$ and $\binom{m}{m}$ terms of the Σ, since we're counting co-occurrences, and these are singletons. On the other hand, including $\mathcal{O}(n)$ singletons has negligible impact on $\mathcal{O}(2^n)$.

$$G_3^{abc} = \{1, a, b, ab\} \times \{1, c\} = \{1, a, b, c, ab, ac, bc, abc\}$$

With G_3^{abc} we get photons, electrons, quarks, protons, neutrons, mesons, gluons - all the familiar members of the Standard Model.

Similarly, $G_2^{ab} \times G_2^{cd}$ and $G_1^d \times G_3^{abc}$ together generate G_4^{abcd} - giving us $\mathcal{H}, \mathcal{M}, \mathcal{D}$, $3+1$ spacetime, mass, gravity, and entropy - at which point we leave quantum mechanics. $G_{4n} \times G_{4n}$ describe higher-order gravitational structures.

However, we have again gotten ahead of our story. In generating G_2 from $G_1 \times G_1$, we can further imagine the co-occurrence and subsequent product of several (say four) G_1's (over, say, a, b, c, d), which will then produce the six bivectors ab, ac, ad, bc, bd, cd.

Once again we recall that co-occurrences always have a lower information content than the singletons that compose them, so entities like $ab + cd$ will again be entropically favored. These are, of course, $\mathcal{T}$'s ($\Rightarrow$ Bell/Magic states and ebits), and so we see that there is an entropically favored route to $\mathcal{H}$ and $\mathcal{M}$. [The same applies to $xy + xz$ (electrons) and $x + yz$ (quarks).] Since, all else seeming equal, there are three times as many $\mathcal{M}$ states as $\mathcal{H}$ states, the tendency here will be for the formation of normal matter.

Similarly, $G_1 + G_3$ will produce co-occurrences like $w + xyz$, the atoms of dark matter, so $\mathcal{D}$ is also an entropically favored outcome. Note that with the exception of ${}^{16}\mathcal{D}_u$, dark matter will be formed preferentially to normal matter, cf. $5.53, 6.87, 5.53$ versus 15.9 in Table 5. [Appendix II continues this discussion of combinatorial expansion.]

In both cases, the expansion is hyperexponential, and, being prior to the actual formation of 3+1d spacetime via the $\mathcal{T}$'s , is also not limited by the speed of light. Thus this combinatorial expansion presumably models the initial inflationary episode of standard cosmology.

Summarizing the cosmological development, both graphs in Figure 1 show the two major pathways to space/mass creation: upward on the left, the creation of $3+1$d space and normal matter, $\delta(\mathcal{H} \cup \mathcal{M}) = abcd$, via the pathway $\delta(\delta(a + b) + \delta(c + d)) = abcd$; and upward on the right, dark matter, via the pathway $\delta\mathcal{D} = \delta(d + \delta(c + \delta(a + b))) = abcd$, but then also for the latter, a "back door" down to $\mathcal{H} \cup \mathcal{M}$ via $\mathcal{D}_q^2$, $\mathcal{D}_u^2$, and $abc\mathcal{D}$ (cf. §6).

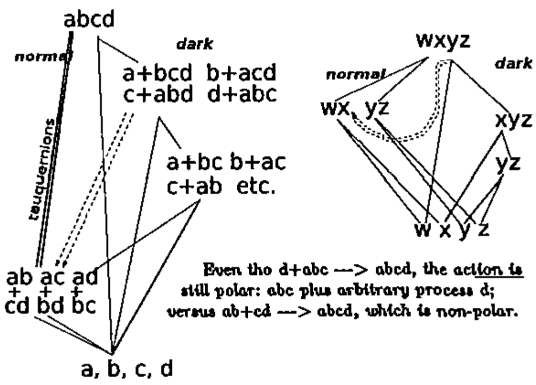

Figure 1: Two equivalent graphs of normal & dark matter creation. Growth (δ) is upward, as the ambient energy falls. The dotted lines symbolize the indirect tauquernion creation from dark-dark interactions.

9. Summary and Conclusions

We have a very promising candidate, the tauquernion $\mathcal{T}$- forms

$$\mathcal{H}, \mathcal{M} = \Sigma \left({}^{\{a,b,c,d\}}_{\;\;\;2} \right), \quad \mathcal{H}^2 = 0, \quad \mathcal{M}^2 = abcd$$

for the long-sought connection between quantum mechanics and 3+1d relativity theory. This connection, which creates space, matter, and time, takes the form of a new, inherently entropic, way to describe 3d space. The conjugate forms of the Higgs bosons $\mathcal{H}$ presumably correspond to the dual polarizations of gravitational waves, and the members of $\mathcal{M}$ are the precursors of the unit mass *abcd*.

The overlap of $\mathcal{H} \cup \mathcal{M}$ and the entanglement states allows the partitioning of our understanding of matter and space into two complementary views: The tauquernion view focuses on the formation of matter, 3 + 1d space, and gravity; whereas the the Bell/Magic view focuses on how the space and the matter all interconnect to form the whole. In hindsight, these two functionalities - the formation of structures and their interconnection - surely do lie best on the *very same* foundation - which turns out to be the largest *even* sub-algebra of $\mathcal{G}_4 = \{1, ab, ac, bc, ad, bd, cd, abcd\}$. But that's hindsight.

A near cousin $\mathcal{D}$ of $\mathcal{H} \cup \mathcal{M}$, the largest *odd* sub-algebra of $\mathcal{G}_4$,

$$\mathcal{D} = \Sigma\left(\{a,b,c,d\} \atop 1,3\right), \; \mathcal{D}^2 \in \{0, xy + xz + yz, (w+x)(y+z)\}$$

offers a uniquely believable candidate for dark matter that also connects to $\mathcal{H} \cup \mathcal{M}$ via secondary $\mathcal{T}$-based connections. Our analysis predicts three types of dark structure, one nilpotent, one space-like (in that these square to quaternion triples), and one material (being 8^{th} roots of unity). This latter has two forms in the proportion $16 : 80$, one (20%) heavy (15.9 bits) and one (80%) light (5.53 bits).

As this last sentence indicates, we have calculated the information content of every expression in $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$. The classification system we developed to do this is based on the observation that an algebraic expression that picks out a single row of its "truth table" uses the most algebraic terms in order to provide this most discriminating specification. The sign-counts (#+'s, #−'s, #0's) associated with an expression, which counts are as well invariant over symbol substitutions, fit this observation exactly. However, because many quite different expressions in $\mathcal{G}_4$ have the same count-signature, giving misleadingly high bin populations, our final classification algorithm therefore uses *both* these counts *and* the number of (non-scalar) terms in the expression - a Euclidean length - to choose a bin. Thus our binning algorithm compactly represents both the state and the algebraic complexity of any expression.

We explicitly iterated through all sign variants of all expressions in $\mathcal{G}_1$ (2 bins), $\mathcal{G}_2$ (4 bins), $\mathcal{G}_3$ (14 bins) and $\mathcal{G}_4$ (86 bins for 43 million expressions) in order to calculate the exact bin populations for each such signature. These in turn yield the highest bit value for the least likely bins (eg. m-vectors and single-row specifiers) and the lowest bit value for the most likely bins (eg. large concurrent expressions).

The biggest surprise was that primitive concurrency (addition of vectors/m-vectors) is easily the primary mechanism for information creation. While multiplication's transformative power is, as we saw (§8), necessary to maintain a supply of novel entities, the hyper-combinatorial state expansion fostered by additive combination of said novelty vastly exceeds the latter's numbers. The potential information so created is ultimately released as energy according to the relation $1 \; \textit{bit} = 4 \; \textit{Planck areas} \; / \ln 2$.

As the state space expands from $\mathcal{G}_1$ thru $\mathcal{G}_4$, the bit value of an individual m-vector grows from 0.58 to 18.9 bits due to the explosion in the size of the state space. This Bit Bang represents *real* bits that are released as *real* energy, the energy that fuels the Big Bang when converted to the (statistically likely) Higgs and mass states at 7.08 bits. Thus, for example, two bivectors at 18.9 bits each, combined concurrently (eg. yielding a $\mathcal{T}$), yield a co-occurrence with

4

an information content of 15.1. This (probably) entangled $\mathcal{T}$-state persists due to irreversibility, and therefore has increased likelihood of forming (with two others) an element of $\mathcal{H} \cup \mathcal{M}$, with an ensuing huge further entropy increase to 7.08 bits. Globally and locally, the expansion process is monotonic due both to the irreversibility of the entangled Bell/Magic states and the entropic expansion in general.

In the standard QM story, the quantum potential Ψ is the home of superpositions, and the transition from superposed to definite states lies at the heart of the quantum mechanical world. Since in the same standard story there is no *mechanism* – it being entirely statistical in content – no finer distinctions were needed. Having with our computational interpretation introduced the missing mechanism, we were able to see the distinction between what can imaginably be ($\mathcal{S}$, superpositions), versus what can *potentially* exist ($G = \Psi$, alternations), versus what actually *is*, $\mathcal{T} \cdot G \mapsto 3+1$d. The distinction between superposition and alternation in turn allowed the formulation of a coherent story of entropic transformation from $\mathcal{S}$ to G to our own $3+1$d spacetime. Our information content calculations, besides being *exact* - a welcome rarity - seem consistent with both observation and standard theory, and as well fill in many details of what happens before the Big Bang bangs.

It seems appropriate now to remind the reader of the *hierarchical* structure of the algebra, and what it might mean when extended beyond $\mathcal{G}_4$. This structure has its foundation in the fact that the algebra's atoms – $a, ab, abc, \ldots$ – whose successive squares are the $+--+$ sequence of powers of i, are also "pure frequencies", since they are the dimensions onto which Parseval's Fourier decomposition projects, and simultaneously they *also* are oscillating co-exclusionary computations. Thus, in a sense, the $\mathcal{G}_3$ particle tables in §6 and Appendix I and their exact fit to the Standard Model are inevitable. At the same time, these m-vectors grow (via δ, cf. "symmetry-breaking") with the size of m in encoded complexity, such that one can only think that the detailed construction of hydrogen, helium, etc. is within reach, with molecular bonding and molecules next. Huygens' principle of secondary sources is a guide in this endeavor.

We note that the discovery of the tauquernions lends strong support to background-independent theories [4,9,13]. The tauquernion foundation for $3+1$d - via both the Higgs mechanism and entanglement - means that cosmological theories need no longer feel forced to *assume* the prior existence of 3+1d, as does eg. string theory. Rather, the availability of the tauquernions should encourage the development of background-free theories, which are for the same reason more conceptually satisfying.

Finally, we note that the Coin Demonstration delivers a decidedly *non-computable* bit of information (in the Turing sense), and would therefore seem to constitute the non-computable element sought by Penrose [18] and others.

All in all, we are very impressed with the deep correspondence of known and/or physically meaningful computational algebraic structures to their calculated information content, and of both of these to the physical phenomena they are meant to model. Even the subtlest processes seem almost to have been anticipated. Together with the present computational interpretation, the power and elegance of the $\mathbb{Z}_3$ geometric algebra can simply not be denied.

- - -

The minimalism of our $\mathbb{Z}_3$ dialect of geometric algebra has effortlessly and incredibly parsimoniously exhibited, via the tauquernions $\mathcal{T}$, virtually all the desired and necessary structures, seamlessly interwoven, to plausibly connect quantum mechanics to 3+1d space-time, both its creation and its content. As a dividend, we also get a detailed structural theory of dark matter. The complete overlap of the $\mathcal{T}$ and entanglement spaces, making entanglement the mechanism of gravity, is a wonderful surprise. The information-theoretical analysis supplies a both detailed and *exact* "null hypothesis" backdrop for experiments. Hopefully, the more detailed formulation of this picture in unchained $\mathbb{Z}$, and its mapping to the body of general relativity and $\mathbb{R}$, will be straightforward, but nevertheless definitely a matter for professional physicists.

In this connection, we think it entirely reasonable that physicists expect, and even require, that the algebraic and interpretive framework that we have introduced provide the actual mechanisms for the physical effects we observe. Call this ***information mechanics***. After all, we have presented a computational theory, and ***mechanism*** – what ***must*** <u>***actually***</u> ***happen*** – is the soul of the computational metaphor.

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Michael Manthey holds a PhD in Computer Science and has taught operating systems, networks, and computer architecture at universities in the USA and Europe. He resigned his tenured associate professorship at Aalborg University (Denmark) to pursue research full-time. He lives in Colorado USA.

Douglas Matzke, a former Senior Member of Technical Staff at Texas Instruments, holds a PhD in quantum computing from UT-Dallas/EE. He both organized and chaired the IEEE Physics and Computation Conferences II and III in 1992 and 1994, and has contributed often to IEEE publications. He lives in Texas USA, and is currently doing contract programming.

References

1. Barrow, John D. "Theories of Everything", in *"Nature's Imagination: A debate on the future of science with contributions by Freeman Dyson, Roger Penrose, John Barrow, Gerald M. Edelman, Oliver Sachs, and others"*, John Cornwell, Editor. Oxford University Press, 1992. Isbn 0 19 851775 0.

2. Bèrut, Arakelyan, Petrosyan, Ciliberto, Dillenschneider & Lutz. "Experimental verification of Landauer's principle linking information and thermodynamics." Nature 483, 187–189 (08 March 2012) | doi:10.1038/nature10872 .

3. Doran, C. and Lasenby, A. *"Geometric Algebra for Physicists"*. Cambridge University Press, 2003. Isbn 978-0-521-71595-9.

4. Dreyer, O. "Background-Independent Quantum Field Theory and the Cosmological Constant Problem". arXiv:hep-th/0409048.

5. Francis, M.R. and Kosowsky, A. "Geometric Algebra Techniques for General Relativity". arXiv: gr-qc/0311007v1 3 Nov 2003.

6. Hestenes, D. *"New Foundations for Classical Mechanics"* (2^{nd} ed). Kluwer Academic Press, 1999. Isbn 0-7923-5514-8.

7. Hestenes, D. and Sobczyk, G. "*Clifford Algebra to Geometric Calculus - A Unified Language for Mathematics and Physics*". Kluwer Academic Press, 1984/87. Isbn 90-277-2561-6.

8. Hestenes, D. "Clifford Algebra and the Interpretation of Quantum Mechanics". In J.S.R. Chisholm/A.K. Commons (Eds.), *Clifford Algebras and their Applications in Mathematical Physics*. Reidel, Dordrecht/Boston (1986), 321-346.

9. Loll, R., Ambjorn, J., and Jurkiewicz, J. "The Universe from Scratch". hep-th/0509010.

10. Lounesto, P. "*Clifford Algebras and Spinors*". London Mathematical Society, Lecture Notes Series #286. Cambridge University Press (1997) 2002. Isbn 0-521-00551-5.

11. Manthey, M. "Synchronization - the Font of Physical Structure". International Journal of Anticipatory Systems, Daniel Dubois, Ed. Issn 1373 - 5411, v. 20-21-22, IJCAS/BCS Symposium, Liege, Belgium. August 6-11, 2007. Isbn 2-930396-09-1.

12. Manthey, M. "A Combinatorial Bit-Bang Leading to Quaternions". Proc. 1997 *Helsinki Conference on Emergence, Hierarchy, and Organization (ECHO III)*; http://arxiv.org/abs/quant-ph/9809033v1.

13. Markopoulou, F. "Towards gravity from the quantum". hep-th/0604120.

14. Matzke, D. Quantum Computation Using Clifford Algebra. PhD thesis, Dept. of Electrical Engineering, 2002. www.utdallas.edu/~cantrell/matzke.pdf. 11.

15. Matzke, D., Manthey, M., and Cantrell, C.D., Quantum Geometric Algebra. Proc. 12. 24th ANPA, 2002.

16. Nielsen, M.A. and Chuang, I.L, "*Quantum Computation and Quantum Information*". Cambridge University Press. 2000.

17. Parseval's Identity (1799). Wikipedia. http://en.wikipedia.org/wiki/Parseval %27s_identity.

18. Penrose, R. "*The Emperor's New Mind*". Oxford University Press, Oxford, New York, Melbourne, 1989. Isbn 0-19-851973-7.

19. Penrose, R. "*The Road to Reality - A Complete Guide to the Laws of the Universe*". Alfred A. Knopf, 2005. Isbn 0-7394-5847-7.

20. Schiffer, M. "The Interplay between Gravitation and Information Theory". Proc. *Workshop on Physics and Computation*, Dallas TX, 1992.

21. Wheeler, J.A. *"Geons, Black Holes, and Quantum Foam"*, pp. 300-301. 1998

22. Vedral, V. "In from the Cold". *New Scientist.* 13 October 2012, p.33 .

23. We would like to give due credit to any who earlier have proposed a link between entanglement and gravity, and would appreciate information that might help.

24. Lluís Masanes, Markus P. Müller, Remigiusz Augusiak, and David Pérez-García. "Existence of an information unit as a postulate of quantum theory". arXiv.org/pdf/1208.0493v2.pdf; 23 October 2013.

25. Ekaterina Moreva, Giorgio Brida, Marco Gramegna, Vittorio Giovannetti, Lorenzo Maccone, Marco Genovese. "Time From Quantum Entanglement: An Experimental Illustration". arxiv.org/abs/1310.4691; 17 Oct 2013

See www.tauquernions.org for other papers by the authors.

Appendix I

The Standard Model in $\mathbb{Z}_3\ \mathcal{G}_3$

The $\mathbb{Z}_3\,\mathcal{G}_3$ Standard Model presented in this Appendix is in support of the preceding text, which provides algebraic context and other necessary details not found here, ie. this Appendix is *not* self-contained.

Our knowledge of the $\mathbb{Z}_3\,\mathcal{G}_3$ algebra has a strong empirical flavor, born of the fact that it takes only about eight seconds to search the entirety of $\mathcal{G}_3$ ($6581 = 3^8$ elements, versus days to weeks with $\mathcal{G}_4$), so instead of isolating abstract groups and proving theorems about their properties and inter-relationships, we just calculate and display all the expressions of interest. We can assure the reader that this Appendix rests on a thorough census of the forms in $\mathcal{G}_3$.

To the reader who would see actual abstract group elements paired off with elements of the algebra in accordance with the well-tested tenets of quarkology, we must plead ignorance. Thus the finer details of particle types and interactions, which *all* work out very nicely, are the algebra's hand at work - we have not attended to such things, nor needed to. While the presentation in the following pages more or less exhausts our knowledge of the subject, given the precision with which the algebra nails all the categories, *and* their details, plus the isomorphism between $\mathcal{G}_3$ and the Pauli algebra, we trust that any discrepancies will turn out to be technical and non-contradictory.

In the classifications that follow, the general reasoning is:

- $\mathbb{Z}_3\,\mathcal{G}$ is an algebra of *distinctions*, and every singleton $xy, xyz, wxyz, \ldots$ expresses a logical *xnor*, the negative of *xor*. Either way, it's the same/different distinction that is effected, and being in $\mathbb{Z}_3 = \{0, 1, -1\}$ ensures a binary classification over ± 1 (since never $x = 0$). This means that the $\mathbb{Z}_3$ algebra implicitly classifies all of its elements as same/different in intricate, yet minimal, combination; eg. unitary elements possess much sameness. This is another way to view an expression's information content.

- Stable particles U, V must be unitary, $U^2 = V^2 = 1$, whence their projectors are the idempotents $-1 \pm U, -1 \pm V$, whence bosons are the nilpotents ω that satisfy $(-1 \pm U)(-1 \pm V) = (-1 \pm U)(\omega)(-1 \pm V)$, thus indicating a causal sequence. Nilpotents and idempotents correspond, respectively, to the wait() and signal() synchronization primitives.

- The other classifications then follow from inner consistency and the Standard Model itself.

Excluding 1-vectors, the only three unitary forms in $\mathcal{G}_3$ are $x+y+xy$, $xy+xz$, and $x+y+z+xy+xz$, and have been found to correspond, respectively, to neutrinos, electrons, and protons (neutrons $= xyz$ protons).

1. Neutrinos:

Name	*Form*	*Vector* ($\mathcal{G}_2$)	*Signature*	*Bits*
ν	$a+b+ab$	$[---0]$	$(0,1,3),3$	1.75
ν_μ	$a-b-ab$	$[--0-]$	"	"
ν_τ	$-a+b-ab$	$[-0--]$	"	"
$\Sigma =$	$a+b-ab$	$[0+++]$	"	"
$\bar\nu$	$-a-b-ab$	$[+++0]$	"	"
$\bar\nu_\mu$	$-a+b+ab$	$[++0+]$	"	"
$\bar\nu_\tau$	$a-b+ab$	$[+0++]$	"	"
$\Sigma =$	$-a-b+ab$	$[0---]$	"	"

Although there are $2^3 = 8$ sign variants here, versus the Standard Model's six neutrinos, it turns out that in each half of the table, ***a fourth neutrino*** can be expressed as the sum of the other three. Indeed, this provides a framework for the mutation of one neutrino type into another, cf. "the solar neutrino problem".

We tentatively identify the nilpotent W and Z bosons as being of the form $x+xy$ (our only 'tentatives'), and one can imagine the sum $(x-xy)+(y-xy) = x+y+xy$, a neutrino. The forms $\imath = \pm 1 + x + xy$, $\imath^{3,6} = 1$, are also relevant.

Electrons can be formed the same way: $e = xy + xz = (x+xy) + (\tilde{x} + xz)$.

2. Electrons:

Name	*Form*	*Vector* ($\mathcal{G}_3$)	*Signature*	*Bits*
e	$ab+ac$	$[-00++00-]$	$(2,2,4),2$	4.70
$\bar e$	$-ab-ac$	$[+00--00+]$	"	"
e^-	$ab-ac$	$[0-+00+-0]$	"	"
$\bar e^-$	$-ab+ac$	$[0+-00-+0]$	"	"
μ	$ab+bc$	$[-0+00+0-]$	"	"
$\bar\mu$	$-ab-bc$	$[+0-00-0+]$	"	"
μ^-	$ab-bc$	$[0-0++0-0]$	"	"
$\bar\mu^-$	$-ab+bc$	$[0+0--0+0]$	"	"
τ	$ac+bc$	$[-+0000+-]$	"	"
$\bar\tau$	$-ac-bc$	$[+-0000-+]$	"	"
τ^-	$ac-bc$	$[00-++-00]$	"	"
$\bar\tau^-$	$-ac+bc$	$[00+--+00]$	"	"

3. Photons: $\pm x \pm y \pm z$. There are four pairs of 2 states γ, γ', which we take to be polarizations. Note that the electron projector $-1+xy+xz$ factors as $x(\tilde{x}+y+z)$; and that $\gamma\gamma' = 1 \pm (xy+xz)$. Also, $-1+xy+xz = (xy+yz+zx)(xyz)(\tilde{x}+y+z)$.

4. Mesons, Gluons, and E/M.

Like electrons, mesons too can be constructed via a 2-sum of the nilpotent $x+xy$ form, and gluons with a 3-sum. The sums that are factorable are nilpotent, and those that are not are roots of unity. We note that quarks have the form $x+yz$, and so mesons can easily consist of two quarks via rearrangement, cf. the first two items below:

- Nilpotent mesons: $\{X \mid X = (x+xz) + (y+yz) = (x+y)(1+z)\ \&\ X^2 = 0\}$
  (24) $= (x + yz) + (y + xz)$
- Massive mesons: $\{X \mid X = (x - xz) + (y + yz)\ \&\ X^2 = \pm xyz\}$
  (24) $= (x + yz) + (y - xz)$
- Gluons (48): $\{\mathfrak{g} \mid \mathfrak{g} = x + y + z + xy + xz + yz\ \&\ \mathfrak{g}^2 = \pm xyz\}$
- Electro-magnetic field: $\{E \mid E = (x + y + z) \pm xyz(x + y + z)\ \&\ E^2 = 0\}$
  (16) $= (1 \pm xyz)(x + y + z)$

Note that $xyz(x+y+z) = xy+xz+yz$ is the 3-space quaternion triple associated with the photon $x + y + z$, while $\pm xyz$ is the charge carrier. The last two items have the same form, differing only via charge vs. nilpotence. All four are eigen forms of xyz.

5. Quarks

The quarks are the only case where the $\mathcal{G}_3$ algebra at first seems insufficient, in that while the $x + yz$ form correctly exhibits three families of 2×2, with spin $(\pm xy, \pm xz, \pm yz)$ and charge ($\pm\frac{1}{3}$ or $\pm\frac{2}{3}$ on x, y, z), in doing so it seems to use up all of its information carrying capacity, and then some, and so be unable to express as well the three colors quarks also can have.

It is appropriate therefore to enquire how a single 1-vector like x might even be ***said*** to carry ***both*** $\pm\frac{1}{3}$ charge ***and*** a color designation, especially since it carries only one bit of information. The answer is that x itself carries only the $\pm$ distinction, one bit. The "$\frac{1}{3}$" is ***our imputation*** of x's contribution to a larger pattern, and indeed the $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ charge-addition business is clearly the space-like non-Shannon information contained in a 3-co-occurrence, cf. the Coin Demonstration, where the answer to the question "is there electro-magnetism" is answered when the third coin is revealed.

Similarly, "color" is ***our*** way of distinguishing x from y from z, which is meaningful only when >1 are present. Since quarks and their colors appear only when there are either two (mesons) or three (hadrons, gluons) quarks present, so then also are the requisite co-occurring x, y, z's present. So we conclude that it is permissable to associate with each of x, y, z both a charge and a color.

We can encode the "colors" red, green, blue (r, g, b) as

$$\begin{array}{ccccccc} r & g & b & r+g & r+b & g+b & r+g+b \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ a & b & c & a+b & a+c & b+c & a+b+c \end{array}$$

Thus both charge and color are emergent, co-occurrence-based, non-Shannon distinctions. The finishing touch is that a particle and its anti-particle must sum to zero, including both charge and color. We then get the following table of quarks: [29]

Name	U	D	$\bar{U}$	$\bar{D}$
Form	$a+bc$	$-a+bc$	$-a-bc$	$a-bc$
Charge	$+\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$+\frac{1}{3}$
Color	r	$\bar{r}$	$\bar{r}$	r

Name	C	S	$\bar{C}$	$\bar{S}$
Form	$b+ac$	$-b+ac$	$-b-ac$	$b-ac$
Charge	$+\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$+\frac{1}{3}$
Color	g	$\bar{g}$	$\bar{g}$	g

Name	T	B	$\bar{T}$	$\bar{B}$
Form	$c+ab$	$-c+ab$	$-c-ab$	$c-ab$
Charge	$+\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$+\frac{1}{3}$
Color	b	$\bar{b}$	$\bar{b}$	b

6. Hadrons; Protons and Neutrons

$\mathcal{G}_3$ contains exactly three compound unitary forms X such that $X^2 = 1$. These are $x+y+xy =$ neutrinos, $xy+xz =$ electrons, and now the largest of these, the 96 hadron forms $x+y+z+xy+xz$, which square to either $1 \pm xyz$ or $+1$, 48 of each. Each 48 divides into three groups of 16, depending on which of the three possibilities $xy+xz$ occurs. By inspection, in the $X^2 = +1$ half, there are three sub-families, made up from the three families of quarks. Of the 16 in one such, the $8+8$ are each two photon polarizations γ and γ', the 8 dividing as $4+4 = 2\times 2\, +\, 2\times 2$, these being the 'conjugate' forms $\gamma \pm (xy+xz)$ and $\gamma \pm (xy-xz)$, and $\gamma' \pm (xy+xz)$ and $\gamma' \pm (xy-xz)$.

[29] Except for U and D, the entries in these tables were not assigned with any particular knowledge of how they are to correspond to real particles.

We saw earlier how the mesons can be constructed from two $x+xy$'s, and in so doing deftly confine the quarks so formed to a minimum presence of two. The same construction can be applied to the hadrons, which are then the sum of three $x+xy$'s, rearranged to make three $x+yz$'s.

In particular, protons are UUD and neutrons UDD, that is, $p = 2U + D$ and $n = 2D+U$. Subtracting these, $p-n = p+\bar{n} = 2U+D - 2D-U = U+\bar{D}$, ie. "$U\bar{D}$", a quark and an anti-quark, ie. a meson. Clearly, $n-p =$ "$\bar{U}D$", which symmetry is appropriate for an exchange particle like a meson. And indeed, the quark model stipulates that mesons be (the sum of) a quark and an anti-quark.

Unfortunately, in our $\mathbb{Z}_3$ algebra, $2U = \bar{U}$, so "count to 2" *also* means the "anti" distinction, and thus we cannot express the UUD vs. UDD distinction as things stand. Fortunately, we can move to $\mathbb{Z}_5 = \{\tilde{2}, \tilde{1}, 0, 1, 2\}$ and still remain in $\mathcal{G}_3$. [30] Being now able to count to 2, the quark model is straightforward. Let $U = a+bc$ and $D = -a+bc$. Then, with $\mathbb{Z}_5$ arithmetic,

$$\begin{gathered}
p = 2U + D = 2(a+bc) + (-a+bc) \\
n = U + 2D = (a+bc) + 2(-a+bc). \\
\text{whence} \\
p - n = (2a + 2bc - a + bc) - (a + bc - 2a + 2bc) \\
= a + 3bc - (-a + 3bc) = 2a = (a+bc) + (a-bc)
\end{gathered}$$

$$= U + \bar{D} \quad = \text{"}U\bar{D}\text{"}$$

just as required; and we note that our proton $p = UUD$ has charge $\frac{4}{3} - \frac{1}{3} = +1$ and our neutron $n = UDD$ has charge $\frac{2}{3} - \frac{2}{3} = 0$. [31]

The success of the shift from $\mathbb{Z}_3$ to $\mathbb{Z}_5$ to clarify the quark model encourages the thought of $\mathbb{Z}_7$ for $\mathcal{G}_4$. This would emphasize the $0\,mod\,4$ cycle, which expands into itself: in the hierarchy of these algebras, they all will be $\mathcal{G}_{0\,mod\,4}$ because, abusing notation, $\delta(\mathcal{G}_{0\,mod4} + \mathcal{G}_{0\,mod\,4}) = \mathcal{G}_{0\,mod\,4}$. We believe this to be a black hole structure in the limit.

But in the first instance this leads to $\mathcal{G}_8$, octonions, and the exceptional Lie group E_8, well-known to string theorists. Perhaps $\mathbb{Z}_{11} = \{\tilde{5}, \tilde{4}, \ldots, 0, \ldots, 5\}$ is the right lens for $\mathcal{G}_8$. [32] The primes $3, 5, 7, 11$ appear initially for their

[30] We defer the interesting foundational question raised here, and instead take the pragmatic view that while Nature knows what it's doing, we need help focusing, and the shift to $\mathbb{Z}_5$ keeps the focus sharp.

[31] It is an interesting exercise to examine how the $\mathbb{Z}_3$ encoding of p (which of course must be equivalent) compensates for its inability to count to two by adding in extra and/or intertwined distinctions. Thus with $p = a+b+c+ab+ac = a+(b+ac)+(c+ab)$, two (non-$U, D$) quarks appear, and the a-distinction is decisive.

[32] On the other hand, we are not fans of octonion multiplicative non-associativity [11].

symmetry around 0, but as well, their self-identifying property correlates with the idempotent forms $\pm 1 \pm x_1 x_2 \ldots x_m$ of the corresponding level m, which in turn are the similarly self-identifying computational primitives ***signal(event)*** [11].

Returning to the $\mathbb{Z}_3$ algebra, we note that the proton form is also the sum of a photon and an electron. Consider now, in idempotent form, an electron $e = -1+xy+xz = x(\tilde{x}+y+z) = x\gamma$, and a proton, $p = -1+(\tilde{x}+y+z)+(xy+xz)$, which factors as $(\tilde{x}+y+z)+x(\tilde{x}+y+z) = (1+x)(\tilde{x}+y+z)$. Then

$$ep = (-1+xy+xz) \times (-1+\tilde{x}+y+z+xy+xz) = -1+xy+xz = e$$

$$= x(\tilde{x}+y+z) \times (1+x)(\tilde{x}+y+z)$$

$$= (\tilde{x}+\tilde{y}+\tilde{z})x \times (1+x) \times (\tilde{x}+y+z)$$

$$= (\tilde{x}+\tilde{y}+\tilde{z}) \times (1+x) \times (\tilde{x}+y+z)$$

$$= (\tilde{x}+\tilde{y}+\tilde{z})(1+x) \times x(\tilde{x}+y+z)$$

$$= (\tilde{x}+\tilde{y}+\tilde{z})(1+x) \times (\tilde{x}+\tilde{y}+\tilde{z})x$$

$$= (-1+\tilde{x}+\tilde{y}+\tilde{z}+xy+xz) \times (-1+xy+xz)$$

$$= p'e \qquad \textit{and} \qquad pe = e'p = p$$

where we note that the phase of the photon in p has changed from $\tilde{x}+y+z$ to $\tilde{x}+\tilde{y}+\tilde{z}$ in p'. So, even though the state $ep = p'e = e$ is nominally fixed (since the idempotents are irreversible) and officially static – it's what *has* happened and no more has happened *yet* – we see [tracing the movement of x] that there is a natural, reversible, electro-magnetic oscillation, or if you like, an indeterminacy of state, in the electron-proton interaction that is consistent with our identification of the photon, electron and proton forms.

Finally, the reader should note that *summing*, using which we have here described the build-up of the Standard Model's structure, ie. *co-occurrence*, is the entropically favored pathway for combining terms. However, the actual expansion is much more complicated than merely summing $x + xy$'s as we have done for expository purposes, which is, rather, simply a limited application of a spectral basis.[33]

[33] The general existence of a spectral basis for $\mathcal{G}$ is an open question.

Appendix II

The Combinatorial Hierarchy

[Continuing from the end of §8:]

There is one last point we wish to make concerning the generation of G_{i+j} from $G_i \times G_j$. Let $A = \{1, a\}$, whence we are in G_1. A-space is $\pm 1 \pm a \Rightarrow 2^2 = \mathbf{4} = 2^{2^1}$ states. Now let $B = \{1, b\}$. Then

$$A \times B = \{1, a, b, ab\}$$

and the resulting space is of size $2^4 = \mathbf{16} = 4^2 = 2^{2^2}$. The next step is

$$\{1, a\}\{1, b\}\{1, c\} = \{1, a, b, c, ab, ac, bc, abc\}$$

which is of size $2^8 = \mathbf{256} = \mathbf{16}^2 = 2^{2^3}$. Next is $\{1, a\}\{1, b\}\{1, c\}\{1, d\} =$

$$\{1, a, b, c, d, ab, ac, bc, ad, bd, cd, abc, abd, acd, bcd, abcd\}$$

which is of size $2^{16} = \mathbf{256^2} = 65538 = 2^{2^4}$.

The sequence of space-sizes increases as the square, $4 \to 16 \to 256 \to 256^2$, because of course $2^{2^n} = 2^{2^{n-1}} 2^{2^{n-1}}$. At the same time, the number of elements in these spaces (a subset of S) is growing even faster, and these two sequences are related. Table 6 shows the generation process, and the intertwining of the two sequences is visible in the related powers of 2 that appear.

In the early ($\mathbb{Z}_2$) analysis [1] of this construction - the *Combinatorial Hierarchy* - it was understood in terms of the state vectors of one level being stacked to make square matrices, which matrices had to be capable of mapping the resulting next-level space onto itself. The intriguing aspect then is that while the matrix, being a stack of basis vectors, exists for $n = 1, 2, 3$, at $n = 4$ the number of co-occurrences explodes, and the $((256)^2)^2 = 2^{32}$ basis vectors are completely swamped by the 2^{127} co-occurrences they should map among. That is, 4 covers 3, 16 covers 7, and 256 covers 127, but then it's over. So the construction halts, or must begin anew, or, at least, something *new* has to happen, seemed to be the message back then.

$Lvl = n = generators \rightarrow$	G_1	G_2	G_3	G_4
# terms	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$	$2^4 = 16$
Full state contents G	$\pm 1 \pm a$	$\pm 1 \pm a \pm b \pm ab$	$\pm 1 \pm a \pm b \pm c \pm$ $\pm ab \pm ac \pm bc \pm abc$	$\pm 1 \pm a \pm b \pm c$ $\pm d \pm ab \pm ac \pm bc$ $\pm ad \pm bd \pm cd \pm abc$ $\pm abd \pm acd \pm bcd \pm abcd$
$\lvert G \rvert = 2^{2^n} = (2^{2^{n-1}})^2$	$2^{2^1} = 2^2 = 4$	$2^{2^2} = 4^2 = 16$	$2^{2^3} = 16^2 = 256$	$2^{2^4} = 256^2 = 65538$
Occurrences $S_G = \sum_1 \binom{k}{j}$	$\sum_1 \binom{2}{j} = 3$	$\sum_1 \binom{3}{j} = 7$	$\sum_1 \binom{7}{j} = 127$	$\sum_1 \binom{127}{j} = 2^{127} - 1 \approx 10^{38}$
	$\{1, a, 1 + a\}$	$\{a, b, a + b \xrightarrow{\delta} ab,$ $a + ab, b + ab, a + b + ab\}$	$\{a, b, c, ab, ac, bc,$ $a + ab, a + ac, \ldots\}$	$\{a, b, c, d, ab, ac, ad, bc, bd,$ $cd, a + ab, a + ac, a + ad, \ldots\}$

Table 6: *The sequence* $3, 7, 127, 2^{127} - 1$ *is the Combinatorial Hierarchy, CH* [1,2].

[Three brief comments: (1) S_G is that part of S that corresponds to G's alternations; (2) the bottom two rows of the table show only + variants because the signature collapses all sign variants to the same bin; and (3) the base of the combinatorics, 2-ary distinctions, is the one that generates the most structure: 3- and 4-ary distinctions cut off sooner, and 5-ary doesn't even get off the ground [1].]

The present ($\mathbb{Z}_3$) perspective sees something *new*: the line that is crossed is the one that separates localizable effects from distributed ones, ie. weak, strong, and electromagnetic from EPR and gravity. Either way, the cut-off occurs with consistent and physically meaningful interpretations, and it seems clear that the two instances of the *CH* ($\mathbb{Z}_2$ and $\mathbb{Z}_3$) are both isomorphic and being imbued with the same physical import.

Finally, the observations that $3+7+127 = 137 \approx \frac{1}{\alpha}$, α being the fine structure constant, and that $3+7+127+2^{127} \approx 10^{38}$ roughly approximates the electromagnetism : gravity ratio, plus the above-described interpretation, led Bastin and Kilmister to refine this purely combinatorial approach to $\frac{1}{\alpha}$. Their most recent result [2] calculates this to 137.036011393. vs. the measured 137.035999710(96). We note that Bagdonaite *et alia.* report [4] that the proton-electron mass ratio has not varied in the past 7 billion years.

References

1. Amson, J.C. (with A.F. Parker-Rhodes), "Essentially Finite Chains", International Journal of General Systems, Vol.27, No.1-3, pp. 81-92 (1998, original paper dated 1965).

2. Bastin, T. and Kilmister, C. *The Origin of Discrete Particles.* World Scientific, Series on Knots and Everything, L. Kauffman, Ed., v. 42, 2009. Isbn-13: 978-981-4261-67-8.

3. Bastin, T. London *Times* obituary. Nov. 17, 2011. http://trove.nla.gov.au/work/159129448.

4. Bagdonaite, J. *et alia.* "A Stringent Limit on a Drifting Proton-to-Electron Mass Ratio from Alcohol in the Early Universe". Science v. 339 4 Jan 2013.

In Memoriam

Parallel with Alan Turing's well-known bio-computational interests, Ted Bastin (and his colleagues Frederick Parker-Rhodes, Clive Kilmister, and John Amson) and Carl Adam Petri (and his colleagues in Germany) were the first (to my knowledge) to devote themselves to a thorough-going *computational* understanding of physical reality. One can rightly say that my work has partaken of both of these efforts. Ted, holder of PhDs in both mathematics and physics from Cambridge University, died on October 15, 2011 at the age of 85. He was very ill when the preceding March I sent the email below, circa two weeks after the discovery.

From: Michael Manthey
Date: Tue, 22 Mar 2011 15:18:19 -0600

Subject: 2 & 3 —> 3D @ lvl 4 !
To: ted.bastin@mistral.co.uk

Dear Ted-

I have found something that I thought you would like to know about, if you still care about such matters. Namely a unique origin for 3D at level four, BUT *surprise*, the 3D space is *dissipative*, ie. any motion in that space costs. So the entropy is built in! The 3D-ness is just like the quaternions, except that the elements are all irreversible, and the "unit element" is the idempotent. I have dubbed these things "tauquernions" !

In my usual Clifford algebra notation where, xx=1 = yy, and xy = -yx, etc:

Let X = ab+cd, Y = ac-bd, Z = ad+bc. These are the epr operators that Doug Matzke uncovered in his thesis. It turns out that

XY = Z, YZ=X, XZ=Y

and XY = -YX YZ = -ZY XZ = -ZX

and XYZ = -ZYX,

and (X+Y+Z)^2 = (ab+ac+ad+bc+bd+cd)^2 = 0

which latter is likely the Higgs boson. The triplet X,Y,Z is unique in the algebra, so each quantum mass unit 'abcd' has its own personal 3D axis system. Such units must now be inter-connected to form the space.

Notice that what we have in X+Y+Z is *not* two sets of quaternions, but rather one set, plus a photon a+b+c: ab+ac+bc + d(a+b+c).

So the emergence of the 3D is *both* at level two, as you have always argued, but as well, when the elements of this level, expanded to contain all 6 level two items of a 4 level system, these six are grouped *again* in three's, two by two, to yield the 3D space (X,Y,Z). [not a sentence!]

I haven't shared this with hardly anyone yet, so please - this is just for you.

My very warmest regards,

-mike

Electro-Magnetism Is

A Fundamentally Fractal Fibonacci Fourier Field

by

Michael Manthey
manthey@acm.org

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Abstract. We offer a purely computational and combinatorial explanation for Coldea *et al.*'s 2010 report of having measured the golden mean in a quantum system. Our method employs the roots of projectors (in the discrete and finite geometric algebra $\mathcal{G}_{5,0}$) to capture both the dimensionality (*3+1*d) and the detailed structure of the electro-magnetic field, including Majorana fermions (with many details). The pattern of growth as the field expands from its source displays the Fibonacci sequence $F = 1,1,2,3,5,\ldots$, where $\lim\limits_{n\to\infty} \frac{F_{i+1}}{F_i} = \varphi$, the golden mean. The Fibonacci sequence and the golden mean are thus guiding principles at and from the very root of our universe.

Keywords: Fibonacci, golden mean, distributed, self-organizing, topological, computational, combinatorial, hierarchical, fractal, systems, Fourier, Parseval, coordinate free, geometric algebra, tauquinion, tauquernion, enquernion, entangled quaternion, Majorana.

0. Introduction - Quantum Systems Are Distributed Systems

A long-standing interest in computer science is how to define and construct *distributed* systems - systems consisting of many more-or-less independent asynchronous computational processes distributed over a number of hosts, that carry out coherent system-wide computations with little or no central coordination. Due to their complexity, there is furthermore much interest in making distributed systems that are *self-organizing*. After all, Nature is unboundedly complex, and we can't write code for every eventuality.

Atoms, crystals, beehives & anthills, ecologies, weather systems, indeed all the works of Nature are such distributed systems, but despite their ubiquity and familiarity, the fundamental principles governing their design and operation are subtle and elusive. It's as if the whole universe were running on an invisible global operating system, one whose goals are namely (like any good operating system) to get everything done, and to be invisible while doing so.

In particular, quantum systems are distributed systems, and in this paper we apply our self-organizing distributed system analysis tools to Coldea *et al*'s finding [1,10], and expose its underlying computational mechanism.

Our algebraic representation of computation - using the real geometric (Clifford) algebras $\mathcal{G}_{n,0} = \mathcal{G}_n$ - shows that the defining property of distributed systems is that they are $wave-like$, in that, conceptually, a wave is everywhere ... and yet, simultaneously, not in any one place in particular. Being wave-like means that distributed systems are *space-like*, viz. rotation around a circle, the *plane* and not the *line* (= individual sequential processes).

Sequential processes, such as those generated by typical programs, are, in contrast, *time-like*. These (so-to-speak) wend their way through the above "distributed space", *a la* relativity theory's reference frames. We have little more to say about sequential processes here, but see [6].

The various special properties of our algebraic representation of distributed computation - its discreteness yields combinatorics and structure, its graded hierarchy collapses structural complexity, the tauquinions (see below) specify field structure, and more - allow us to address the phenomenon observed by Coldea *et al.* in a novel manner.

The computational interpretation that we place on our algebra is simple: *concurrent processes - being independent - are considered to be orthogonal to each other.* Two concurrent 1-bit-of-state processes a,b are written as $a+b$, and it turns out (unobviously from the present explanation) that the ensuing multiplicative anti-commutativity, $ab = -ba$, exactly tracks everything that happens (bit by bit, as 'twere) when two such processes are co-present. We thus build more complex processes (like ab) from these 1-bit primitive processes; that is, every expression in the algebra is either a process or built from same (and some expressions are more interesting than others).

We draw the algebra's scalar coefficients from $\mathbb{Z}_3 = \{0,1,2\} \mapsto \{0,1,-1\}$ for nine reasons (so far):

1. The binary feel of ± 1 is useful, eg. it makes ab into an *xor* (at the scalar level), and dovetails nicely with information theory's requirements;
2. $\mathbb{Z}_3 = \{0, 1, -1\}$ means *no counting*, thus subverting sequentiality at its very root. Counting is replaced by simple *distinction*: same or different. Structure comes from the distinction *co-occur/exclude* on process states [7, §7.2];
3. Generality requires the simplest possible atoms - 1-bit processes;
4. Zero no longer wears two hats - *Void* and "the opposite of 1" - as it does in the usual $\mathbb{Z}_2 = \{0, 1\}$ binary system;
5. The introduction of a minus *number* into the basic algebra (versus $\mathbb{Z}_2$) [1] makes the transition to geometric algebra's vector world easy;
6. Geometric algebra's various physics-relevant isomorphs (eg. the quaternion, Pauli, Grassmann, and Dirac algebras), when expressed in the extremely minimal $\mathbb{Z}_3$ algebra, get a *very* tight fit, eliminating most questions of correct interpretation;
7. This same minimality makes exhaustive searches of $\mathcal{G}_3$ and $\mathcal{G}_4$ possible, and such searches are our main source of data, eg. for entropy calculations [7];
8. The extreme minimalism imposed by $\mathbb{Z}_3$ promotes the exhibition of much symmetry that is implicit and hidden in the turmoil of *multiplicities* of identicals that one finds in larger number systems; [2]
9. Since physical three-ness is both common and deep (three spatial dimensions, three particle generations, three quark pairs, etc.), the match with $\mathbb{Z}_3$ further focuses the algebra's precision of expression.

Further discussion of the algebra appears below.

Finally, because our analysis is purely combinatorial, over a universe of arbitrary processes, it is independent of any particular physical theory.

0.1 *Earlier Work*

This paper is logically an addendum to [7], in that it follows directly from, and fits directly into, this prior work. In that work, we identified a novel set of isomorphs $\mathcal{T} = \{ab - cd, ac + bd, ad - bc\}$, dubbed *TauQuernions*, to the classical *quaternions* $Q = \{ab, bc, ca\}$ - the very definition of 3d space. The tauquernions are novel not only because they are new on the scene, but rather, especially, because they are $time-like$.

That is, tauquernions describe 3d space in *exactly* the same way quaternions do, *but* while $(ab)^4 = +1$, one gets $(ab - cd)^4 = -1 - abcd$ for the corresponding tauquernion. [3]

[1] How do you want your $\mathbb{Z}_2$ negatives - sign-magnitude (oops, *minus* zero); 1's complement (oops, *two* zeroes), 2's complement (huhhh??) ? It's hard to think of any of these as direct representations of anything physical.

[2] Coldea *et al*: "Our results demonstrate the power of symmetry to describe complex quantum behaviors."

[3] Note that $(ab)^4 = +1$ and $ab = -ba$ together imply that $ab \cong \sqrt{-1}$, the imaginary unit i.

$(-1 - abcd)^2 = -1 - abcd$ is idempotent, just like +1, so we're in a dual algebra ... and *in* that dual place, "+1"$= -1 - abcd$ is *also* a projector, a measurement operator. That is, it is $time-like$, so when 3-space is constructed using tauquernions, the quaternions' 3d becomes $3+1$d automatically, with the tauquernions' irreversibility showing up as global entropy growth.

Applying tauquernions to the construction of relativity theory's 3+1d world of space, time, gravity, mass, and entropy yields a very good fit. Furthermore, the tauquernions are *also*, it turns out, the Bell and Magic entanglement operators of quantum mechanics [8], leading to the conclusion that the mechanism of gravity is the quantum entanglement of space itself.

Further investigation of the mathematical structure of the tauquernions eventually produced the realization that is the tool of this paper: that there is *exactly one more* example of a tauquernion-type operator, namely a *TauQuinion*, $ab + cde$.

Like tauquernions, *tauquinions* are also 4^{th} roots of a projector, $-1 \pm abcde$ in this case, and similarly mutually perpendicular. [So we're now in $\mathscr{G}_5$ over $\mathbb{Z}_3$.] Like tauquernions, the tauquinions are also quaternion isomorphs, and so similarly form an implicitly $3+1$d coordinate system - this being the "field" of that for which the projector probes (or, alternatively, the field in which it probes). [We note that for us, a *field* is a coordinate system - which is $\mathscr{G}_5$ itself, it being coordinate-free - having one or more properties assigned to each of its points.]

In the case of the tauquinions, their structure can be reduced to a $3^2 - 1 \Rightarrow 8 \times 8$ group table, and anticipating various other features explained in later sections, we believe that despite apparent differences, the tauquinion group $\mathcal{T}$ is either a representation of $SU(3)$, or contains it. [4]

Appendix I of [7] derives the Standard Model's particle structure almost mechanically from the combinatorics of $\mathscr{G}_3$, which is isomorphic to the Pauli algebra via the mapping $\{iab, iac, ibc\} \mapsto \{\sigma_1, \sigma_2, \sigma_3\}$. This and the smooth way that tauquernions (elements of $\mathscr{G}_4$) build the bridge between QM and GR encourage us to believe that our mathematics connects very well to physics at the level of information.

We show that tauquinions are the *only* other possible quaternion-like form in $\mathscr{G}_n$. The tauquernion field by itself - which yields $3+1$d spacetime, mass, gravity etc. - does not contain any details beyond a photon $(a+b+c)$ and its local coordinate (ie. $abc(a+b+c) = ab+bc+ca$, a quaternion triple). So where else in the algebra could the electromagnetic field be? Spinors (2-vectors) are clearly to be associated with magnetism, and we identify 3-vectors like abc as the carriers of electric charge [7], which $2+3=5$ structure fits tauquinions and $\mathscr{G}_5$ nicely.

Also, one must not forget that this is a *combinatorial* model over *arbitrary* processes, and so it is *in principle* theory-neutral. If $\mathcal{T}$ does *not* contain $SU(3)$, one is forced to consider that it is $SU(3)$ that is off, which is hugely unlikely. Howsoever, either $SU(3)$ is in $\mathcal{T}$ or it is not, and either answer will be interesting.

[4] We recycle our prior use of $\mathcal{T}$ for tauquernions.

A note on terminology. I have always been suspicious of people who make up new words, but it seems justified in the present case. The *tauquernions* $\{ab+cd\}$ are time-like isomorphs of the classical quaternions (which have just one spinor component), and play the starring role in forming $3+1$d spacetime from the quantum mechanical soup [7]. Surely this deserves its own name.

Then some time later, up popped the *tauquinions* $\{ab+cde\}$, quaternion isomorphs with a five-ness (L. *quinque*), that are the star of this paper.

Tauquernions and tauquinions could collectively be called *entangled quaternions*, which indeed they are; and also *entangling*. And since both are irreversible (*tau* $\Rightarrow$ time-like) this could be shortened to *enquernions*, in that the Old Norse root of *quern* is *kværn*, meaning to churn ("the churn of time") or grind ("time's tooth"), which is very apt.

0.2 *Notation*

Our mathematics is that of the canonical *geometric (Clifford) algebras* $\mathscr{G}_{n,0}=\mathscr{G}_n$ over $\mathbb{Z}_3=\{0,1,-1\}$, whence $1+1=-1$. Such an algebra is generated by a set of 1-vectors $\{a,b,c,...\}$ with anti-commutative product $xy=-yx$ to produce an orthogonal space of size $|\mathscr{G}_n|=O(2^n)$ with inner and outer products. The usual distributive and associative laws apply. The dimensions of this [coordinate-free] space are all the possible products $\binom{\{a,b,c,...\}}{0,1,2,...,n}=\{1,a,b,c,\ldots,ab,ac,\ldots,abc,abd,\ldots,\ldots\}$, which are all mutually orthogonal. See [2,3,4,5] for foundations (although our *interpretation* of the algebra is vastly different).

Parseval's Identity [5] applies, so this is a phase space, namely Fourier space. Every expression in $\mathscr{G}$ is thus the Fourier decomposition of some signal entering the system via the concurrent flipping of some set of 1-vectors at the system boundary.

1-vectors denote discrete processes with two states = 1 bit of information; it follows that an m-vector ("pseudo-vector", "singleton") contains 2^m states. A 1-bit process x is necessarily deterministic, since in each state, there is only the other state to change to. [This is where frequency ν_x attaches.]

Generic *concurrent* 1-bit processes x,y (written $x+y$) are considered to be *orthogonal*, which is reflected in the algebra by its anti-commutive product. Time-like sequential processes are represented as products of idempotents and are non-deterministic (as is the entire model). Usually, the expressions we write, eg. $(a+b+ab)^2=1$, are valid for all sign variants $\pm a\pm b\pm ab$, each thus constituting a little theorem in itself.

0.3 Roots of Projectors

Projectors (measurement operators) are idempotents, and have the form $\hat{U}=-1\pm U$, whence $\hat{U}^2=\hat{U}$ if $U^2=+1$, ie. if U is unitary. [6] As with scalar +1, the principal square root of $\hat{U}$ is its negative, ie. $\sqrt{\hat{U}}\mapsto-\hat{U}$.

[5] " Given an orthogonal space S with inner product, the projection of any function $\mathscr{F}$ onto S is the Fourier decomposition of $\mathscr{F}$ ". The Identity is a generalization of the Pythagorean theorem to n dimensions. It is also, for us, both wave-particle duality in a nutshell, and the bridge between the discrete and the continuous.

[6] In $\mathbb{Z}_3$: the -1 in $\hat{U}$ corresponds to decimal $\frac{1}{2}$, which latter one finds in the base-10 version of a projector.

As noted earlier, in our approach $\hat{U}$= "+1" plays a role dual to the scalar $+1$, eg. $(-1+abcde)(ab+cde) = ab+cde$. Likewise, $\sqrt{-\hat{U}} = ab+cde$, being namely an enquernion, plays the role of "i", and $i^2 = -\hat{U} = \sqrt{\hat{U}} = +1 \mp abcde$, which is simultaneously "-1": $(+1-abcde)(ab+cde) = -ab-cde$. Their combination of reversible additive inversion (= 180^o rotation) and multiplicative irreversibility is *the* key property of the enquernions, and is the source of their power.

Table 1 displays this roots-of-projectors structure.

scalar	*quaternion*		*time-like*	*tauquernion*	*tauquinion*
$+1$	$+1$	point	$\text{“}+1\text{”} = -1 \pm U$	$-1 \pm abcd$	$-1 \pm abcde$
$\Updownarrow$	$\Updownarrow$		$\Updownarrow$		
$-1 = \sqrt{+1}$	$-1 = \sqrt{+1}$	line$\pm$	$\text{“}-1\text{”} = \sqrt{\text{“}+1\text{”}}$	$+1 \mp abcd$	$+1 \mp abcde$
$\Updownarrow$	$\Updownarrow$		$\Updownarrow$		
$i = \pm\sqrt{-1}$	$ab = \pm\sqrt{-1}$	plane(s)	$\text{“}i\text{”} = \pm\sqrt{\text{“}-1\text{”}}$	$ab+cd$	$ab+cde$

Table 1: Idempotent Isomorphs of $+1$ and $\hat{U} = -1 \pm U$, $U^2 = +1$.

The first column is the usual $i = \sqrt{-1}$ story. This same vertical progression, in a vector space, yields a spinor/quaternion ab (second column). A 3d rotation in $\mathcal{G}_3$, $abc(a+b+c) = ab+bc+ca$, demonstrates the equivalence of a quaternion triple to a 3d space with axes a, b, c. Like a, b, c, the three quaternion elements anti-commute.

The third column indicates the progression of geometric concepts that appears in this process, and also divides the table in half, in that the i's to the left are reversible, ie. have multiplicative inverses, while those to the right do not, thus making them time-like (in addition to being space-like) rotation operators. The enquernion i's to the right are, spatially speaking, isomorphic to the quaternions' i's (as we will demonstrate later), but provide a new and unique twist, namely that even though having no inverse, they yet perform the requisite anti-commutative space-like rotations that make them quaternion isomorphs.

Another way to look at enquernions is that they connect a quantum level change to an *exactly* equivalent $3+1$d change, via the identity

$$(-1)(ab \pm cde) = (1 \mp abcde)(ab \pm cde)$$

On the left is a reversible change, on the right an irreversible one.

It seems therefore that the roots of projectors can specify the formation, structure and state of the field associated with the unitary entity $abcde$. We pursue this thought in §2.

0.4 Hierarchy

The algebra has the recursive (and hence hierarchical) property, that its semantics cycle exactly like the powers of i: $\{++--++--\ldots\}$ as the grade of its (pseudo-)vectors I increases:

grade	$I \in \mathcal{G}_n$	notation	I^2
...	...	...	...
10	...	A_{10}	-1
9	*abcdefghj*	A_9	$+1$
8	*abcdefgh*	A_8	$+1$
7	*abcdefg*	A_7	-1
6	*abcdef*	A_6	-1
5	*abcde*	A_5	$+1$
4	*abcd*	A_4	$+1$
3	*abc*	A_3	-1
2	*ab*	A_2	-1
1	*a*	A_1	$+1$
0	1	A_0	$+1$

Note particularly that the sequence repeats every four levels, so (eg.) m-vectors of grades 1 and 5 have identical properties. This we exploit in the following, as the means by which a field propagates from its initial locality to a global presence, all the while and everywhere retaining its defining local properties. [7]

Finally, we use the boundary ∂ and co-boundary δ operators to define and build the algebra's graded hierarchy of m-vectors. There is a deep analogy between ∂ as a boundary operator and the differentiation operation of the calculus, and similarly between the co-boundary operator δ and integration.

We take ∂ and δ to be elements *of* the algebra - rather than the usual operators *over* the algebra - this being a less sophisticated but more concrete encoding of the same ideas, thus the definition

$$\delta_Q X = \pm Q \quad \textit{iff} \quad \partial_X Q = XQ \quad \textit{and} \quad XQ \cong X$$

This definition specifies that a boundary X of Q must be an *eigen-form* of Q for the analogy to hold, and it is not difficult to show that it requisitely satisfies $\partial\delta = 1 = \delta\partial$ and $\partial^j = 0 = \delta^k$. We use δ to build hierarchy, eg. $\delta_{ab}(a+b) = ab$, since $\partial_{a+b}ab = (a+b)ab = -a+b \cong a+b$.

See [7] for more.

1. The TauQuinions are isomorphic to the classical quaternions.

The classical quaternions have the following multiplication table:

$\times$	$Q_i = ab$	$Q_j = ac$	$Q_k = bc$
Q_i	-1	$-bc$	ac
Q_j	bc	-1	$-ab$
Q_k	$-ac$	ab	-1

=

$\times$	Q_i	Q_j	Q_k
Q_i	-1	$-Q_k$	Q_j
Q_j	Q_k	-1	$-Q_i$
Q_k	$-Q_j$	Q_i	-1

The respective tauquinions are $\tau_i = ab + cde$, $\tau_j = ac - bde$, $\tau_k = bc + ade$. Their multiplication table is below left; below right is the same table, but with the mapping $1 - abcde \mapsto$ "-1".

[7] There is also a *mod* 8 cycle, over the kinds of spaces spanned, tangential to our present purposes.

$\times$	$\tau_i = ab+cde$	$\tau_j = ac-bde$	$\tau_k = bc+ade$
τ_i	$1-abcde$	$bc+ade$	$-ac+bde$
τ_j	$-bc-ade$	$1-abcde$	$ab+cde$
τ_k	$ac-bde$	$-ab-cde$	$1-abcde$

=

$\times$	τ_i	τ_j	τ_k
τ_i	"-1"	$-\tau_k$	τ_j
τ_j	τ_k	"-1"	$-\tau_i$
τ_k	$-\tau_j$	τ_i	"-1"

Like the Q's, the τ's anti-commute, eg. $\tau_i\tau_j = -\tau_j\tau_i$; close circularly, eg. $\tau_i\tau_k = \tau_j$; and $-\tau_i\tau_j\tau_k = \tau_k\tau_j\tau_i$. We emphasize that these tauquinion relationships are independent of the restriction to $\mathbb{Z}_3$. Clearly, the two tables to the right, quaternion and tauquinion, are isomorphic. So if you can build a field with quaternions, you can build it with tauquinions too.

There are $\binom{5}{2} = \binom{5}{3} = \frac{5 \cdot 4}{2} = 10$ such tauquinion pairs in $\mathcal{G}_5$: [8]

$$\{ab+cde, ac+bde, ad+bce, ae+bcd, bc+ade,$$
$$bd+ace, be+acd, cd+abe, ce+abd, de+abc\}$$

Conjugates to these form another (dual) space

$$\{ab-cde, ac-bde, ad-bce, ae-bcd, bc-ade,$$
$$bd-ace, be-acd, cd-abe, ce-abd, de-abc\}$$

Including negatives, there are 40 tauquinion pairs in all. Most of these form a quaternion triplet with two others, with some products producing some form of -1. [9] A full product table appears later.

2. *The uniqueness of tauquernion* $\{ab+cd\}$ *and tauquinion* $\{ab+cde\}$ *forms.*

Let X, Y be two pseudo-vectors, whence $X^2 = Y^2 = \pm 1$. We wish to determine the conditions under which

$$(X+Y)^2 \;=\; \text{"}-1\text{"} \;=\; 1 \mp XY \;=\; -\sqrt{\text{"}+1\text{"}} \;=\; -\sqrt{-1 \pm XY}$$

where $-1 \pm XY = \text{"}+1\text{"}$ is a projector with unitary element XY. If these be so, then $X+Y$ is an analog to $\sqrt{-1}$, and hence can perform the $4 \times \frac{\pi}{2}$ rotations that $i = \sqrt{-1}$ performs. This in turn implicitly gives two orthogonal dimensions, from which we can then construct a third, *a la* classical quaternions.

Achieving such a $\sqrt{-1}$ analog requires both that $(XY)^2 = +1$ and $(X+Y)^2 = \text{"}-1\text{"}$ as above. Since $(XY)^2 = +1$, XY must have grade $\{0,1\}\, mod 4$, cf. preceding table; and as well, satisfy $\delta(X+Y) = XY$, which means that X and Y must be disjoint. Re $+1 \mp XY$,

$$(X+Y)(X+Y) \;=\; X^2 + XY + YX + Y^2 \;=\; \pm 1 \;+\; (XY+YX) \;\pm\; 1$$

[8] Versus tauquernions: $\binom{4}{2} = 6$, but the disjointness criterion halves this to three tauquernion pairs.

[9] Choosing a particular triple (there are 32 in the 10×10 table given later) constitutes an arbitrary choice of coordinate system orientation, cf. the "right hand rule" in 3d.

so to get the desired unitary XY, X and Y must commute, the right-most yielding

$$= \pm 1 \; - \; XY \; \pm \; 1$$

since $XY+XY=-XY$ in $\mathbb{Z}_3$. This commutativity means that at least one of X,Y has grade $\{0,2\}\,mod4$. Say $X_2=ab$. Then Y can be of either even or odd grade.

To get a $+1$ (in "-1" $=+1\pm XY$), both X and Y must square to -1, which means that X,Y must both have grade $\{2,3\}\,mod4$. So $X_2=ab$ still holds.

If we then choose Y to also be of *even* grade, $Y_2=cd$, we get the tauquernion family $\{ab+cd\}$. Choosing X,Y to be both of grade 4 yields $(X_4+Y_4)^2=(-1\pm X_4Y_4)$, which is "$+1$", not "$-1$". Choosing grades 2 and 4 yields grade 6, whence $(X_2Y_4)^2=-1$, not $+1$. [10] So the base case is $2+2=4$, the tauquernions.

Instead, if for *odd* Y we choose grade $=1$, then $Y_1=c$ and we would have $X_2+Y_1=ab+c$, which leads to the quark family [7]; so we choose grade $=3$, say $Y_3=cde$. This then yields the tauquinion family $\{ab+cde\}$. Choosing $odd=1$ and $even=4$ is in the tauquinion table automatically - see below. So the base case is $2+3$, the tauquinions.

Thus, due to the algebra's $mod4$ cycle, any pair with the grade structure $2mod4+3mod4$ will have tauquinion properties. Similarly, any pair with the grade structure $2mod4+2mod4$ will have tauquernion properties. But no others are possible, since the tauquernion and tauquinion forms exhaust the algebra's possibilities in this regard.

Note that $1+3=4$ appears nowhere. Cf. [7], this corresponds to dark matter. Because $(1mod4+3mod4)^2$ is always zero, yielding no "$\pm 1\pm U$" at all, dark matter is therefore out of the structure game except as it is associated with tauquernions. We return to this later.

Summarizing, there are then *just these two groups*, tauquernions and tauquinions, of fundamental field-generators derived from pairs of pseudo-vectors $X+Y$ such that $(X+Y)^2=$ "-1", which ("-1")^2 is the idempotent operator "$+1$"$=\hat{U}=-1\pm XY$.

3. How the tauquinion field is organized and propagated.

Both the tauquernion and tauquinion fields achieve their coverage in two ways:

1. Lateral recombination at/across the same two grade-levels, eg. $(ab+cde)+(fg+hij) \rightsquigarrow (ab+hij)+(fg+cde)$. Note that this recombination is $\mathcal{O}(n^2)$, and so convenient to associate a dilution effect of $\mathcal{O}(\frac{1}{r^2})$ [once the tauquernions have built $3+1$d]. { "$\rightsquigarrow$" means "leads to"}
2. Hierarchical consolidation, eg. $\delta(ab+cde)=abcde$.

The latter, which models local-to-global propagation (vs. Lateral's local-to-local), occurs via the co-boundary sequence

[10] But note that $6+6=12=0mod4$, which works. Tauquernion space (like tauquinion space, as we shall see) thus expands in two ways, $4+2=6=2mod4 \rightsquigarrow 6+6=12=0mod4$ and $1+1=2 \rightsquigarrow 2+2=4=0mod4$. But we keep our focus on tauquinion space.

pairs	$\delta(pair)$	*newlevel*		
$2mod4+3mod4$	$\rightsquigarrow$	5	$=1mod4$	$\searrow$
$2mod4+2mod4$	$\rightsquigarrow$	4	$=0mod4$	$\searrow$
$1mod4+2mod4$	$\rightsquigarrow$	3	*charge*	$.\downarrow$
$1mod4+1mod4$	$\rightsquigarrow$	2	*spin*	$.\downarrow$
$0mod4+1mod4$	$\rightsquigarrow$	1	*existence*	$\leftarrow.$

wherein we see the formative role to be played by the algebra's telescoping of its semantic levels $mod4$. In particular, the $2+3=5 \xrightarrow{mod4} 1$ loop propagates its form in every cycle of hierarchical consolidation, always being created from previous levels $\{2,3\}mod4$. In this way a field is propagated "up" to macroscopic size, whence its basic character is the same at all scales.

The table below displays, in the left-most six columns, the algebraic buildup of higher and higher grade pseudo-vectors in this $2+3=5 \mapsto 1$ fashion.

pair	δ*(pair)*	*pair*	δ*(pair)*	*pair*	δ*(pair)*	*grade g*
				...	...	...
				C_2+C_3	C_5	50+75=125
				C_2+C_2	C_4	50+50=100
				C_1+C_2	C_3	25+50=75
				C_1+C_1	C_2	25+25=50
		B_1+B_4	B_5	$\mapsto C_1$		5+20=25
		B_2+B_3	B_5	$\mapsto C_1$		10+15=25
		B_2+B_2	B_4			10+10=20
		B_1+B_2	B_3			5+10=15
		B_1+B_1	B_2			5+5=10
A_1+A_4	A_5	$\mapsto B_1$				1+4=5
A_2+A_3	A_5	$\mapsto B_1$				2+3=5
A_2+A_2	A_4					2+2=4
A_1+A_2	A_3					2+1=3
A_1+A_1	A_2					1+1=2
A_1						0+1=1
A_1						1
A_0						0

The rightmost column, *grade g*, is the grade of the m-vector created by $\delta(pair)$.

pair	*pair*	*pair*	*grade g*	*mod*4	*i*	F_i	$F_i\,mod4$
		...	...	**0**	***12***	144	**0**
		C_2+C_3	50+75=125	**1**	***11***	89	**1**
		C_2+C_2	50+50=100				
		C_1+C_2	25+50=75	**3**	***10***	55	**3**
		C_1+C_1	25+25=50	**2**	***9***	34	**2**
	B_1+B_4	$\mapsto C_1$	5+20=25				
	B_2+B_3	$\mapsto C_1$	10+15=25	**1**	***8***	21	**1**
	B_2+B_2		10+10=20				
	B_1+B_2		5+10=15	**3**	***7***	13	**1**
	B_1+B_1		5+5=10	**2**	***6***	8	**0**
A_1+A_4	$\mapsto B_1$		1+4=5				
A_2+A_3	$\mapsto B_1$		2+3=5	**1**	***5***	5	**1**
A_2+A_2			2+2=4				
A_1+A_2			2+1=3	**3**	***4***	3	**3**
A_1+A_1			1+1=2	**2**	***3***	2	**2**
A_1			0+1=1	**1**	***2***	1	**1**
A_1			1	**1**	***1***	1	**1**
A_0			0	**0**	***0***	0	**0**

Note now the next column, labelled *mod*4, ie. $g\,mod\,4$. For example, the grade of the *m*-vector made by $\delta(A_2+A_3)=A_5\mapsto B_1$ is the sum of the grades of its two constituents, namely $2+3=5$, and $5\,mod\,4=1$; and similarly the next octave up, $B_2+B_3\mapsto 10+15=25$, we get C_1, and $25\,mod\,4=1$. So $B_2+B_3\mapsto C_1$ is just like $A_2+A_3\mapsto B_1$.

The last three columns are, respectively, a counter *i*, the corresponding element F_i of the Fibonacci series, and finally $F_i\,mod\,4$. Note that the two *mod*4 columns correspond closely; see the footnote *re* the differences at steps 6 and 7. [11] Furthermore, the ratio of successive entries in the *grade g* column approximates the golden mean φ, eg. $g_{14}/g_{13}=525/325=1.6153\ldots$ *vs.* $\varphi=(1+\sqrt{5})/2=1.6180\ldots$, as one would expect. That is, the tauquinion field has an underlying deep Fibonacci structure! [12] [13]

It is at this level of abstraction that we connect to Coldea *et al*'s finding [1].

The $\mathbb{Z}_3\,\mathscr{G}_n$ picture is a bit like an x-ray photograph of the quantum mechanical world - it shows the overall bone structure, but the details of the flesh - of which there are *many* - must be provided from elsewhere (ie. known physics). In fact, one could argue that our analysis shows that much of the mathematical thicket that is quantum mechanics is (apparently) concerned with expressing process, structure, and their interrelationship. Because in our algebra, vectors are processes and structure derives from

[11] The differences originate in $5+5=(1+1)+(4+4)=10=2\,mod\,4$ versus F's $3+5=8=0\,mod\,4$. Since $4+4=0\,mod\,4$, the two differ by $1+1=2$; step 7 then adds 1 to get 3. The two extra 1's ultimately stem from δ's *doubling* of 5 (vs. doubling 4). The two sequences are identical $mod\,2\rightsquigarrow(0,1,1)^{1*}$, and hence rejoin explicitly every other F-cycle, next at $F_6+F_7=F_8$. [^{1*} means one or more instances]

[12] Actually, *any* such accumulating sequence with positive numbers will converge to φ, but the subject of §5 shows that this version of the sequence has its origin in Fibonacci's.

[13] As with tauquernions, there are two tauquinion sequences: $2+3=5\mapsto 1$ and $3+3=6\rightsquigarrow 6+6=12\mapsto 0$. I speculate that the paired Fibonacci numbers in (eg.) pinecones and sunflowers arise from this.

their co-occurrence ("+"), these two are completely integrated, and the algebra's minimality wraps the physics tightly.

Stepping back, it is apparent that the emergence of the Fibonacci sequence at the quantum level is a mathematical inevitability that Coldea et al.'s experimental result confirms. It is the dual physical *and* computational, process-oriented interpretations that we *simultaneously* lay on the algebra that give this conclusion conceptual heft.

Herein also lies a prediction, since the $\mathcal{T}$ sequence does differ slightly ($mod\,4$ vs. $mod\,2$) from the mathematical ideal, which might perhaps be measurable. This measurement would tell us if the universe is fundamentally built on the rationals (the $mod\,4$ sequence) or the reals (the $mod\,2$ sequence, which approximates the irrational φ better).

Furthermore, the correspondence we have found is not just the usual numerical sequence - it is also, uniquely, *exact operators and actual states*, a gift from $\mathcal{G}$'s graded structure. That is, one would expect that the Fibonacci properties of macroscopic entities like flowers, pinecones, and sea shells are brought about by the operation of tauquinions, steering the growth. As well, since electro-magnetism itself is very well characterized, one can inquire directly. These are predictions. [14]

We needn't restrict $F_i\, mod\, n$ to $n = 4$: any n produces a pattern *a la* $(011231)^{1*}$ above, but more jumbled (ie. longer); $mod\,2$ produces $(011)^{1*}$. Indeed, the fact that these patterns are all (more and less) simultaneously present means that the hierarchy, and the patterns themselves, are fundamentally fractal in nature.

It is interesting to see what the significant neighbor to Fibonacci's sequence, namely Lucas' sequence, says:

$$L_n = F_{n+1} + F_{n-1} = (F_n + F_{n-1}) + F_{n-1}$$

i	L_i	$(L_{i-2}+L_{i-1})\,mod\,4$
11	123	$3+0 \mapsto 3$
10	76	$1+3 \mapsto 0$
9	47	$2+1 \mapsto 3$
8	29	$3+2 \mapsto 1$
7	18	$3+3 \mapsto 2$
6	11	$0+3 \mapsto 3$
5	7	$3+0 \mapsto 3$
4	4	$1+3 \mapsto 0$
3	3	$2+1 \mapsto 3$
2	1	1
1	2	2

As noted earlier, we identify the form $w + xyz =$ "$1+3$" as dark matter. This form is nilpotent in all sign variants, and so cannot be the "-1"-like root of a projector $-1 \pm$

[14] Even though the Fibonacci imprint is present from the very bottom of the hierarchy, this same mechanism's effect or appearance at higher levels is not reductionistically dependent on this prior presence. Rather, the same forms can emerge spontaneously at any given level as a collective (ie. emergent) property of the very [growth!] processes that are taking place there. Maybe Fibonacci cities are the future [9].

$wxyz$ that we seek, even though $\delta(w+xyz) = wxyz$. Therefore, it cannot be the basis for a coordinate system.

Thus dark matter is no part of the tauquinion field, nor its putative electro-magnetic properties.

4. A Closer Look at the TauQuinion Field

Here is a full tauquinion set of ten, chosen such that their sum is nilpotent, ie. field-like.
NB: In the table below, "..−1.." $= +1 - abcde$:

×	$ab+cde$	$ac-bde$	$ad+bce$	$ae-bcd$	$bc+ade$	$bd-ace$	$be+acd$	$cd+abe$	$ce-abd$	$de+abc$
$ab+cde$	..−1..	$-bc-ade$	$-bd+ace$	$-be-acd$	$ac-bde$	$ad+bce$	$ae-bcd$	$e-abcd$	$-d-abce$	$c-abde$
$ac-bde$	$bc+ade$	..−1..	$-cd-abe$	$-ce+abd$	$-ab-cde$	$-e+abcd$	$d+abce$	$ad+bce$	$ae-bcd$	$-b-acde$
$ad+bce$	$bd-ace$	$cd+abe$	..−1..	$-de-abc$	$e-abcd$	$-ab-cde$	$-c+abde$	$-ac+bde$	$b+acde$	$ae-bcd$
$ae-bcd$	$be+acd$	$ce-abd$	$de+abc$	..−1..	$-d-abce$	$c-abde$	$-ab-cde$	$-b-acde$	$-ac+bde$	$-ad-bce$
$bc+ade$	$-ac+bde$	$ab+cde$	$e-abcd$	$-d-abce$	..−1..	$-cd-abe$	$-ce+abd$	$bd-ace$	$be+acd$	$a-bcde$
$bd-ace$	$-ad-bce$	$-e+abcd$	$ab+cde$	$c-abde$	$cd+abe$	..−1..	$-de-abc$	$-bc-ade$	$-a+bcde$	$be+acd$
$be+acd$	$-ae+bcd$	$d+abce$	$-c+abde$	$ab+cde$	$ce-abd$	$de+abc$	..−1..	$a-bcde$	$-bc-ade$	$-bd+ace$
$cd+abe$	$e-abcd$	$-ad-bce$	$ac-bde$	$-b-acde$	$-bd+ace$	$bc+ade$	$a-bcde$	..−1..	$-de-abc$	$ce-abd$
$ce-abd$	$-d-abce$	$-ae+bcd$	$b+acde$	$ac-bde$	$-be-acd$	$-a+bcde$	$bc+ade$	$de+abc$	..−1..	$-cd-abe$
$de+abc$	$c-abde$	$-b-acde$	$-ae+bcd$	$ad+bce$	$a-bcde$	$-be-acd$	$bd-ace$	$-ce+abd$	$cd+abe$	..−1..

Note that there are five groups, defined by their particular $v + wxyz =$ '-1'. To focus on one of these groups, taking only those 2-vectors that belong to $\mathscr{G}_4$ on$\{a,b,c,d\}$ reduces this 10×10 table to 6×6:

$\times$	$ab+cde$	$ac-bde$	$ad+bce$	$-bc-ade$	$bd-ace$	$-cd-abe$
$ab+cde$	$1-abcde$	$-bc-ade$	$-bd+ace$	$-ac+bde$	$ad+bce$	$-e+abcd$
$ac-bde$	$bc+ade$	$1-abcde$	$-cd-abe$	$ab+cde$	$-e+abcd$	$-ad-bce$
$ad+bce$	$bd-ace$	$cd+abe$	$1-abcde$	$-e+abcd$	$-ab-cde$	$ac-bde$
$-bc-ade$	$ac-bde$	$-ab-cde$	$-e+abcd$	$1-abcde$	$cd+abe$	$bd-ace$
$bd-ace$	$-ad-bce$	$-e+abcd$	$ab+cde$	$-cd-abe$	$1-abcde$	$bc+ade$
$-cd-abe$	$-e+abcd$	$ad+bce$	$-ac+bde$	$-bd+ace$	$-bc-ade$	$1-abcde$

This particular set of tauquinions was chosen so that the 2-vectors form a Higgs boson $\mathscr{H}$ (ie. nilpotent). The 3-vector electric component $\mathscr{E}$ is also nilpotent, as is $\mathscr{H} + \mathscr{E}$. [15]

Noting that *both* $1 - abcde =$ "-1" *and* $-e + abcd =$ '-1', which we take to be the magnetic and electrical polarity indicators, respectively, the preceding table can be rewritten a bit more clearly:

$\times$	$ab+cde$	$ac-bde$	$ad+bce$	$-bc-ade$	$bd-ace$	$-cd-abe$
$ab+cde$	"-1"	$-bc-ade$	$-bd+ace$	$-ac+bde$	$ad+bce$	'-1'
$ac-bde$	$bc+ade$	"-1"	$-cd-abe$	$ab+cde$	'-1'	$-ad-bce$
$ad+bce$	$bd-ace$	$cd+abe$	"-1"	'-1'	$-ab-cde$	$ac-bde$
$-bc-ade$	$ac-bde$	$-ab-cde$	'-1'	"-1"	$cd+abe$	$bd-ace$
$bd-ace$	$-ad-bce$	'-1'	$ab+cde$	$-cd-abe$	"-1"	$bc+ade$
$-cd-abe$	'-1'	$ad+bce$	$-ac+bde$	$-bd+ace$	$-bc-ade$	"-1"

This table is then, presumably, the entire field situation in 3-space at a single point - as specified by the tauquernion subset, including the electro-magnetic field - as specified by the tauquinion relationships, which latter are thus automatically constrained in $3+1$d by their tauquernion components.

An electro-magnetic field interplays two polarities - magnetic and electric - which polarities are ultimately specified by the orientations of the associated spinors. The 2-spinor configuration defines the *magnetic* field in 3-space, and its minus-sign is indicated by the NW-SE diagonal, "-1" $= 1 \pm abcde$.

Similarly, the 3-spinor configuration defines the *electric* field $\mathscr{E}$, and its polar indicator '-1' $= \pm(e - abcd)$ is the NE-SW diagonal; note that electric-plus is $(e \pm abcd)^2 = -1 \pm abcde =$ "$+1$" as it should. However, it's "minus" with a twist: $(e - abcd)(ab + cde) = -cd - abe$, so not only are both charges reversed, but the electric and magnetic components do a dosey-do as well; fans of Maxwell's equations will recognize this. More straightforwardly, $(1 - abcde)(-e + abcd) = e - abcd$.

The following relationships hold for the above-specified field. However, due to the extreme symmetry of $\mathscr{G}_n$ over $\mathbb{Z}_3$, one can view them as true for all field-type states; other states are roots of unity.

[15] Such nilpotent forms are a real minority - only 240, versus 13,200 that are not.

Tauquernions $\mathcal{T}_i = ab - cd$; $\mathcal{T}_j = ac + bd$; $\mathcal{T}_k = ad - bc$. $\mathscr{H}$ is the Higgs boson.

$$\mathscr{H} = \mathcal{T}_i + \mathcal{T}_j + \mathcal{T}_k \qquad \mathscr{H} * \mathscr{H} = 0$$

$$\mathscr{E} = cde - bde + bce - ade - ace - abe \qquad \mathscr{E} * \mathscr{E} = 0$$

$$= (cd - bd + bc - ad - ac - ab)e = -\mathscr{H}e$$

$$\mathscr{H} + \mathscr{E} = \mathscr{H} - \mathscr{H}e = \mathscr{H}(1 - e) \qquad (\mathscr{E} + \mathscr{H})^2 = 0$$

Since $\mathscr{E} = -\mathscr{H}e$, $\mathscr{E}$ is compatibly entangled with the gravitational field formed by $\mathscr{H}$.

Finally, the Appendix describes Majorana fermions, which are of great interest in quantum computing, and which have recently [possibly] been observed [11].

5. *A Final Puzzle*

If one lists all the unitary elements U in the algebra $\mathscr{G}_3$ (isomorphic to the Pauli algebra), one finds the following ($U^2 = +1$):

$$1, \quad a, \quad ab + ac, \quad a + b + ab, \quad a + b + c + ab + ac$$

with dimensionalities (= terms) respectively 1,1,2,3,5. Until now, it has been entirely opaque as to whether this was chance or a (maybe) instance of a Fibonacci progression. Now, the problem is reversed: how to understand these given the preceding analysis.

The unitary elements U we have discussed in earlier sections have always been a singleton term, and these constructed via δ from pairs of same. Now, however, our idempotents look like

$$-1 + a + b + ab, \quad -1 + ab + ac, \quad -1 + a + b + c + ab + ac$$

Where do these multi-term U's come from? Our usual constructor, the co-boundary operator δ, fails. Adding suspense to the story, these $-1 + U$'s we have elsewhere identified as the neutrino, electron, and proton projectors, respectively [7]. Do they nevertheless have the roots we require? [see Table 2.]

time-like	neutrino	electron	proton
"$+1$" $= -1 \pm U$	$-1+a+b+ab$	$-1+ab+ac$	$-1+a+b+c+ab+ac$
$\Updownarrow$			
"-1" $= \sqrt{\text{"}+1\text{"}}$	$1-a-b-ab$	$1-ab-ac$	$1-a-b-c-ab-ac$
$\Updownarrow$			
"i" $= -\sqrt{\text{"}-1\text{"}}$	none	$\pm(b-c-abc)$ $= abcU$	$\pm(b-c+ab-ac+bc-abc)$ $= abcU; (b-c+ab-ac+bc)^4 = U$

Table 2: Time-like roots of stable particles.

Clearly, from the table, we can try to apply our same reasoning with electrons and protons - they at least *have* fourth roots - but neutrinos will need a different treatment (next ¶+1). If we are to remain faithful to our earlier interpretation of $\mathscr{H}$ and $\mathscr{E}$ as field generators, then $b-c-abc$ too should be the actual field element associated with the electron projector $-1+ab+ac$, and namely charged: $-abc$; and similarly for the proton. Certainly, $b-c-abc$ is, like an electron, very nearly a geometric point: an oriented volume $-abc$ with mostly missing sides ($b-c$ only determines a plane bc); the proton's fourth root is similarly missing various faces.

Do the multiple i's (for a given particle form e or p) anti-commute with each other *a la* quaternions and enquernions? Ie. make tiny $3+1$d spaces that tile up into something macroscopic? No, they remain isolated. Instead of anti-commuting, these products produce each other's additive inverse. That is, they do not form quaternion triples, and so no field. In fact, these fourth root "imaginaries" are just abc rotations of the original idempotent form, not at all what is needed.

Regarding the neutrino, the form $\pm1\pm a\pm b\pm ab$ always factors into one of the products $(\pm1\pm a)(\pm1\pm b)$ or $(\pm1\pm b)(\pm1\pm a)$, of which there are sixteen in all. Those with -1 are idempotents, and as before, their negatives (ie. with $+1$) are their square roots (sqerts). But alas, the neutrino's sqerts themselves have no square roots at all, so our game is stymied again.

Despite the fact that the neutrino's structural dynamic differs from that of electrons and protons, which themselves aren't simple pseudo-vectors either, *some* juggling act nevertheless ends up creating the first five members of the Fibonacci sequence, even though what's going on is, by definition, completely uncoordinated co-occurrences of small 1- and 2-vector "atoms" ... in an entirely non-deterministic process frenzy that is nevertheless self-synchronized and convergent ... to namely $1,1,2,3,5$. How might this come to be?

The three complex oscillations - engendered ultimately by the Bit Bang's entropy creation [7] - are stable because they are unitary (and entropically favored). Their structures are inter-connected:

	ab		$\cancel{ac}$	
a	$+$	b		$\cancel{c}$

	ab	$+$	ac	
$\cancel{a}$		$\cancel{b}$		$\cancel{c}$

	ab	$+$	ac	
a	$+$	b	$+$	c

The transition $a+b\overset{\delta}{\longmapsto}ab$ results in the unitary entity $a+b+ab$, which is the simplest possible non-trivial stable oscillatory structure, simplest because it derives from the simplest possible structure-generating distinction: two 1-bit states that co-occur/exclude. $a+c\overset{\delta}{\longmapsto}ac$ proceeds similarly. The co-occurrences $ab+ac$ and $a+b+c+ab+ac$ are unitary already, and do not engender a co-boundary transition.

Their behavior is wave-like, and as the independent elements (the 1-vectors a,b,c) change, so will the spins of ab,ac oscillate accordingly. Note that while particular frequencies ν_a,ν_b,ν_c can be directly associated with a,b,c, the unitarity of $a+b+ab$ and the others is namely *not* dependent on their values.

In conclusion, the actual unitaries $\{ab+ac,\ a+b+ab,\ a+b+c+ab+ac\}$ are the very first generation of the underlying recursive Fibonacci structure, born in the inevitable unitarity of their oscillation.

So why is the Fibonacci sequence the convergent and not something else? Our answer is that the uniqueness of the two enquernion forms, along with the algebra's *mod*4 cycle, allow very little room for Nature to experiment in. If there are other possible Fibonacci-like sequences, they apparently all fizzle out, eg. the closely related Lucas' sequence, leaving the Fibonacci sequence the only surviving possibility.

One last, mathematical, remark: the golden ratio derives from the proportion $\frac{1}{\varphi} = \frac{\varphi}{1+\varphi}$, leading to the roots (golden ratios) $\varphi = \frac{1+\sqrt{5}}{2} = 1.618033989$ and $\varphi' = \frac{1-\sqrt{5}}{2} = -0.618033989$. It is easy to show that $\varphi = -\frac{1}{\varphi'}$ and vice versa. Consider now the following (left-to-right): [Recall that $U = U^{-1}$ when $U^2 = +1$]

$$\varphi = -\frac{1}{\varphi'} \qquad i = -\frac{1}{i} \qquad \tau_i = \text{“}-\frac{1}{\tau_i}\text{”} = -(\tau_j\tau_k)^{-1} = -\tau_k\tau_j$$

or

$$\varphi\varphi' = -1 \qquad ii = -1 \qquad \tau_i\tau_i = \tau_i(\tau_j\tau_k) = \text{“}-1\text{”}$$

That is, the multiplicative inverse of φ includes a sign inversion ... just like i ... just like tauquinions. That is, the golden ratios φ and φ', *and* $i = \sqrt{-1}$, *and* $\tau = ab + cde$, *and* $\tau = ab + cd$, are actually all in the same family: fourth roots of unity (ie. of its particular version of "+1"). So it is not quite so surprising that the Fibonacci sequence should show up in our analysis of tauquinions. [They are also examples of Jacobian theta/modular functions.]

6. Re *Coldea* et al.*'s findings.*

Coldea *et al.*'s theoretical and experimental considerations lie well outside our expertise. This, we think, supports our case, since we have shown in the preceding that the build-up of structure up in $\mathscr{G}_3$ (= the Standard Model) inevitably produces the Fibonacci sequence, in ignorance of their work.

Considering the $\mathscr{G}_3$ build-up to be the base case in an inductive proof, we then proceeded to show that the $2+3=5 \mapsto 1$ pattern continues into higher grades without bound. Had we not discovered Coldea *et al.'s* result via literature search, we would have predicted that the golden mean would be found in quantum systems, from the simplest to the most complex, and indeed, beyond.

As regards the exceptional Lie group E_8, we do not at this point know if it is present here, or whether we have solved its problem without it.

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References.

1. R. Coldea, *et al.* "Quantum Criticality in an Ising Chain: Experimental Evidence for Emergent E_8 Symmetry." Science 327, 177 (2010); DOI: 10.1126/science.1180085

2. Doran, C. and Lasenby, A. *"Geometric Algebra for Physicists"*. Cambridge University Press, 2003. Isbn 978-0-521-71595-9.

3. Hestenes, D. *"New Foundations for Classical Mechanics"* (2^{nd} ed). Kluwer Academic Press, 1999. Isbn 0-7923-5514-8.

4. Hestenes, D. and Sobczyk, G. *"Clifford Algebra to Geometric Calculus - A Unified Language for Mathematics and Physics"*. Kluwer Academic Press, 1984/87. Isbn 90-277-2561-6.

5. Lounesto, P. *"Clifford Algebras and Spinors"*. London Mathematical Society, Lecture Notes Series #286. Cambridge University Press (1997) 2002. Isbn 0-521-00551-5.

6. Manthey, M. "Synchronization - the Font of Physical Structure". International Journal of Anticipatory Systems, Daniel Dubois, Ed. Issn 1373 - 5411, v. 20-21-22, IJ-CAS/BCS Symposium, Liege, Belgium. August 6-11, 2007. Isbn 2-930396-09-1. *Tauquernions.org*

7. Manthey, M. and Matzke, D. "TauQuernions τ_i, τ_j, τ_k: 3+1: Dissipative Space out of Quantum Mechanics". *Tauquernions.org April 2013.*

8. Matzke, D. *Quantum Computation Using Clifford Algebra.* PhD thesis, Dept. of Electrical Engineering, 2002. www.utdallas.edu/~cantrell/matzke.pdf.

9. Bettencourt, Luis M.A. "The Origins of Scaling in Cities". Science v340, p. 1438. 21 June 2013.

10. Borthwick, D. and Garibaldi, S. "Did a 1-Dimensional Magnet Detect a 248-dimensional Lie Algebra?". arXiv:1012.5407v3 [math-ph] 25 Mar 2011

11. http://physicsworld.com/cws/article/news/2012/apr/19/first-hint-of-majorana-fermions-spotted-in-nanowires

The author, a computer scientist, has taught at universities in Europe and the USA. He is now an independent researcher living in Colorado, USA.

Appendix: Majorana Fermions in $\mathscr{G}_5$ over $\mathbb{Z}_3$

Majorana fermions are characterized by having spin 1/2 and being self-conjugate, meaning that they are nilpotent, and thus are field-like forms of otherwise material particles.

We find that all four fermions - protons, neutrons, electrons and neutrinos - all have at least one Majorana operator M that transforms them into a self-conjugate form. That is, M is a *form* whose variants match corresponding variants in the fermions.

$M = a + bcde$ is one such form in the case of an electron E:

$$E = -1 + ab + ac = -(a+b+c)a$$

where $(ab+ac)^2 = 1$ and $a+b+c$ is a photon. Then

$$ME = (-a+b+c) + abde - acde - bcde$$

$$EM = -(a+b+c) - abde + acde - bcde$$

and $(EM)^2 = 0 = (ME)^2$.

EM (for example) can be rewritten as

$$EM = -(a+b+c) - (ab+bc+ca)de$$

Note that $ab+bc+ca$ is a quaternion triplet. And because $abc(ab+bc+ca) = -(a+b+c) = (ab+bc+ca)abc$ and $-abc*abc = 1$, we can write

$$= -(a+b+c) - (ab+bc+ca)(-abc*abc)de$$

$$= -(a+b+c) - (a+b+c)abcde$$

$$= (a+b+c)(-1-abcde)$$

wherein we see that the Majorana electron's photonic aspect consists of a "fundamental" $-(a+b+c)abcde$ and a "*mod4*-octave" overtone $-(a+b+c)$. [16] The "*mod4*-octave" refers to geometric algebra's pseudo-vectors, whose squares cycle as powers of $i = \sqrt{-1}$ as their grade increases, ie. $mod4$. Therefore 1-vectors like $a+b+c$ and 5-vectors like $abcde$ have the same (grammatical) algebraic properties, whence the "octave" designation. [17]

Or we can factor a out and in, and see the two concurrent electron phases:

[16] Or, if you prefer, a fundamental $a+b+c$ and a "*mod4*-octave" undertone $-(a+b+c)abcde$.

[17] "...This description applies to the ground-state rotational band of carbon-12, but it also has significance for the Hoyle state. This is because *the spectrum of the Hoyle-state rotational band appears to be similar to that of the ground-state band* – with two of the five spin states measured already. However, the Hoyle state appears to have a *larger moment of inertia* than the ground state." [My italics] From http://physicsworld.com/cws/article/news/2014/jul/08/carbon-nucleus-seen-spinning-in-triangular-state.

$$= (-1+ab+ac)a + (-1+ab+ac)bcde$$

$$EM = (-1+ab+ac)(1+abcde)a$$

which tells us that a Majorana electron is an a-rotation of that electron's combined fundamental and *mod4*-octave frequencies.

The $\mathscr{G}_5$ Majorana operator M has two basic forms, $\Sigma(v+wxyz)$ and $\Sigma(vw+xyz)$, where the latter is a tauquinion field element, and the former one of the minus 1's in the tauquinion group. In particular, this minus 1 form is the linchpin connector between the magnetic and electric fields, ensuring that any true change in one results in a corresponding change in the other.

For example, in the case of $M = e + abcd$ and another field element, say $F = ab - cde$, we find that $MF = cd - abe = FM$, so M inverts the signs of both the magnetic (2-vector) and electric (3-vector) components, *and* it interchanges them ($ab \leftrightarrow cd$), thus linking the two fields utterly. [Recall that the forms $vw+xyz$ are all quaternion isomorphs.]

In the case of the electron $E = -1+ab+ac$, the $1+4$ Majorana operators are:

$$M = a+bcde \qquad EM == (a+b+c)(-1-abcde)$$

$$M = (b+acde)+(c-abde) \quad EM = (a+b+c)(-1+abcde)$$

$$M = (b+acde)+(-c+abde)+(d+abce)$$

$$EM = (b-c+d)(-1+e+abcd-abcde)$$

$$M = (a+bcde)+(b+acde)+(-c+abde)+(e-abcd)$$

The electron's $2+3$ Majorana operators are:

$$M = ad+bce$$

$$M = (bd-acd)+(cd-abe)$$

$$M = (-ab+cde)+(ac+bde)+(-ae+bcd)$$

$$M = (-ab-cde)+(ac-bde)+(ae+bcd)+(bc-ade)$$

So we see that the electron's Majorana operators are linear combinations of tauquernion group elements. Similarly for the proton $P = -1+a+b+c+ab+ac$, the corresponding Majorana operator is three $2+3$'s:

$$M = (-ae+bcd)+(-bd+ace)+(-be-acd) \qquad PM = -ae-be-ce+abd-acd+bcd$$

and likewise for the neutron $N = abcP = b-c+ab-ac+bc-abc$,

$$M = (-ae+bcd)+(-bd+ace)+(-be-acd) \qquad NM = -ad-bd-cd-abe+ace-bce$$

whose M is identical.

In the case of the neutrino $n = -1+a+b+ab$, [18] the corresponding Majorana operators are:

$$M = (a - bcde) + (-b - acde)$$
$$M = (-c + abde) + (-d - abce) + (-e + abcd)$$
$$M = (ad + bce) + (bd - ace)$$
$$M = (cd - abe) + (ce + abd) + (de - abc)$$
$$M = (-ac - bde) + (-ae + bcd) + (cd + abe) + (-de + abc)$$

Thus there are Majorana operators in $\mathscr{G}_5$ for all four fermions, and all of these operators are linear combinations of elements of the tauquinion group.

This in turn prompts the question, Are there Majorana fermions in the gravitational (ie. tauquernion) field too? Yes. And indeed, every category has at least one valid Majorana example except $2+2$ for protons and neutrons. We list a few examples:

$\mathscr{G}_4$ Tauquernion Majorana Operators

1+3

Electron	$d - abc$
	$-b+c-abd-acd$
	$-a+b+d-abc-acd+bcd$
	$-a+b-c+d-abc-abd-acd-$
	$-a+b-c+d-abc-abd-acd-bcd$
Neutrino	$-c-d-abc-abd$
	$a-b+c+abd+acd+bcd$
Proton	$-b+c-d+abc+abd+acd$
Neutron	$-b+c+d-abc+abd+acd$

2+2

Electron	$ad+bc$
Neutrino	$-ac-bd$ + $-ad-bc$
Proton	*None*
Neutron	*None*

In general, while there are several hundred tauquinion (*1+4* and *2+3*) variants altogether, the tauquernion variants (*1+3* and *2+2*) are fewer than a hundred; details are available on request. We note that [7] identifies the $1+3$ forms $w+xyz$ as dark matter.

One can hope that the present computationally distributed and *combinatorially exact* description of quantum mechanics, which is as well consistent with the Standard Model (but doesn't need SUSY), will be useful in *topological* quantum computation.

Since the algebra is a *literal* representation of *actual* events (ie. state changes), each of the various possible factorizations represents a different pattern of actions leading to

[18] I note that Appendix I in [7] supports the existence of a fourth neutrino as a linear combination of three basic phases, and strictly combinatorially there could be three more phases via pairwise combination.

the Majorana particle, eg. EM. Furthermore, products express actual processes, just as sums express their concurrency.

Thus these factorizations are not just mere mathematical manipulations, but rather, due to the algebra's literality, they are *different structural views* of the same object, just as one gets different views of a house from various directions. Since the algebra is both discrete and finite, the statistics of these patterns and processes should match those of actual experiment.

BiEntropy – The Approximate Entropy of a Finite Binary String

v2.33

Grenville J. Croll
grenvillecroll@gmail.com

We design, implement and test a simple algorithm which computes the approximate entropy of a finite binary string of arbitrary length. The algorithm uses a weighted average of the Shannon Entropies of the string and all but the last binary derivative of the string. We successfully test the algorithm in the fields of Prime Number Theory (where we prove explicitly that the sequence of prime numbers is not periodic), Human Vision, Cryptography, Random Number Generation and Quantitative Finance.

1 INTRODUCTION

The purpose of this paper is to illustrate the means by which a finite binary string of arbitrary length can be compared against another in terms of the relative order and disorder of all of its digits. We do this using a simple function called BiEntropy, which is based upon a weighted average of the Shannon Entropies of all but the last binary derivative of the string.

This paper is organised as follows: First, we briefly cover the historical background regarding the development of tests and measures of order, disorder, randomness, irregularity and entropy. We show that binary derivatives have previously been used in the measurement of disorder and have also been used in cryptographic applications and attacks.

Second, we discuss our intuitive understanding of order and disorder, Shannon Entropy and Binary Derivatives in more detail. We outline a number of issues that were important in formulating the BiEntropy function, following which we formally define it together with some simple variations.

Third, we apply the BiEntropy method in a number of diverse application areas including:

a) *Prime Number Theory* – We show that BiEntropy empirically determines that the sequence of early prime numbers is not periodic. This result is implicit due to the Prime Number Theorem. We prove this non-periodicity explicitly through two simple corollaries.

b) *Human Vision* - We demonstrate the use of BiEntropy in powerfully discriminating between the geometrical layouts of some standard ISO dot matrix characters compared to some randomly produced strings, the Braille character set and two specially designed high and low entropy character sets.

c) *Random Number Generation* - We use BiEntropy to evaluate the decimal expansions of some well known random, normal and irrational numbers using a simple decimal to binary coding scheme. We compare these expansions with similar output from the Random Number Generators within Excel 2003 and Excel 2010.

d) *Cryptography* - We use BiEntropy to simply and easily reveal significant differences between the encrypted and unencrypted binary files of some real and synthetic spreadsheets.

e) *Quantitative Finance* - We use BiEntropy to examine 10 years worth of UK daily stock market prices to show that the BiEntropy of historical price changes is strongly correlated with future stock market prices.

Finally, we provide a summary and identify some areas for future work.

2 HISTORICAL BACKGROUND

There are very many tests and algorithms for the determination or measurement of randomness, regularity, irregularity, order, disorder and entropy for binary and other strings [Marsaglia, 1968][Gao, Kontoyiannis & Bienenstock, 2008]. There are several measures which derive a scalar linked to the randomness, disorder or entropy of finite strings such as Binary Entropy [Shannon, 1948], Approximate Entropy [Pincus, 1991] [Pincus & Singer, 1996][Rukhin, 2000a,b], Sample Entropy [Richman &Moorman, 2000] and Fuzzy Entropy [Chen, Wu & Yang, 2009]. Existing tests and algorithms appear to be divided into two classes: those which use a sliding window technique to examine substrings of the original string [Marsaglia & Zaman, 1993] and those which are related to the length of algorithms used to generate the entire string [Kolmogorov, 1965] [Chaitin, 1966]. Some of these functions have been applied in critically important domains [Pincus & Viscarello, 1992].

Whilst there have been a number of attempts to use binary derivatives in randomness tests [McNair, 1989], cryptographic applications [Carroll, 1989, 1998] and attacks [Bruwer, 1995], we believe that the use of a weighted average of the Shannon entropies of the binary derivatives of a string is unique.

3 INTUITIVE INSIGHT INTO BINARY ORDER & DISORDER

Table 1 suggests how we might intuitively regard the order and disorder of some 8 bit binary strings.

Table 1 - Intuitive Insight into some short binary strings

Binary String	Description	Reason
11111111	Perfectly ordered	All 1's
00000000	Perfectly ordered	All 0's
01010110	Mostly ordered	Mostly 01's
01010101	Regular, not disordered	Repeating 01's
11001100	Regular, not disordered	Repeating 1100's
01011010	Mostly ordered	0101 then 1010
01101011	Somewhat disordered	No Apparent Pattern
10110101	Somewhat disordered	No Apparent Pattern

Determination of the relative order and disorder of the 256 possible 8-bit binary strings is an obvious example problem that does not appear to have been previously addressed in the literature. There are 256! ($\approx 8.58 * 10^{508}$) differing ways of ordering the 8 bit binary strings.

We require an algorithm which will determine the relative and/or absolute degree of order and disorder of binary strings such as the above, for arbitrarily long binary strings. The algorithm will return 0 for perfectly ordered strings and 1 for perfectly disordered strings.

4 SHANNON ENTROPY, BINARY DERIVATIVES & WEIGHTING METHODS

4.1 Shannon Entropy

Shannon's Entropy of a binary string $s = s_1, \ldots, s_n$ where $P(s_i=1) = p$ (and $0 \log_2 0$ is defined to be 0) is:

$$H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$$

For perfectly ordered strings which are all 1's or all 0's i.e. $p = 0$ or $p = 1$, $H(p)$ returns 0. Where $p = 0.5$, $H(p)$ returns 1, reflecting maximum variety. However, for a string such as 01010101, where $p = 0.5$, $H(p)$ also returns 1, ignoring completely the periodic nature of the string.

4.2 Binary Derivatives & Periodicity

The first binary derivative of s, $d_1(s)$, is the binary string of length n - 1 formed by XORing adjacent pairs of digits. We refer to the kth derivative of s $d_k(s)$ as the binary derivative of $d_{k-1}(s)$. There are n-1 binary derivatives of s.

Some years ago [Nathanson, 1971], following the work of [Goka, 1970] defined the notions of *period* and *eventual period* within arbitrary binary strings and outlined the related properties of the derivatives both individually and collectively. Amongst a number of useful results we find that: a) if the derivative of a binary string is eventually periodic with a period P then the binary string is also eventually periodic with a period P or $2P$; b) if a derivative is all zero's then the string has a period 2^m for some m, $0 \leq m \leq n$; c) if a derivative has eventual period P, the string has eventual period $2^m P$ for some m satisfying $0 \leq m \leq n$.

Adapting Nathanson's definitions for finite strings, a binary string s of length n is *periodic* if, for some least positive integer P, $s_{i+P} = s_i$ for all $1 \leq i \leq n-P$. A binary string s of length n is *eventually periodic* if, for some least positive integer P and some least positive nonnegative integer k, $s_{i+P} = s_i$ for all $k \leq i \leq n-P$. Note that a finite binary string can be read left to right or right to left such that we *may* need to refer to a string as being either *right* or *left reading eventually periodic* and adapt our notation accordingly.

For example, the first binary derivative of 01010101 (with *period*, $P = 2$) is 1111111 ($P = 1$), following which all the higher derivatives are all 0's. The third derivative of 00010001 ($P = 4$) is 11111, following which again all the higher derivatives are 0. The sixth derivative of 00011111 (with *right reading eventual period* $P = 1$ from the fourth digit) is 10.

By calculating all the binary derivatives of s we can discover the existence of repetitive patterns in binary strings of arbitrary length. If a binary string is *periodic* its last derivative is zero. A binary string is *aperiodic* if its last derivative is 1 (else its last but one derivative is periodic and the string itself is therefore eventually periodic). A binary string is *nperiodic* if its last derivative is 0, but is not periodic.

Although Nathanson's definitions (and our own adapted definitions) of periodicity and eventual periodicity are useful, in this paper we rely solely upon the binary derivatives of a finite string to resolve the issue of the periodicity within the string.

[Davies et al, 1995] outline some further properties of binary derivatives. Let $p(k)$ denote the observed fraction of 1's in $d_k(s)$ where $p(0)$ denotes the fraction of 1's in s. Let $\pi(k)$ denote the corresponding population proportions. Provided $p(0) = 0.5$, $p(1)$ is not correlated with $p(0)$. Likewise, where $\pi(k) = 0.5$, $p(k+1)$ is not correlated with $p(k)$, the sample proportions are independent. By induction, these properties apply to the higher derivatives.

4.3 Weighting Methods

Thus there are number of important factors to consider in designing a function $\mathcal{F}$ to compute the approximate entropy of a finite binary string:

i) The Shannon entropy of a binary string does not give a complete picture regarding the order and disorder of the string due to the failure to accommodate periodicity.
ii) The derivatives of a binary string determine the existence of periodicity in the string.
iii) Determination of any periodicity may require the evaluation of all n-1 derivatives.
iv) The proportion of 1's & 0's in the binary derivatives are or can be independent.
v) Differing consideration may have to be given to higher or lower derivatives
vi) The function is required to be effective on strings of arbitrary length.

The Shannon entropy of a binary derivative is $H(p(k))$. The approximate entropy of a binary string – the BiEntropy – could therefore be some function $\mathcal{F}(H(p(k)))$ for $0 \leq k < n$.

We must decide how to combine or weight the $H(p(k))$ in order to arrive at a function $\mathcal{F}$ that is likely to have some utility. The field of time series analysis [Makridakis et al, 2008] provides comprehensive guidance on a variety of methods (including moving averages & exponential smoothing) which are used to extract information from sometimes noisy historical time series.

The $H(p(k))$ of a string for increasing or decreasing k is clearly a progression, possibly noisy, though not temporal. Exponential weighting would therefore be a first choice of weighting method. Exponential methods have the advantage that they can accommodate numerical series of arbitrary length – no matter how long the string all the $H(p(k))$ would make some contribution.

A competing consideration is that each of the $H(p(k))$ is potentially independent of any other and so any weights must discriminate clearly between each $H(p(k))$. Although we could manipulate an exponential method to do this with an additional parameter, a simpler method would be to assign a polynomial weight such as 2^k to each of the $H(p(k))$ (which vary between 0 and 1) thereby clearly separating the influence of each from the other.

Since the $H(p(k))$ are not temporal, we could weight the $H(p(k))$ from the highest ($k = n$-1) to the lowest ($k = 0$) derivative or vice versa. For higher periods P, the d_k only fall to zero at a higher k. For some strings the k-1th derivative does not fall to zero at all, indicating that there is no periodicity in the binary string. Since we are attempting to measure the order and disorder of a binary string, if no order (i.e. periodicity) has emerged following the calculation of the n-1th derivative we should assign the highest weight to that derivative thereby indicating that that string is more disordered than other strings where the derivative falls to zero earlier (i.e. at a lower k) and is either periodic or eventually periodic.

Other weighting methods could include *none* where the $H(p(k))$ are simply averaged and *linear*, where the $H(p(k))$ is a constant proportion of k. We have not evaluated either of these latter two methods.

5 BIENTROPY

BiEntropy, or BiEn for short, is a weighted average of the Shannon binary entropies of the string and the first n-2 binary derivatives of the string using a simple power law. This version of BiEntropy is suitable for shorter binary strings where $n \leq 32$ approximately.

$$\mathrm{BiEn}(s) = \Big(1 / (2^{n-1} - 1)\Big)\Big(\sum_{k=0}^{n-2} ((-p(k) \log_2 p(k) - (1-p(k)) \log_2 (1-p(k)))) \, 2^k\Big)$$

The final derivative d_{n-1} is not used as there is no variation in the contribution to the total entropy in either of its two binary states. The highest weight is assigned to the highest derivative d_{n-2}.

If the higher derivatives of an arbitrarily long binary string are periodic, then the whole sequence exhibits periodicity. For strings where the latter derivatives are not periodic, or for all strings in any case, we can use a second version of BiEntropy, which uses a Logarithmic weighting, to evaluate the complete set of a long series of binary derivatives.

$$\mathrm{Tres\ BiEn}(s) = \Big(1 / \sum_{k=0}^{n-2} \log_2 (k+2)\Big)\Big(\sum_{k=0}^{n-2} (-p(k) \log_2 p(k) - (1-p(k)) \log_2 (1-p(k))) \log_2 (k+2)\Big)$$

The logarithmic weighting or (TBiEn for short) again gives greater weight to the higher derivatives. Depending upon the application, other weightings could be used.

The BiEntropy algorithm evaluates the order and disorder of a binary string of length n in $O(n^2)$ time using $O(n)$ memory.

6 BIENTROPY OF THE 2-BIT STRINGS

The BiEntropy of a 2-bit string is given in Table 2.

Table 2 - The BiEntropy of a 2-bit string

String	Description	BiEntropy
00	Perfectly ordered	0
01	Perfectly disordered	1
10	Perfectly disordered	1
11	Perfectly ordered	0

Table 2 depicts the XOR operation and the computation of the binary derivative of a 2-bit string.

7 BIENTROPY OF THE 4-BIT STRINGS

We show in Tables 3A & 3B the layout of some simple Excel spreadsheets to compute the BiEn and TBiEn of a 4-bit string. We used a simple =IF statement to compute each bit of the derivatives. We show a graphic of the BiEn of the 4-bit strings in Figure 1. The TBiEn graphic is very similar though the values of BiEn and TBiEn differ slightly.

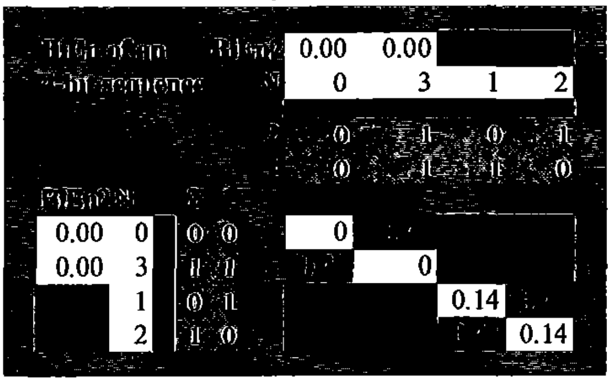
Table 3A - Computing the BiEn of a 4-bit string

				1's	***n***	***p***	***(1-p)***	**-*p*log(*p*)**	**-(1-*p*)log(1-*p*)**	**BiEn**	***k***	**2^*k***	**BiEn*2^*k***
1	0	1	1	3	4	0.75	0.25	0.31	0.50	0.81	0	1	0.81
1	1	0		2	3	0.67	0.33	0.39	0.53	0.92	1	2	1.84
0	1			1	2	0.50	0.50	0.50	0.50	1.00	2	4	4.00
												7.00	6.65
												BiEn(s)	**0.95**

Table 3B - Computing the TBiEn of a 4-bit string

				1's	***n***	***p***	***(1-p)***	**-*p*log(*p*)**	**-(1-*p*)log(1-*p*)**	**BiEn**	***k***	**log(*k*+2)**	**BiEn*log(*k*+2)**
1	0	0	1	2	4	0.50	0.50	0.50	0.50	1.00	0	1.00	1.00
1	0	1		2	3	0.67	0.33	0.39	0.53	0.92	1	1.58	1.46
1	1			2	2	1.00	0.00	0.00	0.00	0.00	2	2.00	0.00
												4.58	2.46
												TBiEn(s)	**0.54**

Figure 1 - BiEn of the four-bit strings

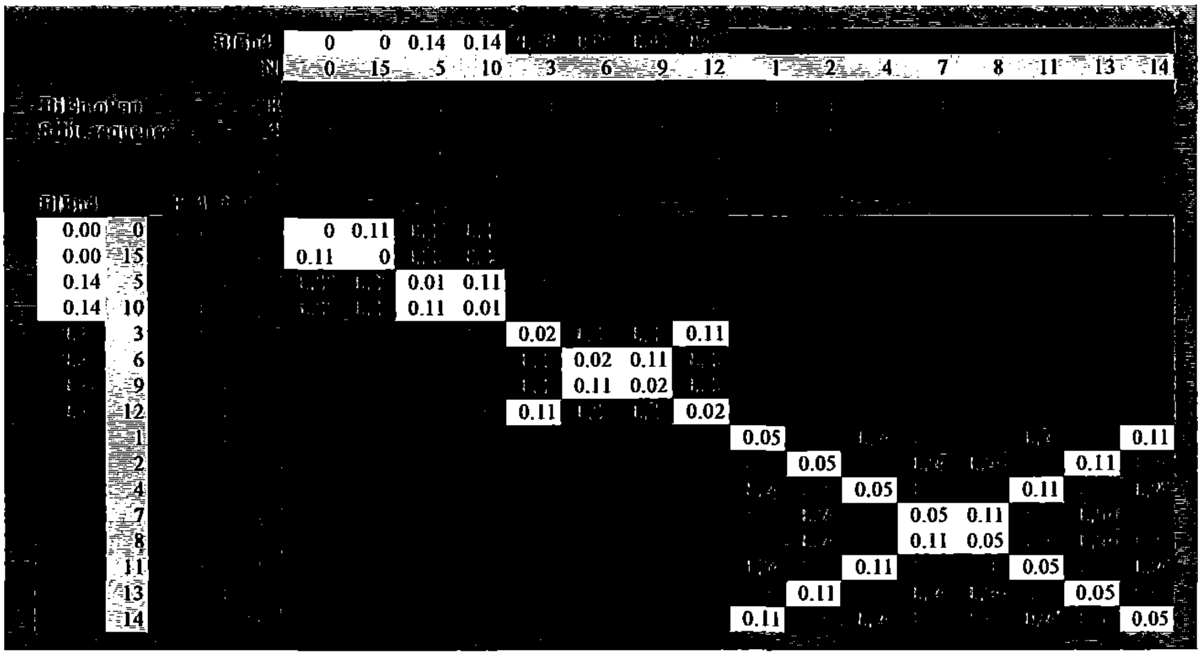


There are two perfectly ordered strings 0000 & 1111, two nearly ordered, periodic strings 0101 and 1010, four intermediately disordered (nperiodic) strings where the left two bits are the 1's complement of the right two bits and eight disordered (aperiodic) strings where either a single 1 or a single 0 transits a four bit field. Note the general XOR structure of the table. Mean BiEn for the 4-Bit strings is 0.594, standard deviation 0.389. Mean TBiEn is 0.644, standard deviation 0.355.

8 BIENTROPY OF THE 8-BIT STRINGS

We show in Figure 2 the BiEn of all 256 8-Bit strings. They are colour coded such that the *periodic* strings are white, *nperiodic* strings are light and dark yellow and *aperiodic* strings are dark orange. The diagram is structured such that the X and Y axes show the 4 bit strings of which each 8 bit string is comprised. The X and Y axes are sorted so that low BiEn or ordered 4 bit strings appear towards the top and left of the table and high BiEn or disordered 4-bit strings appear to the bottom and right. The Y-Axis corresponds to the first four bits of the string. Note that Figure 2 also has the general configuration of the XOR function. The white diagonal shows the zero and lower BiEn of the 16 repeated 4-bit strings. The TBiEn diagram is identical in its main XOR partitions but differs slightly in the other two partitions.

Figure 2 BiEn of the 4 and 8-bit strings



Using the BiEn (and TBiEn) metrics, exactly half of the 8-bit strings are classified as being nearly perfectly disordered (BiEn > 0.90). The last binary derivative of each of these strings is 1. They are the *aperiodic* 8 bit strings. 16 strings, which comprise two repetitions of the same 4-bit string, are nearly perfectly ordered (BiEn < 0.10). The last derivative of each of these strings is 0. These are the *periodic* strings. The remainder are neither ordered nor disordered to a greater or lesser degree. These are the *nperiodic* strings – the last derivative is 0 but the entire string has no single period. BiEntropy is fractal from the self-similarity exhibited in Figures 1 & 2. Mean BiEn for the 8-Bit strings is 0.625, standard deviation 0.340. Mean TBiEn is 0.747, standard deviation 0.209. The BiEn and TBiEn of the 8 bit strings are strongly correlated (Adjusted $R^2 = 0.85$).

We show in Figure 3 the distribution of BiEn and TBien for the 8-bit strings. BiEn & TBiEn do not reach 1.0. Were they to do so, the $p(k)$ for all $k \leq n$-2 would have to be exactly 0.5, which is impossible as k is odd at least once for all $n \geq 3$. Note that in the absence of a closed form solution for determining the BiEntropy of a finite binary string, BiEntropy has to be determined empirically.

Figure 3 - Distribution of the BiEntropy of the 8-bit Strings

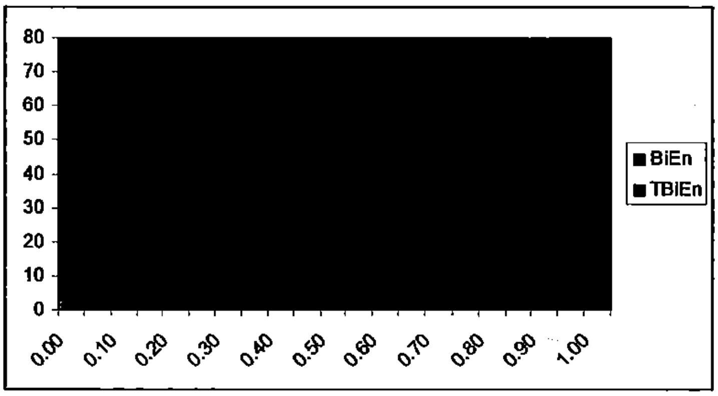
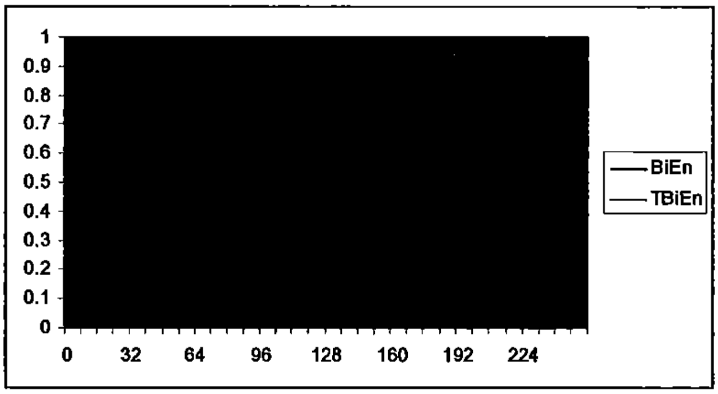


Figure 4 shows the BiEntropies of the 8-bit strings in ascending order and further illustrates the fractal nature of BiEntropy.

Figure 4 BiEntropy of the 8-bit strings in ascending order



BiEn and TBiEn differ somewhat in their distributions and the way in which they order the nperiodic 8 bit strings. They order the periodic and aperiodic 8 bit strings in the same way. The comparative utility of BiEn and TBiEn will have to be determined experimentally.

9 BIENTROPY OF THE PRIME NUMBER SEQUENCE

9.1 The Binary Encoded Primes (BEP's)

Consider the q natural numbers starting from 2:

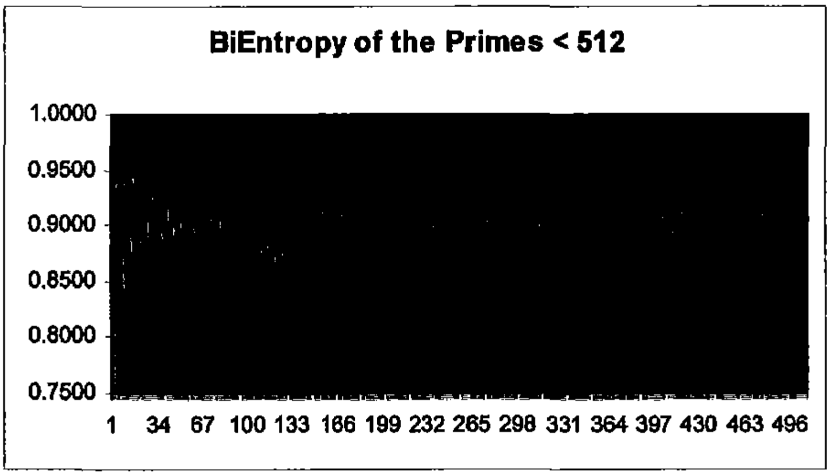
2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.....

We can encode them in a binary string B of length n ($1 \leq n \leq q$) such that the primes are encoded as 1 and the composites as 0. B_i is the ith digit of B ($i \geq 1$).

1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1.....

Thus B_1, corresponding to the natural number 2 is 1 and B_4, corresponding to the natural number 5 is also 1. We can easily compute the logarithmic BiEntropy of strings B for all n ($2 \leq n \leq q$) which we show in Figure 2 for $q = 512$.

Figure 5 – The Logarithmic BiEntropy (TBiEn) of binary encoded primes < 512



We can see that the logarithmic BiEntropies of the binary strings corresponding to the primes < 512 are close to 1.0 and that these strings are mostly (but not exclusively) aperiodic. This result is implicit due to the Prime Number Theorem. The BiEntropy dip around 114-136 corresponds to a long sequence of composites broken by only two primes.

There are some simple corollaries regarding the periodicity of the primes which follow directly from Nathanson's definitions of periodicity for infinite binary strings.

COROLLARY ONE – THE SEQUENCE OF PRIME NUMBERS IS NOT PERIODIC

Consider a binary string B of even length n ($n \geq 4$) containing the binary encoding of the primes as above starting from 2. B_i is the ith digit of B ($i \geq 1$). The binary string B is periodic if, for some positive integer p, $B_{i+p} = B_i$ for all $1 \leq i \leq n$.

This is impossible for $p = 1$ because the even numbers are composite. This is also impossible for $p \geq 2$ because:

a) both $B_1 = 1$ and $B_2 = 1$ (because both 2 and 3 are prime) *and*
b) no further pairs of natural numbers which are adjacent primes can occur because the even numbers are composite.

Hence the binary string B corresponding to the primes is not periodic for all even values of n ($n \geq 4$) for all $p \geq 1$. The binary string B of length $n = 2$ corresponding to the first two primes 2 & 3 is periodic with $p = 1$. We choose not to compare strings of unequal length which would be necessary if n was odd.

9.2 The Prime Encoded Non-Negative Integers (PENNI's)

Consider the r non negative integers:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.....

We can encode them in a binary string E (after Eratosthenes) of length n ($1 \leq n \leq r$) such that the primes are encoded as 1 and the non primes (0 and 1) and composites are encoded as 0. E_j is the jth digit of E ($j \geq 1$).

0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0.....

Thus E_1, corresponding to 0 is 0 and E_4, corresponding to the natural number 3 is 1. We can again compute the logarithmic BiEntropy of strings E for all n ($2 \leq n \leq 512$) with results almost identical to those depicted in Figure 2 and omitted for brevity.

There follows a second simple, but differing corollary, which demonstrates that the PENNI's (which includes the primes) are not periodic, but in a way that emphasises the complete absence of all periodicity.

COROLLARY TWO – THE PENNI'S ARE NOT PERIODIC

Consider a binary string E of even length n ($n \geq 4$) containing the binary encoding of the primes and the non-negative integers as above starting from 0. E_j is the jth digit of E ($j \geq 1$). The binary string E is periodic if, for some positive integer p, $E_{j+p} = E_j$ for all $1 \leq j \leq n$. Let $e_{p,k,j}$ be the jth character of the kth period of E for period of length p (p, k, $j \geq 1$). We show in Figure 6 the periodicities of the primes and the composites for various small p starting from the origin, 0. There is no other natural number which is the origin of all the primes and all of the composites. We choose not to compare strings of unequal length which would be necessary if n was odd.

Figure 6 The Periodicities of the Primes and the Composites

Note that for each period p where p is prime and $p \geq 2$, $e_{p,2,1}$ is not equal to $e_{p,3,1}$ because $e_{p,2,1}$ marks the first occasion where p is known to be prime, after the algorithm of Eratosthenes, and $e_{p,3,1}$ marks the first non prime multiple of p which occurs in the first occasion of the following period. Where p is even $e_{p,1,3}$ is not equal to $e_{p,2,3}$ because 2 is the only even prime. For $p = 1$, there is an absence of periodicity not least because both the even numbers and the odd numbers are periodic with period 2.

Hence the binary string E corresponding to the PENNI's is not periodic for all even n ($n \geq 4$) for all $p \geq 1$. Corollary One proves explicitly that the sequence of prime numbers is not periodic. Corollary Two is necessary to demonstrate explicitly the total absence of periodicity in the sequence of prime numbers for all even sequences for all periods from the origin 0.

10 BIENTROPY OF SOME 7*5 DOT MATRIX PRINTER CHARACTERS

We obtained [Mitchell, 2008] the binary patterns corresponding to the 96 Alphabetic (upper and lower case), Numeric and Punctuation characters of the US ISO 646 7-bit character set. We arranged each of these characters in a 7*5 array and computed the Horizontal and Vertical Binary BiEntropies of these strings using Horizontal and Vertical Raster scans, both of which were 35 bit binary strings.

By way of comparison, we used the random number generator within Microsoft Excel 2003 to create a set (RANDOM) of 96 randomly generated 7*5 dot matrix characters where $p(0)$ of the 35 bit array was 0.5. We also used the 6 bit Braille [Jiménez et al, 2009] dot-pattern (BRAILLE), arranged as 3*2 array superimposed on the central 5*3 bits of a 7*5 grid. Finally, we designed twelve 7*5 dot matrix characters to exhibit High Entropy (HECS) and a further twelve to exhibit Low Entropy (LECS). We show samples from each character set in Table 4A & their average BiEntropies in Table 4B.

Table 4A – Samples from five character sets with differing BiEntropies

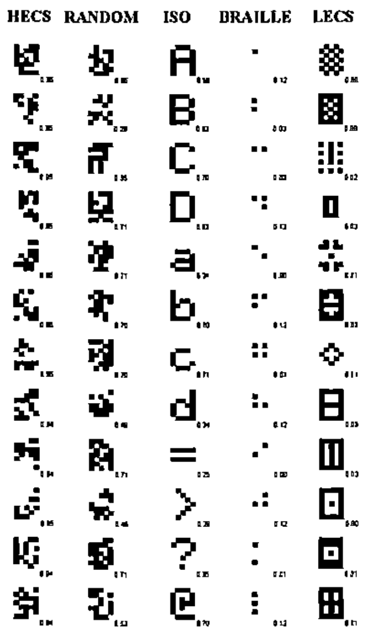


Table 4B - The BiEntropy of Some 7*5 Dot Matrix Character Sets

Character Set	BiEn Mean	BiEn Stdev	TBiEn Mean	TBiEn Stdev	N
HECS	0.945	0.006	0.947	0.011	12
RANDOM	0.634	0.240	0.876	0.168	96
ISO	0.494	0.291	0.844	0.117	96
BRAILLE	0.068	0.053	0.661	0.124	64
LECS	0.014	0.012	0.622	0.205	12

The BiEntropies of each of the five character sets are distinguished from each other ($p < 0.01$) by the BiEn metric, though note that HECS and LECS were designed to be such. For the TBiEn metric, HECS & ISO, ISO & BRAILLE and ISO & LECS were distinguished from each other ($p < 0.01$). BiEn penalises zero upper derivatives more heavily than TBiEn but note that both metrics place the groups in the same order. Note that binary strings with a length ≥ 35 may contain these characters during the evaluation of their 35th last derivative.

11 BIENTROPY OF SOME IRRATIONAL, NORMAL AND PSEUDORANDOM NUMBERS

We obtained the first million digits of the decimal expansions of π, e, $\sqrt{2}$ and $\sqrt{3}$ from the internet [Nemiroff & Bonnell, 1994][Andersson, 2013]. We also obtained a set of one million random decimal digits produced by the RAND corporation almost 60 years ago [RAND, 2001]. We obtained one million random decimal digits from Excel 2003 (Service Pack 2) and Excel 2010 using 200,000 consecutive calls to the RAND() function formatted as a five digit integer with leading zeroes. We created the first million digits of the [Champernowne, 1933] number, CHAMP, which is proven normal. For each expansion we computed BiEn(s) for the first 1,000 then the first 10,000 and then the first 30,000 consecutive sections each of length $s = 32$ bits starting at the first digit. From the decimal expansions, we encoded digits 0-4 as 0 and 5-9 as 1. We show the mean and standard deviation of BiEn for each set of strings in Table 5.

Table 5 - The BiEntropy of Some Irrational, Random and Normal Numbers

Number/Set	Mean N=1,000	Stdev N=1,000	Mean N=10,000	Stdev N=10,000	Mean N=30,000	Stdev N=30,000
PI	0.6190	0.3367	0.6302	0.3346	0.6283	0.3361
EXCEL10	0.6260	0.3422	0.6286	0.3381	0.6281	0.3374
CHAMP	0.6707	0.3763	0.6180	0.3443	0.6263	0.3353
SQRT(2)	0.6171	0.3400	0.6267	0.3384	0.6254	0.3384
EXCEL03	0.6271	0.3310	0.6237	0.3364	0.6250	0.3370
SQRT(3)	0.6408	0.3321	0.6307	0.3356	0.6247	0.3375
E	0.6425	0.3341	0.6243	0.3388	0.6235	0.3376
RAND	0.6288	0.3382	0.6179	0.3385	0.6217	0.3374

The BiEntropy of the RAND string is significantly different from (i.e. lower than) the BiEntropy of the decimal expansions of π and the Excel 2010 random digit string ($p < 0.05$) for 10,000 and 30,000 trials and from Champernowne's number for 30,000 trials. The BiEntropies of the irrational, Excel RNG and Champernowne strings are not significantly different from each other for 30,000 trials. Note that there are 2^{32} potential entropy states for the 32 bit BiEn function. The sample sizes used in this study constitute a minute proportion of this state space.

We performed a further range of BiEntropic analysis on this data using strings varying in length from 16 bits up to 256 bits. We used BiEn for the shorter string lengths and TBiEn for string lengths > 32. We used in addition an alternative encoding whereby each decimal digit 0-7 generated a bit string 000-111, decimal 8 generated 0 and decimal 9 generated 1. We did not find any statistically significant differences between the BiEntropies of any of these strings save the expected occasional spurious result. Except that we found the result for the RAND string compared with π, Excel 10 and Champernowne's number confirmed ($0.1< p < 0.001$) in the alternative binary expansion when measured by the 32 bit BiEn function for 75,000 trials.

We show in Figure 5 the frequency distribution of BiEntropy for all eight digit strings for N=1,000 trials. We show in Figure 6 the initial slow progress of convergence by averaging BiEntropy for the first 2, 3, 4 ... 1000 values for each series. Other work has explored deficits from maximal irregularity for the same irrationals [Pincus & Kalman, 1997].

Figure 5 Frequency distribution of BiEn for the first 1,000 32-bit strings

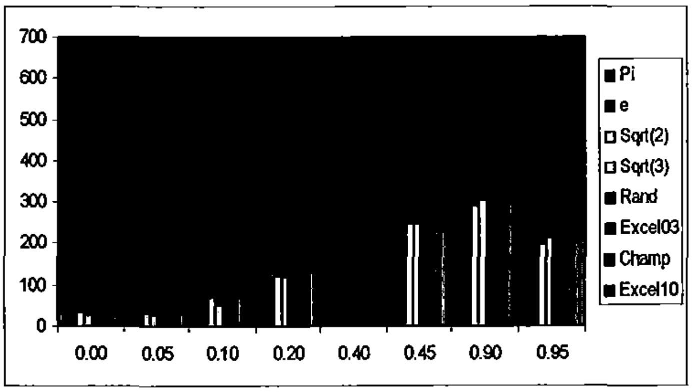
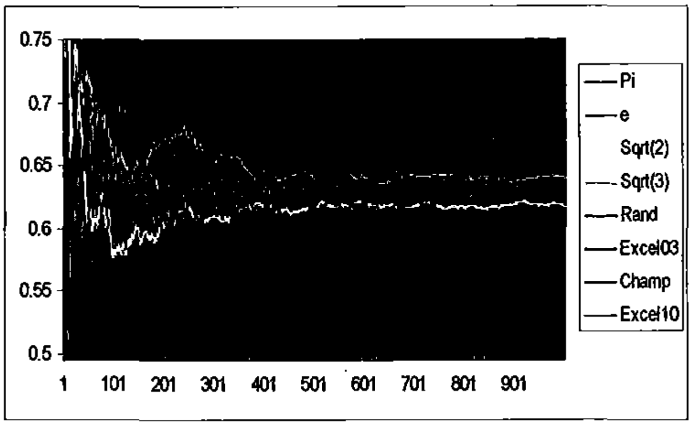


Figure 6 Mean BiEntropy for the first 1000 32-bit strings



12 BIENTROPY OF SOME PLAIN AND ENCRYPTED SPREADSHEETS

We created and/or obtained four large Excel Spreadsheets each of which had a file length of approximately 175 Kilobytes. We encrypted each spreadsheet using the weak Office 97 and strong 128 bit AES algorithms supported in Excel 2003. We computed TBiEn, using a simple C routine, for 1,000 sections each of length 1,024 bits for each encrypted or unencrypted spreadsheet file using the unmodified raw binary data of each file. The contents of each spreadsheet and the values of TBiEn are given in Table 6.

Table 6 - The Logarithmic BiEntropy of Some Plain and Encrypted Spreadsheets

Spreadsheet	**Encryption**	**TBiEn (1024 bit) N=1,000**	
		Mean	**Stdev**
Numbers (all cells = 123)	None	0.8980	0.0732
Numbers (all cells = 123)	Office 97	0.9913	0.0545
Numbers (all cells = 123)	AES	0.9913	0.0545
Random Numbers	None	0.9857	0.0559
Random Numbers	Office 97	0.9913	0.0545
Random Numbers	AES	0.9914	0.0545
Address Database	None	0.9428	0.1770
Address Database	Office 97	0.9913	0.0545
Address Database	AES	0.9913	0.0545
Financial Model	None	0.9450	0.1736
Financial Model	Office 97	0.9912	0.0545
Financial Model	AES	0.9911	0.0545

The BiEntropies of all the encrypted files differ from their unencrypted counterparts ($p < 0.01$). All the unencrypted spreadsheets are distinguished from each other ($p < 0.01$) except for the Address Database / Financial model pairing. The BiEntropy reflects the file contents. The unencrypted file with the lowest entropy had the numeric constant 123 in every cell. The unencrypted file with the highest entropy had random number values in every cell. The entropy of the address database and the financial model were similar, lying between the other two extremes. BiEntropy has not distinguished between the two encryption methods. BiEntropy has, of course, no knowledge of the Excel file structure. Figure 2 implies that only half of all binary strings are fully aperiodic, which may have implications in the cryptographic security of key bit strings.

13 BIENTROPY OF UK STOCK MARKET PRICE CHANGES

We obtained [Yahoo, 2012] the UK FTSE daily closing prices for the near ten year period 1/1/2003 – 22/8/2012. These were provided in an Excel spreadsheet with the rows representing the days and the columns representing the 100 largest companies in the FTSE index. We deleted rows corresponding to weekends and bank holidays, and columns where the company had not been FTSE quoted for the entire period. The resulting spreadsheet had 2,452 rows i and 71 columns j containing FTSE data in a contiguous array $P_{i,j}$ with no zero values.

We created a second binary array $T_{i,j}$ which recorded absolute price changes between one day and the next, starting at the second day (first day = 0). We used a threshold R, ($0.00 \leq R \leq 0.03$) such that:

$$\textbf{\textit{if}}\ \mathrm{ABS}\,(P_{i,j}\ /\ P_{i-1,j} - 1) > R\ \textbf{\textit{then}}\ T_{i,j} = 1\ \textbf{\textit{else}}\ T_{i,j} = 0$$

We used the Excel RAND () function and an Excel Data Table [Tyszkiewicz & Balson, 2012] to facilitate the repeated selection of 1000 start prices S_k from $P_{i,j}$ where i and j were uniform random ($1 \leq i \leq 2200$ and $1 \leq j \leq 71$) and a corresponding set of closing prices C_k, where $C_k = P_{i+d,j}$ ($32 \leq d \leq 63$).

We calculated a set of holding returns

$$H_k = C_k\ /\ S_k$$

And a corresponding set of 32 bit BiEntropies B_k using $T_{i,j}$ through $T_{i+31,j}$.

We implemented the spreadsheet such that we created values of B_k and H_k separately for each of 32 values of d and 7 values of R. We sorted the Holding Returns H_k by descending BiEntropy B_k and computed the mean holding return for the upper and lower BiEntropy deciles (by summing the 100 closing prices and dividing by the sum of the 100 opening prices for each of the two deciles).
When analysing the Mean Holding Returns and the BiEntropies for each of the 7 individual values of R, we were unable to find any statistical significance for d. We show in Table 7 the Holding Return observed for all $R \geq 0$ for all d for the upper and lower BiEntropy deciles.

Table 7 High & Low BiEntropy - Mean Holding Return for various R

	Upper BiEntropy Decile		Lower BiEntropy Decile			
R	**Mean Holding Return**	**S.D.**	**Mean Holding Return**	**S.D.**	**p (n=32)**	**Sparsity% $T_{i,j}$**
0.000	1.0249	0.0328	1.0393	0.0406	-	49.04
0.005	1.0442	0.0401	1.0238	0.0437	< 0.10	36.55
0.010	1.0552	0.0513	0.9964	0.0196	< 0.01	25.44
0.015	1.0593	0.0573	1.0000	0.0141	< 0.01	17.45
0.020	1.0543	0.0443	1.0030	0.0155	< 0.01	11.95
0.025	1.0452	0.0509	1.0054	0.0115	< 0.01	8.38
0.030	1.0342	0.0301	1.0098	0.0101	< 0.01	6.00

We replaced a very small number of outliers where the Holding return was more than 3 S.D. from the mean for $R < 0.03$. For $R = 0.03$ the Holding Return for the lower BiEntropy decile was probably bimodal as 7 closely spaced results with a Holding Return << 1.00 were replaced. The significance and sense of the reported result was unchanged. For $R > 0.02$, the sparsity of $T_{i,j}$ was such that the lower BiEntropy decile contained more than 100 zero entries.

The mean holding return for FTSE stocks held for an average period of 48 days was approximately 4% higher for stocks which had exhibited more disordered (higher BiEntropic) behaviour at the beginning of the observed period compared with stocks which had exhibited more periodic (lower BiEntropic) behaviour ($p \ll 0.01$). The effect was not observed without a threshold $R \geq 0.005$.

Subsequent analysis of the data for all $R > 0$ showed that d also had a small positive effect of 1.3% for $d = 32$ on the holding return ($p < 0.05$).

The above analysis was performed using the 32 bit logarithmic TBiEn function. There were no statistically significant results to report using the 32 bit power law version of BiEn.

14 SUMMARY

We have described the BiEntropy algorithm and investigated its basic performance on 2, 4 and 8 bit binary strings. We have demonstrated that we can rank binary strings of arbitrary length in terms of the relative order and disorder of all of their digits. Our method is very simple and is based upon the use of the Exclusive Or function and some arithmetic weights. We show that BiEntropy is fractal such that the ordering method is consistent across strings of arbitrary length.

We used the BiEntropy function to investigate the order and disorder in the sequence of prime numbers. We showed that BiEntropy determined that the sequence of early prime numbers was disordered. We proved explicitly that the sequence of prime numbers is not periodic.

We then evaluated some 5*7 ISO dot matrix printer characters and some randomly generated characters on a similar 5*7 grid. Using the insight gained from BiEn, we designed a Low Entropy Character Set (LECS) and compared these characters with others from a High Entropy Character Set (HECS) which we also designed. We measured the BiEntropy of the characters of the Braille character set and showed for the first time that Braille is also a low entropy character set. The LECS character set is visually distinctive and easily extensible. It may provide a rational basis for extending the Braille methodology to a significantly wider character set and user base.

We then used BiEntropy to determine the relative order and disorder of consecutive 32 digit sections of some long expansions of some well-known irrational numbers including *pi* and *e* which we compared with Champernowne's normal number and outputs from the Excel 2003 and Excel 2010 RNG's. Despite the well known problems [McCullough, 2008] with the Excel RNG's we show that we cannot distinguish between the Excel RNG output and normal and suspected normal numbers using the 32 bit BiEntropy method.

We examined some plain and encrypted Excel spreadsheets and show that BiEntropy is able to clearly distinguish between them in the absence of any knowledge of the Excel file structure.

We examined ten years worth of daily UK stock market prices. We showed that investors prefer – i.e. pay more for – stocks that exhibit prior disordered behaviour. This result, if confirmed, will have significant implications for global financial markets [Croll, 2007 & 2009]. It is possible that entropy options may emerge as a means of profiting from and then neutralising the illustrated effects. There is in principle hidden information in any time series. Statistical time series methods may need to be re-examined in the light of these results.

15 FURTHER WORK

Given the successful application of BiEntropy in the five diverse fields outlined above, it is likely that BiEntropy will be widely applicable. We are working on the use of BiEntropy in Bit String Physics [Noyes, 1997]. We keenly anticipate any results which may follow from BiEntropic analysis of the binary encoding of the four bases of the human genome.

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□□ □□ □□*m*

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Andersson, E.A (2013) One Million Digits of Pi http://www.eveandersson.com/pi/digits/1000000 Accessed 10th Feb 2013 *m*

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Bruwer, C. S., (2005) Correlation attacks on stream ciphers using convolutional codes, Masters Thesis, University of Pretoria*m*

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Carroll, J.M., (1989) The binary derivative test: noise filter, crypto aid, and random-number seed selector, Simulation, 53 pp129-135 http://sim.sagepub.com/content/53/3/129.full.pdf+html

Carroll, J.M., Sun, Y., (1998) The Binary Derivative Test for the Appearance of Randomness and its use as a Noise Filter Tech Report No. 221 University of Western Ontario. Dept. of Computer Science

Chaitin, G. J. (1966). On the length of programs for computing finite binary sequences. Journal of the ACM (JACM), 13(4), 547-569.

Chen, W., Zhuang, J., Yu, W., & Wang, Z. (2009). Measuring complexity using FuzzyEn, ApEn, and SampEn. Medical Engineering & Physics, 31(1), 61-68.

Champernowne, D. G. (1933). The construction of decimals normal in the scale of ten. Journal of the London Mathematical Society, 1(4), 254-260. http://www.uea.ac.uk//~h720/teaching/dynamicalsystems/champernowne1933.pdf

Croll, G. J. (2007). The importance and criticality of spreadsheets in the city of London. Proc. EuSpRIG. arXiv preprint arXiv:0709.4063.

Croll, G. J. (2009). Spreadsheets and the financial collapse. Proc. EuSpRIG. arXiv preprint arXiv:0908.4420.

Davies, N., Dawson, E., Gustafson, H., Pettitt, A.N., (1995) Testing for randomness in stream ciphers using the binary derivative Statistics and Computing, Vol 5, pp307-310

Gao, Y., Kontoyiannis, I., Bienenstock, E., (2008) Estimating the Entropy of Binary Time Series: Methodology, Some Theory and a Simulation Study, Entropy, 10, pp71-99

Goka, T., (1970) An operator on binary sequences, SIAM Rev., 12 (1970), pp. 264-266

Haralick, R. M., Shanmugam, K., & Dinstein, I. H. (1973). Textural features for image classification. Systems, Man and Cybernetics, IEEE Transactions on, (6), 610-621.

Hawker, A. (2000). Building Financial Accuracy into Spreadsheets. Proc. EuSpRIG. arXiv preprint arXiv:0805.4219.

Jiménez, J., Olea, J., Torres, J., Alonso, I., Harder, D., & Fischer, K. (2009). Biography of Louis Braille and invention of the braille alphabet. Survey of ophthalmology, 54(1), 142-149.

Kolmogorov, A. N. (1965). Three approaches to the quantitative definition of information'. Problems of information transmission, 1(1), 1-7.

Makridakis, S., Wheelwright, S. C., & Hyndman, R. J. (2008). Forecasting methods and applications. Wiley. com.
Marsaglia, G.,"Random numbers fall mainly in the planes", Proc. Natl. Acad. Sci. 61(1), 25–28 (1968).

Marsaglia, G., & Zaman, A. (1993). Monkey tests for random number generators. Computers & mathematics with applications, 26(9), 1-10.

McCullough, B. D. (2008). Microsoft Excel's 'not the Wichmann–Hill'random number generators. Computational Statistics & Data Analysis, 52(10), 4587-4593.

McNair, J., (1989) The Binary Derivative: a new method of testing the appearance of randomness in a sequence of bits. MSc Thesis report, University of Western Ontario.

Mitchell, S., (2008) Open Source Urbanism http://www.openobject.org/opensourceurbanism/Bike_POV_Beta_4

Nathanson, M. B. (1971). Derivatives of binary sequences. SIAM Journal on Applied Mathematics, 21(3), 407-412.

Nemiroff, R., Bonnell, J. (1994) RJN's More Digits of Irrational Numbers Page, http://apod.nasa.gov/htmltest/rjn_dig.html accessed 10 Feb 2013 13:40.

Noyes, H. P. (1997). A Short Introduction to Bit-String Physics. arXiv preprint hep-th/9707020.

Pincus, S., (1991) Approximate entropy as a measure of system complexity. Proc. Natn. Acad. Sci. USA Vol 88 pp 2297-2301

Pincus, S. & Kalman, R.E., (1997) Not all (possibly) "random" sequences are created equal Proc. Natn. Acad. Sci. USA Vol 94 pp 3513-3518

Pincus, S., & Kalman, R. E. (2004). Irregularity, volatility, risk, and financial market time series. Proceedings of the National Academy of Sciences of the United States of America, 101(38), 13709-13714.

Pincus, S. & Singer, B. H., (1996) Randomness and Degrees of Irregularity, Proc Natl Acad Sci U S A. March 5; 93(5): 2083–2088.

Pincus, S. M., & Viscarello, R. R. (1992). Approximate entropy: a regularity measure for fetal heart rate analysis. Obstetrics & Gynecology, 79(2), 249-255.

RAND, (2001) A Million Random Digits With 100,000 Normal Deviates, http://www.rand.org/pubs/monograph_reports/MR1418.html Accessed 10th Febraury 2013, 14:25

Richman, J. S., & Moorman, J. R. (2000). Physiological time-series analysis using approximate entropy and sample entropy. American Journal of Physiology-Heart and Circulatory Physiology, 278(6), H2039-H2049.

Rukhin, A.L., (2000a) Approximate Entropy for Testing Randomness, J. Appl. Prob. Volume 37, Number 1, 88-100

Rukhin, A.L., (2000b) Testing Randomness: A Suite of Statistical Procedures, Theory of Probability and its Applications, 45:1, 111–132

Shannon, C. E., (1948), A Mathematical Theory of Communication, Bell System Technical Journal 27 (3): 379–423

Tyszkiewicz, J., & Balson, D. (2012). User Defined Spreadsheet Functions in Excel. Proc. EuSpRIG. arXiv preprint arXiv:1211.6781.

Yahoo Finance (2012) http://uk.finance.yahoo.com/

Duality, Chirality and Singularity

Peter Rowlands

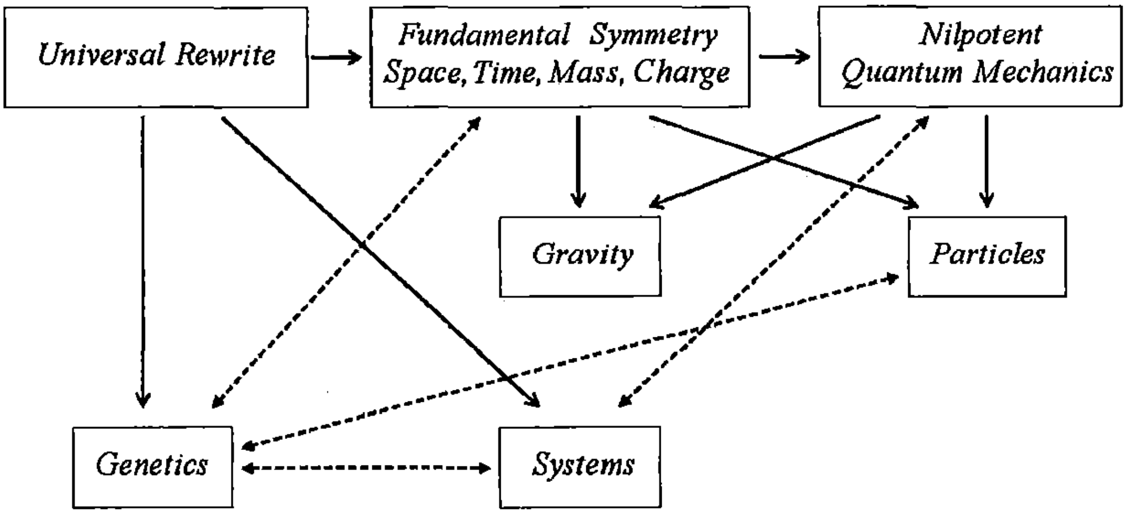
Department of Physics, University of Liverpool, Oliver Lodge Laboratory, Oxford St, Liverpool, L69 7ZE, UK. email p.rowlands@liverpool.ac.uk

Abstract. Duality, chirality and singularity are identified as necessary consequences of a nilpotent structure emerging from a universal rewrite system. The fundamental requirement incorporates two vector spaces, each containing the same information but differently organized because the minimal structure requires that the rotation symmetry of one space is preserved while that of the other is broken. Many applications follow.

Universal rewrite as a source of fundamental theory

A long series of researches presented at successive ANPA meetings and to a large extent summarised in Rowlands (2007) has indicated that a universal rewrite system can be seen as a foundation or basis for aspects of mathematics, physics, chemistry, biology, computer science and systems theory. It appears to be a natural information-processing theory which can be described on a very general basis and is a strong candidate for the driving process which is responsible for the behaviour of self-organizing systems. The rewrite structure was devised in collaboration with Bernard Diaz (Rowlands and Diaz, 2002, Diaz and Rowlands, 2005), and one of its most significant consequences appears to be a fundamental symmetry between the physical parameters space, time, mass and charge, long explored by the author, and explicable on the basis of an evolution of a Clifford-type algebra through successive stages representing real numbers, complex numbers, quaternions and multivariate vectors, and so organized as to lead to a zero-totality universe, as specified in the rewrite process. In effect, 'physical' properties seem to be entirely explicable in terms of algebraic ones.

When the information from this symmetry is compactified into a minimal form, the result appears to be a form of relativistic quantum mechanics, based on *nilpotent* or zero-norm wavefunctions. Many aspects of particle physics and of gravity seem to emerge as particular consequences of either the fundamental symmetry or the nilpotent quantum mechanics, or some combination of them. The rewrite mechanism, however, would appear to have an even wider application, and results obtained in genetics, in collaboration with Vanessa Hill (Hill and Rowlands, 2008, 2011a and b), and in systems theory, in collaboration with Peter Marcer (Marcer and Rowlands, 2010a, b and c), can be seen as either emerging from the rewrite mechanism or massively extending it into new domains. In addition these results often parallel similarly-structured results obtained in fundamental physics. The thing that emerges most strongly is that various fundamental ideas emerge at each stage in the rewrite process and in each of the separate areas of investigation. The same patterns are seen at different levels of complexity, and it is largely because of this that complex beings such as ourselves are able to understand simpler foundations, though only after multiple iterations from our starting assumptions. Three particularly important ones are duality, chirality and singularity.



Universal Rewrite and Fundamental Theory

direct connections   *parallel developments*

Rewrite systems: conventional and universal

Rewriting is a general process involving strings and alphabets, and is classified according to what is rewritten – strings, terms, graphs, etc. A rewrite system is a set of equations that characterises a system of computation that provides one method of automating theorem proving and is based on the use of rewrite rules. In rewriting we are concerned with much the same components as in languages and grammars, namely an alphabet (finite or not) with symbols, strings or words, sentences and expressions; and also rules governing what is rewritten and how it is rewritten. All computer programming based on the Turing machine can be described in terms of a rewrite (or production) system. A conventional rewrite system requires 4 fixed components:

alphabet
rewrite rules (productions)
a start 'axiom' or symbol
stopping criteria

The generation of Fibonacci numbers provides a convenient example. Here we have:

alphabet:	A B
stopping criteria	none
start axiom	A
rules (productions)	p1: $A \to B$
	p2: $B \to AB$

We start with generation 0, and the single symbol A. Apply rule p1, and replace A with B. In generation 2, B becomes AB. Now, A becomes B, and B becomes AB ... etc.

N=0	A	length	1
N=1	→B		1
N=2	→ AB		2
N=3	→ BAB		3
N=4	→ ABBAB		5
N=5	→ BABABBAB		8
N=6	→ ABBABBABABBAB		13
N=7	→ BABABBAB...		

We may notice how $A \to B$ and $B \to AB$ reproduce the structure of 3-dimensional algebra

$$\boldsymbol{i} \to \boldsymbol{j} \qquad \boldsymbol{j} \to \boldsymbol{i}\boldsymbol{j} = \boldsymbol{k}$$

A string like ABBABBABABBAB seems to be creating a fractal-like structure in 3-dimensional space, but in the plane. The logarithmic spiral continually extends the structure along the 2 dimensions of the plane because it is unable to extend into the third dimension.

In addition to conventional alphabets, it is possible to create a *universal* alphabet – an alphabet used in a universal rewrite system. A universal rewrite system is one which allows all four elements – alphabet, start object, rules, stopping criteria – to be varied. At the beginning of such a system the alphabet is often the same as the start object. Perhaps the alphabet is also the rules. One universal alphabet allows us to do this if we assume the universe or any alphabet within it is always entirely nothing. The principal and only assumption is a zero totality state, with no unique description, that is infinitely degenerate. In other words, we have to keep describing the alphabet in a way that ensures that it is always new, but still zero.

The start position is zero – but is any such state and not unique; and there is no limit or stopping criteria – because the final state is also always zero. We can conceive it as defining a zero attractor. Any nonzero deviation from 0, say R, necessarily incorporates an automatic mechanism for recovering the zero, say 'conjugate' R^*. But the zero totality (R, R^*), does not define a *unique* zero, and must always define a new structure. We know that the new structure is new because it defines the position of the previous structure, in this case (R, R^*), within it. The process then continues indefinitely. In effect, we define a series of *cardinalities*, but ones based on zero, rather than on infinity.

The process is most conveniently displayed (though not defined) by a 'concatenation' or placing together, with no algebraic significance, of the alphabet with respect to either its components ('subalphabets') or itself. If the alphabet describes a cardinality or totality, then anything other than itself will necessarily be a 'subalphabet' and the concatenation will yield nothing new. The only other option will be concatenation with itself, which, to ensure that the cardinality is not unique, must yield a new cardinality or zero totality alphabet.

The condition *create* symbolised by $\Rightarrow$ means that every alphabet produces a new one which subsumes itself as a component. The condition *conserve* symbolised by $\rightarrow$ means that nothing new is created except by extending the alphabet. Creation is always of a new zero. The process can be recursive, creating everything E (all symbols) at once, or iterative, creating one symbol only. In fact it is both iterative and recursive. The process is fractal and can begin or end at any stage. Self-similarity exists at all stages. It exists before time – in fact, it creates time. That is, the condition of non-unique cardinality requires that

(subalphabet) (alphabet) $\rightarrow$ (alphabet) *there is nothing new*
(alphabet) (alphabet) $\Rightarrow$ (new alphabet) *the zero totality not unique*

The nature of the new alphabet produced by $\Rightarrow$ will always be determined by the need to satisfy $\rightarrow$ in all possible cases.

We can only find out what a new alphabet will look like when we have worked out all the ways in which concatenation with its subalphabets will yield only itself. Suppose, then, that our first zero totality alphabet has the form (R, R^*). Applying the conserve mechanism $(\rightarrow)$ by concatenating it with its subalphabets should produce nothing new. So

$$(R)\,(R,R^*) \rightarrow (R,R^*)$$
$$(R^*)\,(R,R^*) \rightarrow (R^*,R) \equiv (R,R^*)$$

No concept of 'ordering' is required by concatenation, but each term must be distinct, so we can easily show that these concatenations lead to rules of the form:

$$(R)\,(R) \rightarrow (R)\,;\,(R^*)\,(R) \rightarrow (R^*)\,;\,(R)\,(R^*) \rightarrow (R^*)\,;\,(R^*)\,(R^*) \rightarrow (R)$$

Any given zero-totality alphabet such as (R, R^*), however, cannot be unique, and concatenation with itself (or 'create'), symbolised by $\Rightarrow$, must produce a new conjugated alphabet, and the new alphabet can only be guaranteed to be new if it also incorporates the old. Something like (A, A^*), with the terms undefined, would be not be distinguishable from (R, R^*), but (R, R^*, A, A^*) would be if (A, A^*) was distinguishable from (R, R^*). In addition, (R, R^*, A, A^*) must be defined in such a way that the conserve mechanism still applies, so concatenation with the subalphabets yields nothing new.

$$(R)\,(R,R^*,A,A^*) \rightarrow (R,R^*,A,A^*)$$
$$(R^*)\,(R,R^*,A,A^*) \rightarrow (R^*,R,A^*,A)$$
$$(A)\,(R,R^*,A,A^*) \rightarrow (A,A^*,R^*,R)$$
$$(A^*)\,(R,R^*,A,A^*) \rightarrow (A^*,A,R,R^*)$$

The listing or order of the terms is different, but there are no new ones. We guarantee that (R,R^*) and (A,A^*) can only be different by making

$$AA = R^*, \text{ etc., while } RR = R$$

At the next stage, we have a new problem, for

$$(R, R^*, A, A^*)\,(R, R^*, A, A^*) \Rightarrow (R, R^*, A, A^*, B, B^*)$$

would produce new concatenated ***terms*** like AB, AB^*, which are not in the alphabet, when we apply the conserve mechanism (→). So we include these in advance, as in

$$(R, R^*, A, A^*)\,(R, R^*, A, A^*) \Rightarrow (R, R^*, A, A^*, B, B^*, AB, AB^*)$$

but we have to check that the conserve operation with (R), (R^*), (A), (A^*), (B), (B^*), (AB), (AB^*) successively concatenated with this alphabet leaves the alphabet unchanged. The process is straightforward for the first six operations:

$$\begin{aligned}
&(R)\,(R, R^*, A, A^*, B, B^*, AB, AB^*) \rightarrow (R, R^*, A, A^*, B, B^*, AB, AB^*)\\
&(R^*)\,(R, R^*, A, A^*, B, B^*, AB, AB^*) \rightarrow (R^*, R, A^*, A, B^*, B, AB^*, AB)\\
&(A)\,(R, R^*, A, A^*, B, B^*, AB, AB^*) \rightarrow (A, A^*, R^*, R, AB, AB^*, B, B^*)\\
&(A^*)\,(R, R^*, A, A^*, B, B^*, AB, AB^*) \rightarrow (A^*, A, R, R^*, AB^*, AB, B^*, B)\\
&(B)\,(R, R^*, A, A^*, B, B^*, AB, AB^*) \rightarrow (B, B^*, AB, AB^*, R^*, R, A, A^*)\\
&(B^*)\,(R, R^*, A, A^*, B, B^*, AB, AB^*) \rightarrow (B^*, B, AB^*, AB, R, R^*, A^*, A)
\end{aligned}$$

However, when we come to the operations of the concatenated terms, such as (AB) and (AB^*) on themselves and on each other, we have to choose between the 'commutative' and 'anticommutative' options:

$$\begin{array}{llll}
(AB)\,(AB) & \rightarrow & (R) & (\textit{commutative})\\
(AB)\,(AB) & \rightarrow & (R^*) & (\textit{anticommutative})
\end{array}$$

It quickly becomes apparent that the decision has already been made, for *only the anticommutative option* leads to something new. The commutative option leaves A and B indistinguishable and so does not extend the alphabet. We therefore have no option but to default on the anticommutative option, and the last two concatenations become:

$$\begin{aligned}
&(AB)\,(R, R^*, A, A^*, B, B^*, AB, AB^*) \rightarrow (AB, AB^*, B, B^*, A, A^*, R^*, R)\\
&(AB^*)\,(R, R^*, A, A^*, B, B^*, AB, AB^*) \rightarrow (AB^*, AB, B^*, B, A^*, A, R, R^*)
\end{aligned}$$

The process cannot be repeated by introducing new terms, such as (C), (D), etc., when the alphabet is extended stages because some inconsistency will always emerge at some point in the analysis. Anticommutativity produces a closed 'cycle' with components (A, B, AB) and their conjugates, and excludes any further terms from anticommuting with them. However, the extension can be made if (C), (D), etc., constitute a *new* anticommuting cycle, and the repetition of this process can continue to infinity. The terms in anticommuting combinations such as (C), (D); (E), (F); (G), (H), etc., become uniquely distinguishable *because each has a unique partner.*

The alphabets which emerge from this create process ($\Rightarrow$) are constructed from an indefinitely extended series of identically structured closed anticommutative cycles, each of which commutes with all others. Essentially the same structure is familiar to us as the infinite series of finite (binary) integers of conventional mathematics. The closed cycles constitute an infinite ordinal sequence, establishing the meaning of both the number 1 and the binary symbol 1 as it appears in classical Boolean logic as a conjugation state of 0, and the alphabets structure themselves as an infinite series of binary digits. The general process can be represented in symbolic form:

$$
\begin{array}{ll}
\Delta_a & (R) \\
\Delta_b & (R, R^*) \\
\Delta_c & (R, R^*, A, A^*) \\
\Delta_d & (R, R^*, A, A^*, B, B^*, AB, AB^*) \\
\Delta_e & (R, R^*, A, A^*, B, B^*, AB, AB^*, C, C^*, \\
 & AC, AC^*, BC, BC^*, ABC, ABC^*) \; \ldots
\end{array}
$$

And the algebraic structure generated by universal rewrite is a Clifford algebra constructed out of an infinite sequence of quaternionic units:

$$
\begin{array}{llll}
(1,-1) & & & \\
(1,-1) & \times (1, \boldsymbol{i}_1) & & \\
(1,-1) & \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) & & \\
(1,-1) & \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) & \times (1, \boldsymbol{i}_2) & \\
(1,-1) & \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) & \times (1, \boldsymbol{i}_2) \times (1, \boldsymbol{j}_2) & \\
(1,-1) & \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) & \times (1, \boldsymbol{i}_2) \times (1, \boldsymbol{j}_2) & \times (1, \boldsymbol{i}_3) \\
\ldots & & &
\end{array}
$$

Repetition is clearly established at the fourth stage. At this point we have what can be recognised as a Clifford algebra – the algebra of 3-D space. However, the independent existence of all four stages as descriptions of the universe requires a combination equivalent to the sixth stage, which is itself equivalent to a double Clifford algebra, or the algebra created by two vector spaces.

Some key numbers

The universal rewrite structure is based on zero totality, which automatically requires duality, and the number 2 (the conserve process). It is infinitely degenerate in its generation of zeros, and new ones are only generated when 'products' are anticommutative, which introduces the number 3 (the create process). The groups and structures which are significant in nature seem to involve symmetries based on just these two numbers – singly or repeated. The number 5, which emerges in the combinations, is always a symmetry-breaker and introduces chirality and singularity. The way these basic integers interact to produce the spectrum of the fundamental particles can be seen in the following table, adapted from Hill and Rowlands (2011 b). Here the second column represents quarks q, each of which comes in 3 colours, the third leptons l, while the fourth gives the total number of fermions (quarks + leptons) f. Adding this to the total number of bosons (b) in the fifth column produces the total number of fermions + bosons.

Table of fundamental particles based on basic symmetries

	q	l		f	b							
1	3	1	=	4	1	=	5					
2	6	2	=	8	2	=	10	S				
3	9	3	=	12	3	=	15					G
4	12	4	=	16	4	=	20	S	I			
5	18	6	=	24	6	=	30	S				G
6	24	8	=	32	8	=	40	S	I	A		
7	36	12	=	48	12	=	60	S	I			G
8	48	16	=	64	16	=	80	S	I	A	V	
9	72	24	=	96	24	=	120	S	I	A		G
10	144	48	=	192	48	=	240	S	I	A	V	G

Successive rows are multiplied by factors of 2 or 3 as we supply the symmetries *S*, *I*, *A*, *V* and *G*. *S* refers to 2 states of spin, *I* refers to 2 weak isospin states (up / down quark, electron neutrino / electron, etc.), *A* refers to the duality of particle and antiparticle, *V* to the duality of particle and vacuum. *G* is the generation. So, in the first row, $q = 3$ stands for the 3 colours of up quark and $l = 1$ for the associated lepton, the electron neutrino. In the first row, there will be just 3 colours of the up quark and 1 lepton, the electron neutrino, each in just one spin state. The first doubling comes from assuming 2 spin states (*S*). This gives us the second row. Alternatively, we can take one spin state of the equivalent of the quarks in each generation (up, charm, top), along with the 3 neutrinos. This is the meaning of the third row. Ultimately, the final row will give us 3 generations (*G*), each with quarks and leptons with 2 states of isospin (*I*) (up / down, electron neutrino / electron; charm / strange, muon / muon neutrino; top / bottom, tau / tau neutrino), each with 2 states of spin (*S*); it will also provide the equivalent antiparticles (*A*), and both the real and vacuum versions of the particle states (*V*). In addition, there will be one boson for every four fermions / antifermions, so leaving us with a total of 240 possible states, as in a model based on the root vectors of the E8 group. While the factors of 3 come from the 3 colour states and *G*, and the factors of 2 from *S*, *I*, *A* and *V*, the multiples of 5 come from adding boson states to fermion states, a natural symmetry-breaking. There are, in all, 4 dualities, relating, respectively, to space (*S*), charge (*I*), time (*A*) and mass (*V*); and two 3-dimensionalities, one for the quark colours and one for the generations, relating to the respective 3-dimensionalities of space and charge. As we will see for the nilpotent structures of the particles themselves, one of these (the generations) relates to the entire set of the states, while the other (the quark colours) forms a self-contained structure within it. Combining two 3-dimensionalities in this way always creates such an asymmetry.

Though the table here represents fundamental particle structures, it becomes a natural consequence of anything based on 2 interacting spacelike structures. The numbers in the table include all the integers, for example, which are necessary to describe the Platonic solids in any number of dimensions up to 8, for Platonic solids can be considered as inhabiting a double space, in which one structure can be thought of as inhabiting the observed space, and the dual structure as inhabiting an unobserved dual space. Exactly the same numbers appear to be significant in describing the genetic code, which previous work has shown as being based on geometric structures related to the Platonic solids (Hill and Rowlands, 2008, 2011a and b).

The rewrite system in fundamental physics

The universal rewrite structure seems to be applicable, in general, to self-organizing, self-governing, self-replicating systems. In physics at an even more fundamental level than the structures of particles, it manifests itself as the generator of the fundamental *parameters*, mass, time, charge and space, as these emerge at successive levels of the algebra:

$(1, -1)$	mass	scalar	1
$(1, -1) \times (1, \boldsymbol{i}_1)$	time	trivector = pseudoscalar	i
$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1)$	charge	bivector = quaternion	$\boldsymbol{i}\ \boldsymbol{j}\ \boldsymbol{k}$
$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) \times (1, \boldsymbol{i}_2)$	space	vector	$\mathbf{i}\ \mathbf{j}\ \mathbf{k}$

We can imagine the structure continuing to infinity as with algebra and arithmetic, but the repetition allows us to choose the fourth level in such a way that a zeroing occurs at this level, and occurs automatically thereafter at all higher levels. In a way it is analogous to the process of the rewrite structure itself, as well as being generated by it. *A*, *B*, etc. could be commutative or anticommutative. Commutative relations can be organized an infinite number of times. Anticommutative relations between *A* and *B* exclude any other term except *AB*, and so are reduced to a finite number. Significantly, complex numbers, in the rewrite process, emerge simply as incomplete quaternion sets, a fact which affects the nature of the physical quantities with which they are associated.

The universal rewrite structure, as applied to physics at this most basic level generates the successive parameters, with their algebras and subalgebras:

		algebra	*subalgebras*		
mass	scalar	1			
time	pseudoscalar	i			1
charge	quaternion	$\boldsymbol{i}\ \boldsymbol{j}\ \boldsymbol{k}$			1
	≡ pseudovector ≡ bivector	$\equiv i\mathbf{i}\ i\mathbf{j}\ i\mathbf{k}$			1
space	vector	$\mathbf{i}\ \mathbf{j}\ \mathbf{k}$	$\boldsymbol{i}\ \boldsymbol{j}\ \boldsymbol{k}$	i	1
		$\equiv \mathbf{i}\ \mathbf{j}\ \mathbf{k}$	$i\mathbf{i}\ i\mathbf{j}\ i\mathbf{k}$	i	1

The first three have the structure of subalgebras of the last. The physical properties can be derived from the algebraic ones.

A group of order 4

The key result for deriving significant consequences from this algebraic structure is the symmetry relation which it generates between the four parameters. This has the structure of a Klein-4 group or D2. It appears to be absolutely exact and has withstood all challenges since it was first proposed. It is based on the principle that each parameter has 3 defining properties / antiproperties, which can be thought of 'physically' in a variety of different ways, but which are ultimately pure expressions of the algebraic terms by which they are described:

mass	conserved	real	continuous
	identity	norm 1	nondimensional
	translation asymmetric	commutative	global
time	nonconserved	imaginary	continuous
	no identity	norm –1	nondimensional
	translation symmetric	commutative	global
charge	conserved	imaginary	discrete
	identity	norm –1	dimensional
	translation asymmetric	anticommutative	local
	rotation asymmetric		
space	nonconserved	real	discrete
	no identity	norm 1	dimensional
	translation symmetric	commutative	local
	rotation symmetric		

The real / imaginary and commutative / anticommutative distinction are obviously purely algebraic, but even the conserved / nonconserved one seems to be attributable to the fact that nonconserved quantities, unlike conserved ones, incorporate incomplete quaternion sets. At this level, it would appear that physics is not only described by algebra but actually *is* algebra. The properties / antiproperties may also be conveniently represented using more conventional algebraic symbolism, for example:

mass	x	y	z
time	$-x$	$-y$	z
charge	x	$-y$	$-z$
space	$-x$	y	$-z$

In algebraic terms, this is a conceptual zero. Physically, it produces a set of parameters that can produce a dual to the only measurable quantity: space. In principle, any one of the parameters can be seen as the dual of the other three (and any two as the dual of the other two).

time	from	space,	charge,	mass
mass	from	time,	space,	charge
charge	from	time,	space,	mass

However, space is the only parameter that we can physically measure and so this allows us to make a division between an 'epistemology' of measurement and an 'ontology' created by the other parameters. Also, the other three parameters cumulatively create the algebra of space.

In fact, the first three parameters produce a combined vector-like structure, even though there is no physical vector quantity associated with them.

mass	scalar	1		
time	pseudoscalar	*i*		1
charge	quaternion	***i j k***		1
	≡ pseudovector	*i***i** *i***j** *i***k**		1
	≡ bivector			
COMBINED	vector	**i j k**	***i j k***	*i* 1
STRUCTURE		**i j k**	*i***i** *i***j** *i***k**	*i* 1

Physically, this combined structure becomes a dual to space – an 'antispace'. So, we have

i	**j**	**k**	SPACE

Alternatively, we can derive the vectors from a combination of:

$i\mathbf{I}$	$i\mathbf{J}$	$i\mathbf{K}$	CHARGE
i			+ TIME
1			+ MASS
I	**J**	**K**	= ANTISPACE

Taken together, the space and the antispace structures produce the zero totality required without taking the rewriting to infinity. All higher structures are then automatically zeroed. Duality comes built into the system. Combining

	i j k	***i j k***	i	1
with	**I J K**	i**I** i**J** i**K**	i	1

allows the immediate zeroing which produces singularity, and the 5-fold symmetry-breaking which produces chirality.

The vector algebra of 3-D space

The vector units, **i**, **j**, **k**, i, are effectively complexified quaternions (i***i***) = **i**, (i***j***) = **j**, (i***k***) = **k**, (i1) = i, and follow the multiplication rules:

$$\begin{gathered}\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -i \;\; \mathbf{ijk} = 1 \\ \mathbf{ij} = -\mathbf{ji} = i\mathbf{k} \\ \mathbf{jk} = -\mathbf{kj} = i\mathbf{i} \\ \mathbf{ki} = -\mathbf{ik} = i\mathbf{j}\,.\end{gathered}$$

They are isomorphic to Pauli matrices. If we complexify *this* algebra, we revert to quaternions, so (i**i**) = ***i***, (i**j**) = ***j***, (i**k**) = ***k***, etc

The vectors in this algebra have a full (algebraic) product **ab** = **i.b** + i **a** × **b**, from which all the rules concerning unit vector multiplication may be derived. Terms like i**i**, i**j**, i**k** are pseudovectors (e.g. area, angular momentum) and i is a pseudoscalar (e.g. volume). For comparison, the four quaternion units, ***i***, ***j***, ***k***, 1, follow the multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$$
$$\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}$$
$$\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

The full Clifford algebra of 3D space requires:

i	**j**	**k**	vector				
*i***i**	*i***j**	*i***k**	bivector	≡	pseudovector	≡	quaternion
i			trivector	≡	pseudoscalar		
1			scalar				

Pseudovectors and pseudoscalars give us areas and volumes, etc. The intrinsic complexification produces a kind of 'doubling' of the elements.

I	**J**	**K**	vector				
*i***I**	*i***J**	*i***K**	bivector	≡	pseudovector	≡	quaternion
i			trivector	≡	pseudoscalar		
1			scalar				

Here, we are saying here is that this is a composite quantity, made from everything in the physical world that isn't space – mass, time, charge.

If we combine the two algebras commutatively in a tensor product, or alternatively take the algebraic product of the eight base units, 1, **i**, **j**, **k**, *i*, **I**, **J**, **K**, we obtain 64 terms, which are + and – versions of:

i	**j**	**k**	*i***i**	*i***j**	*i***k**	*i*	1
I	**J**	**K**	*i***I**	*i***J**	*i***K**		
iI	**iJ**	**iK**	*i***iI**	*i***iJ**	*i***iK**		
jI	**jJ**	**jK**	*i***jI**	*i***jJ**	*i***jK**		
kI	**kJ**	**kK**	*i***kI**	*i***kJ**	*i***kK**		

We can describe this as a double vector algebra or a double Clifford algebra of 3D space.

Alternatively, we can take the algebraic product of the four quaternion units, 1, ***i***, ***j***, ***k***, and the four vector units *i*, **i**, **j**, **k**, to obtain + and – versions of:

$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$i\boldsymbol{i}$	$i\boldsymbol{j}$	$i\boldsymbol{k}$	i	1
$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{i}$	$i\mathbf{k}$		
$\mathbf{i}\boldsymbol{i}$	$\mathbf{i}\boldsymbol{j}$	$\mathbf{i}\boldsymbol{k}$	$i\mathbf{i}\boldsymbol{i}$	$i\mathbf{i}\boldsymbol{j}$	$i\mathbf{i}\boldsymbol{k}$		
$\mathbf{j}\boldsymbol{i}$	$\mathbf{j}\boldsymbol{j}$	$\mathbf{j}\boldsymbol{k}$	$i\mathbf{j}\boldsymbol{i}$	$i\mathbf{j}\boldsymbol{j}$	$i\mathbf{j}\boldsymbol{k}$		
$\mathbf{k}\boldsymbol{i}$	$\mathbf{k}\boldsymbol{j}$	$\mathbf{k}\boldsymbol{k}$	$i\mathbf{k}\boldsymbol{i}$	$i\mathbf{k}\boldsymbol{j}$	$i\mathbf{k}\boldsymbol{k}$		

This is exactly isomorphic to the previous algebra and can be described as a vector quaternion algebra.

A third version of the same algebra could be obtained by complexifiying the algebraic product of two commutative sets of quaternion units 1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, and 1, $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$. This algebra has + and – versions of:

$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$i\boldsymbol{i}$	$i\boldsymbol{j}$	$i\boldsymbol{k}$	i	1
$\boldsymbol{I}$	$\boldsymbol{J}$	$\boldsymbol{K}$	$i\boldsymbol{I}$	$i\boldsymbol{J}$	$i\boldsymbol{K}$		
$\boldsymbol{iI}$	$\boldsymbol{iJ}$	$\boldsymbol{iK}$	$i\boldsymbol{iI}$	$i\boldsymbol{iJ}$	$i\boldsymbol{iK}$		
$\boldsymbol{jI}$	$\boldsymbol{jJ}$	$\boldsymbol{jK}$	$i\boldsymbol{jI}$	$i\boldsymbol{jJ}$	$i\boldsymbol{jK}$		
$\boldsymbol{kI}$	$\boldsymbol{kJ}$	$\boldsymbol{kK}$	$i\boldsymbol{kI}$	$i\boldsymbol{kJ}$	$i\boldsymbol{kK}$		

This can be described as a complexified double quaternion algebra.

We now have three completely isomorphic algebras. The units form a group of order 64, with a minimum of 5 generators. Their physical significance is that they are also isomorphic to the gamma algebra of the Dirac equation, based on 4 × 4 matrices. In fact all possible gamma matrices can be derived from the products of two commuting sets of Pauli matrices, say σ_1, σ_2, σ_3 and Σ_1, Σ_2, Σ_3. Relativistic quantum mechanics, it seems, requires a dual vector space.

We can choose the 5 generators of the group in many different ways. From the 60 elements other than + and – 1 and i, we can choose 12 sets of 5 generators at any one time.

$\boldsymbol{i}$	$\boldsymbol{j}$*	$\boldsymbol{k}$	$i\boldsymbol{i}$	$i\boldsymbol{j}$	$i\boldsymbol{k}$*	i	1
$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{i}$	$i\mathbf{k}$		
$\mathbf{i}\boldsymbol{i}$*	$\mathbf{i}\boldsymbol{j}$	$\mathbf{i}\boldsymbol{k}$	$i\mathbf{i}\boldsymbol{i}$	$i\mathbf{i}\boldsymbol{j}$	$i\mathbf{i}\boldsymbol{k}$		
$\mathbf{j}\boldsymbol{i}$*	$\mathbf{j}\boldsymbol{j}$	$\mathbf{j}\boldsymbol{k}$	$i\mathbf{j}\boldsymbol{i}$	$i\mathbf{j}\boldsymbol{j}$	$i\mathbf{j}\boldsymbol{k}$		
$\mathbf{k}\boldsymbol{i}$*	$\mathbf{k}\boldsymbol{j}$	$\mathbf{k}\boldsymbol{k}$	$i\mathbf{k}\boldsymbol{i}$	$i\mathbf{k}\boldsymbol{j}$	$i\mathbf{k}\boldsymbol{k}$		

All the sets of 5 which contain the 8 base units have exactly the same structure. (It is, of course, possible to generate the algebra using **i**, **j**, **I**, **K**,

i or $\mathbf{i}$, $\mathbf{j}$, $\boldsymbol{i}$, $\boldsymbol{j}$, i, or even $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{I}$, $\boldsymbol{K}$, i, where some of the base units and component alphabets would be hidden.)

The 5 generators of the group can be matched to the 5 gamma matrices in a number of ways, for example:

$$\gamma^0 = i\boldsymbol{k} \qquad \gamma^1 = \mathbf{i}\boldsymbol{i} \qquad \gamma^2 = \mathbf{j}\boldsymbol{i} \qquad \gamma^3 = \mathbf{k}\boldsymbol{i} \qquad \gamma^5 = i\boldsymbol{j}$$

There are many ways of doing this but the overall structure is always the same. Always one of the two 3-dimensional structures ends with its symmetry preserved (here, the vectors) while the symmetry of the other is broken (here, the quaternions).

Physically, what we are doing here is to create new physical quantities.

time	space	mass	charge
i	$\mathbf{i}$ $\mathbf{j}$ $\mathbf{k}$	1	$\boldsymbol{i}\boldsymbol{j}\boldsymbol{k}$

become

energy	momentum	rest mass
$i\boldsymbol{k}$	$\mathbf{i}\boldsymbol{i}$ $\mathbf{j}\boldsymbol{i}$ $\mathbf{k}\boldsymbol{i}$	$\boldsymbol{j}$

as we combine aspects of the original time, space and mass with one each of the three 'dimensions' of charge, also breaking the symmetry between these in the process.

Charge also changes in the combination as well as time, space and mass. The changes to charge give us the symmetry-breaking of the Standard Model.

$i\boldsymbol{k}$	$\mathbf{i}\boldsymbol{i}$ $\mathbf{j}\boldsymbol{i}$ $\mathbf{k}\boldsymbol{i}$	$\boldsymbol{j}$
weak	strong	electric
$SU(2)$	$SU(3)$	$U(1)$
pseudoscalar	vector	scalar

Returning to

energy	momentum	rest mass
$i\boldsymbol{k}$	$\mathbf{i}\boldsymbol{i}$ $\mathbf{j}\boldsymbol{i}$ $\mathbf{k}\boldsymbol{i}$	$\boldsymbol{j}$
E	$p_x\,p_y\,p_z$	m

we see that the characteristic of the new parameters are determined by the algebraic operators, not the coefficients (E, p_x, p_y, p_z, m), which are merely scalar multiples.

Nilpotent quantum mechanics

All we need to do now is to write down the combined structure incorporating the 5 generators and recognise that to determine that it is a structure representing all of physical reality, it must be a nilpotent. That is,

$$(ikE + \mathbf{ii}p_x + \mathbf{ij}p_y + \mathbf{ik}p_z + jm)\,(ikE + \mathbf{ii}p_x + \mathbf{ij}p_y + \mathbf{ik}p_z + jm) = 0$$

or

$$(ikE + \mathbf{i}\mathbf{p} + jm)\,(ikE + \mathbf{i}\mathbf{p} + jm) = 0.$$

Multiplying this out gives us Einstein's energy-momentum equation:

$$E^2 - p^2 - m^2 = 0$$

If we apply a canonical quantization procedure to the first bracket in the squared expressions, to replace the terms E and $\mathbf{p}$ by the operators $E \to i\partial / \partial t$, $\mathbf{p} \to -i\nabla$ (with units where $\hbar$ to 1), and assume that the operators act on the phase factor for a free fermion, $e^{-i(Et - \mathbf{p.r})}$, we obtain the nilpotent Dirac equation for a free fermion:

$$\left(\mp k\frac{\partial}{\partial t} \mp i\mathbf{i}\nabla + jm\right)(\pm ikE \pm \mathbf{i}\mathbf{p} + jm)e^{-i(Et-\mathbf{p.r})} = 0$$

If we use a multivariate vector for the $\mathbf{p}$ or ∇ term it automatically includes spin (through the extra $\times$ term in the full product) (Hestenes 1966). So, here, $\mathbf{p}$ is interchangeable with $\boldsymbol{\sigma}.\mathbf{p}$ and ∇ with $\boldsymbol{\sigma}.\nabla$.

Nilpotent quantum mechanics produces all the standard results of conventional relativistic quantum mechanics, which can easily be obtained by replacing

$$\left(\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z} + im\right)\psi = 0$$

with $$-i\gamma^5\left(\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z} + im\right)\psi = 0,$$

but it also produces many new results which are not accessible by conventional methods. The expression

$$(\pm\, ikE \pm \mathbf{i}\mathbf{p} + jm)\,(\pm\, ikE \pm \mathbf{i}\mathbf{p} + jm) \to 0$$

can be used in a flexible way in which the terms in the brackets are either operators or amplitudes, to give the classical expression, the Dirac equation, Klein-Gordon equation, or a combination of fermion and vacuum. All expressions for fermions in any state, subject to any number of interactions, can be reduced to this form.

Since physics is only concerned with fermions and their interactions (bosons), this means, in principle, that the only information necessary to define physics on a fundamental basis is that provided by the two 'spaces' whose units are

$$\mathbf{i}\,\mathbf{j}\,\mathbf{k} \qquad \text{and} \qquad \mathbf{I}\,\mathbf{J}\,\mathbf{K}$$

Every physical variable or conserved quantity is merely defined by a coefficient applied to the units of one 'space' or another, or their combination. In effect, the physics of fermions is reduced to the mathematics of defining a real point in space by invoking a dual space.

Local and nonlocal

Nilpotent quantum mechanics (NQM) gives us a very precise definition of the boundary between local and nonlocal, defined effectively by the two 'spaces'. It must, of course, be relativistic, as the Lorentzian space-time or energy-momentum connection is needed to define the local, and without a proper account of the local, we cannot specify what we mean by nonlocal.

In principle, when we define a fermion using a nilpotent operator, we also create a description of the universe, for the rest of the universe becomes the vacuum which would make a zero total. The key aspect of NQM, is the fact that an operator of the form $(\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ automatically generates a phase term on which it operates to produce a nilpotent amplitude of the form $(\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, that is, one that squares to zero. We don't really need an equation.

The fermion needn't be free. We can incorporate field terms or covariant derivatives into the operator, with, for example, $E \to i\partial / \partial t + e\phi + \ldots$, and $\mathbf{p} \to -i\nabla + e\mathbf{A} + \ldots$ We can still represent the operator as $(\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, but the phase term will no longer be $e^{-i(Et - \mathbf{p.r})}$. It will be whatever is needed to create an amplitude of the general form $(\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, which squares to zero, with the eigenvalues E and $\mathbf{p}$ representing more complicated expressions that will result from the presence of the field terms.

In NQM the total structure of the universe is exactly zero. Pauli exclusion, a fundamentally nonlocal phenomenon, is an immediate consequence. If we imagine creating a fermion wavefunction of the form $\psi_f = (\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ from *absolutely nothing*, then we must simultaneously create the dual term, $\psi_v = -(ikE + i\mathbf{p} + jm)$, which negates it both in superposition and combination:

$$\psi_f + \psi_v = (ikE + i\mathbf{p} + jm) - (ikE + i\mathbf{p} + jm) = 0$$
$$\psi_f \psi_v = -(ikE + i\mathbf{p} + jm)(ikE + i\mathbf{p} + jm) = 0$$

As the Klein-4 group connecting the parameters may suggest, 'local' and 'nonlocal' are more general terms than would be suggested by confining them to quantum mechanics. It is interesting, here, to compare the arithmetical processes of addition and multiplication. The addition $A + B$ provides a global or single connection between A and B. The multiplication $A \times B$ makes this a *local* or multiple connection. Every single part of which A is or can be composed is connected to every single part of which B is or can be composed. Because both local and nonlocal descriptions are always needed, addition and multiplication are often needed simultaneously, as are doubling and squaring. Often they are versions of the same process.

The nilpotent structure immediately gives us a *formal* way of separating the local from the nonlocal. The bracketed term representing the fermion creation operator or wavefunction determines how conservation of energy applies to that fermion, as squaring the wavefunction and equating to zero gives us back the energy-momentum equation, and, of course, it is local, as the required Lorentzian structure is intrinsic. However, the addition and multiplication of nilpotent wavefunctions construct the nonlocal processes of superposition and combination, and these processes do not require a Lorentzian structure. In effect, anything inside the fermion bracket is local and anything outside it is nonlocal.

We have identified the dual term as the vacuum appropriate to that fermion state, in principle, the rest of the universe needed to maintain a fermion in that particular state. Pauli exclusion then tells us that no two fermions can have the same quantum numbers because the combination state would be zero. It also implies that no two fermions can share the same vacuum. Vacuum is intrinsically nonlocal. Because the fermion is localized, then the rest of the universe is necessarily nonlocalized. If the fermion is a point, as experiments suggests that it may be, then the rest of the universe is defined as everything outside that point. So the nonlocal connection which makes Pauli exclusion possible can be thought to occur through the vacua for each fermion.

The nilpotent structure clearly demands a holistic approach to physics. When we write down an operator or amplitude in the form ($\pm$ ***ikE*** $\pm$ ***i*p** + ***jm***), the brackets may suggest that we have created a closed system, but in fact the E and **p** terms may contain an unlimited number of potentials. We have created a system but it is open. Closure or energy conservation is only maintained over the entire universe, and requires the second law of thermodynamics as well as the first. So, though the bracket may define locality, locality does not imply a closed system.

The components of the NQM spinor

As we have seen in earlier presentations (summarised in Rowlands, 2007), 4 simultaneous solutions are required for the wavefunction: 2 for fermion / antifermion $\times$ 2 for spin up / spin down. Rather than a 4×4 matrix differential operator and a column vector wavefunction, we use a row vector operator and a column vector wavefunction, each of which may be represented in abbreviated form by ($\pm$ ***ikE*** $\pm$ ***i*p** + ***jm***). In the nilpotent formalism, the four solutions can be represented as, say:

(***ikE*** + ***i*p** + ***jm***)	fermion spin up
(***ikE*** – ***i*p** + ***jm***)	fermion spin down
(–***ikE*** + ***i*p** + ***jm***)	antifermion spin down
(–***ikE*** – ***i*p** + ***jm***)	antifermion spin

Negative energy, as we have argued elsewhere (Rowlands, 2007), represents vacuum rather than the local quantized state, and the apparent disparity between matter and antimatter in the universe is really a result of the fact that, as in the four-component Dirac wavefunction, one set of states exists in obervable real space, and the other in an unobservable 'vacuum space', as required by zero totality. The dual nature of $\pm$ *i* is also a factor in creating the existence of two states of helicity. Once we have decided on a sign convention for **p**, the spin state of the particle (or, more conventionally, the helicity or handedness **σ.p**) is determined by the ratio of the signs of E and **p**. So ***i*p** / ***ikE*** has the same helicity as (– ***i*p**) / (–***ikE***), but the opposite helicity to ***i*p** / (–***ikE***).

Negative energy also corresponds to reversed time. When we specify a fermion state as existing in real space, we are effectively specifying the corresponding antifermion state as existing simultaneously in vacuum space. This unobservable state operates in reverse time and with the opposite causality. In principle, the vacuum state which would completely cancels the fermion state contains the fermion's entire future causality at any instant; but this does not require us to accept determinism, because our ability to define this causality is limited by our ability to define backward causality by making a measurement.

The observed particle state in any specification of 4 simultaneous states in the spinor is the first in the column, while the others are the accompanying vacuum states, or states into which the observed particle could transform by respective P, T and C transformations:

$$
\begin{array}{ll}
P & \boldsymbol{i}\,(\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{i} = (\boldsymbol{k}E - \mathbf{i}\mathbf{p} + \boldsymbol{j}m) \\
T & \boldsymbol{k}\,(\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{k} = (-\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m) \\
C & -\boldsymbol{j}\,(\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{j} = (-\boldsymbol{k}E - \mathbf{i}\mathbf{p} + \boldsymbol{j}m)
\end{array}
$$

Replacing the observed fermion state spin up with any of the others would simultaneously transform all four states by P, T or C. It is often convenient to specify just the first term, with the others assumed to be automatic consequences.

Among the most important results which are seemingly unique to NQM are the descriptions of three different boson-type states, which are combinations of the fermion state which any of the P, T or C transformed ones, the result being a scalar wavefunction.

$$
\begin{array}{ll}
(\pm\, \boldsymbol{k}E \pm \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,(\mp\, \boldsymbol{k}E \pm \mathbf{i}\mathbf{p} + \boldsymbol{j}m) & \text{spin 1 boson} \\
(\pm\, \boldsymbol{k}E \pm \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,(\mp\, \boldsymbol{k}E \mp \mathbf{i}\mathbf{p} + \boldsymbol{j}m) & \text{spin 0 boson} \\
(\pm\, \boldsymbol{k}E \pm \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,(\pm\, \boldsymbol{k}E \mp \mathbf{i}\mathbf{p} + \boldsymbol{j}m) & \text{fermion-fermion}
\end{array}
$$

One of the most significant aspects of this formalization is that a spin 1 boson can be massless, but a spin 0 boson cannot, as then $(\pm\, \boldsymbol{k}E \pm \mathbf{i}\mathbf{p})$ $(\mp \boldsymbol{k}E \mp \mathbf{i}\mathbf{p})$ would immediately zero: hence Goldstone bosons must become Higgs bosons in the Higgs mechanism. Another way of looking at this is to say that the fermion and antifermion cannot both be purely left-handed (or both purely right-handed) – or massless – and act via a weak interaction to produce a bosonic state.

$$
\begin{array}{ll}
(\boldsymbol{k}E + \mathbf{i}\mathbf{p}) & \text{LH fermion} \\
(-\,\boldsymbol{k}E - \mathbf{i}\mathbf{p}) & \textit{LH antifermion}
\end{array}
$$

This chirality is a direct consequence of the structure of the Dirac equation even in the conventional formalism, but is an immediate consequence of the nilpotent version. A spin 1 boson, structured as

$$(\boldsymbol{i}kE + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\ (-\boldsymbol{i}kE + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)$$

LH fermion RH antifermion

can be massless as ($\boldsymbol{i}kE$ + $\mathbf{i}\mathbf{p}$) (– $\boldsymbol{i}kE$ + $\mathbf{i}\mathbf{p}$), but, with a spin 0 boson, structured as

$$(\boldsymbol{i}kE + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\ (-\boldsymbol{i}kE - \mathbf{i}\mathbf{p} + \boldsymbol{j}m)$$

LH fermion *LH antifermion*

or

$$(\boldsymbol{i}kE - \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\ (-\boldsymbol{i}kE + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)$$

RH fermion RH antifermion

one half or the other (as italicised) is not weakly interacting, which forces it to acquire mass – possibly maximally in the case of the Higgs boson.

We can also devise structures for baryons, by separating out three components of **p** in the form

$$(\boldsymbol{i}kE \pm \mathbf{i}\mathbf{i}p_x + \boldsymbol{j}m)\ (\boldsymbol{i}kE \pm \mathbf{i}\mathbf{j}p_y + \boldsymbol{j}m)\ (\boldsymbol{i}kE \pm \mathbf{i}\mathbf{k}p_z + \boldsymbol{j}m)$$

We then have a nonzero combination. Spin is defined in only one direction at a time, so, at any given instant, the wavefunction will reduce (after normalization) to

$$\begin{aligned}
(\boldsymbol{i}kE + \mathbf{i}\mathbf{i}p_x + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE + \mathbf{i}\mathbf{i}p_x + \boldsymbol{j}m)\\
(\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \mathbf{i}\mathbf{j}p_y + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE - \mathbf{i}\mathbf{j}p_y + \boldsymbol{j}m)\\
(\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \mathbf{i}\mathbf{k}p_z + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE + \mathbf{i}\mathbf{k}p_z + \boldsymbol{j}m)
\end{aligned}$$

The change of sign in the second case is significant.

This leads to baryonic structure. Since we need to maintain the symmetry between the three directions of momentum, there will be six possible outcomes, using both + and – values of momentum terms, that is, a superposition of six combination states:

$$\begin{aligned}
(\boldsymbol{i}kE + \mathbf{i}\mathbf{i}p_x + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE + \mathbf{i}\mathbf{i}p_x + \boldsymbol{j}m)\\
(\boldsymbol{i}kE - \mathbf{i}\mathbf{i}p_x + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE - \mathbf{i}\mathbf{i}p_x + \boldsymbol{j}m)\\
(\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \mathbf{i}\mathbf{j}p_y + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE - \mathbf{i}\mathbf{j}p_y + \boldsymbol{j}m)\\
(\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE - \mathbf{i}\mathbf{j}p_y + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE + \mathbf{i}\mathbf{j}p_y + \boldsymbol{j}m)\\
(\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \mathbf{i}\mathbf{k}p_z + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE + \mathbf{i}\mathbf{k}p_z + \boldsymbol{j}m)\\
(\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE + \boldsymbol{j}m)\ (\boldsymbol{i}kE - \mathbf{i}\mathbf{k}p_z + \boldsymbol{j}m) &\rightarrow (\boldsymbol{i}kE - \mathbf{i}\mathbf{k}p_z + \boldsymbol{j}m)
\end{aligned}$$

Vacuum and the separation of local and nonlocal

The creation of the fermion state is the creation of a local region in phase space, to which everything else becomes nonlocal; the creation of the two regions is simultaneous. Any subsequent change inside the bracket (a rewriting of the structures of E and $\mathbf{p}$) also affects everything else outside it, and vice versa.

Nilpotent quantum mechanics shows there is no such thing as an isolated system, and so a complete local description, which originates in the individual particle, still requires knowledge of the contents and disposition of the whole universe. Using nilpotent version of quantum mechanics, it is possible to show that both the local and nonlocal aspects of the strong, weak and electric interactions contribute to their special characteristics.

In principle, the difference between local and nonlocal is not in the phenomena they describe, but in the method of description, essentially whether we use an iterative or recursive computational paradigm. Local interactions are determined by the collective nonlocal effect of the rest of the universe. Gravity has a special significance among the four known interactions, in this respect, in providing the most directly observable aspect of nonlocality. Its negative energy marks it out as a *nonlocal* vacuum property, the source of instantaneous quantum correlation. The nonlocal character immediately leads to a prediction of dark energy.

In the nilpotent formalism, the vacuum can be structured, directly reflecting that of matter. If we take $(\pm\, \boldsymbol{i}\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and post-multiply it by the idempotent $\boldsymbol{k}(\pm\, \boldsymbol{i}\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ any number of times, the only effect is to introduce a scalar multiple, which can be normalized away.

$$(\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\, \boldsymbol{k}(\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\, \boldsymbol{k}(\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \ldots$$
$$\to (\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

The same thing happens with $\boldsymbol{j}(\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ or $\boldsymbol{i}(\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, the extra vector terms produced in the latter case cancelling each other out after every other bracket.

The three idempotent terms have the mathematical characteristics of vacuum operators. They also correspond to the respective transformations of alternate brackets by T, C and P, or to the production of the spin 1, spin 0 and fermion-fermion bosonic states. The character of the original fermionic state remains unchanged, while vacuum versions of the three bosonic states are created.

In principle, the quaternionic operators $\boldsymbol{k}, \boldsymbol{j}, \boldsymbol{i}$, like T, C and P, which are associated respectively with E, m and $\mathbf{p}$, split the continuous vacuum into discrete units, and we can to some extent regard these units as the respective 'charges' or sources of the weak, electric and strong interactions, acting to create the vacuum necessary for these forces to act

What this means is that the quaternionic operators $\boldsymbol{k}, \boldsymbol{j}, \boldsymbol{i}$, effectively define the axes of a *vacuum* space, dividing it into parts which respond respectively to the weak, electric and strong sources or 'charges', and that they can also be used to represent these charges. The sum total, the continuous vacuum that exists before we impose the '3D' structure upon it, responds to the force we know as gravity. String theorists have a name for this connection – gravity / gauge theory correspondence. In effect, the gravitational response acts like the totality of the three gauge forces *but in a negative sense* (negative energy), as was apparent from the beginning in this theory.

I have referred to the vacuum space in various ways over the years because different aspects of it reveal different characteristics. Vacuum space refers to the fact that the 3 axes represent a partitioning of the vacuum. Charge space refers to the fact that it separates out the 3 components of charge – electric, strong and weak, and through its combination with ordinary space gives them the $U(1)$, $SU(3)$ and $SU(2)$ characteristics – their character as symmetries of 3-dimensionality are evidence that they can't be explained by enlarging space to higher dimensions. Antispace refers to the fact that it is the dual to ordinary space, effectively zeroing it in fermion creation.

Duality and the nilpotent structure

The two 'spaces', though seemingly different, are truly dual, each containing the same information, and the duality manifests itself directly in many physical forms.

Pauli exclusion by antisymmetric wavefunctions	uses **i, j, k**
Pauli exclusion by nilpotency	uses **I, J, K**

Both sets of coordinates yield information about the same physical quantity: angular momentum.

spin ½ from anticommuting aspects of **p** components	uses **i, j, k**
spin ½ using Thomas precession (relativity)	uses **I, J, K**

velocity addition law from 2 D of space	uses **i**, **j**, **k**
same using relativistic space-time	uses **I**, **J**, **K**
holographic principle – bounding 'area' defined by	
two spatial coordinates	uses **i**, **j**, **k**
or one of space and one of time	uses **i**, **j**, **k**
Pauli exclusion by antisymmetric wavefunctions	uses **i**, **j**, **k**
but by nilpotency	uses **I**, **J**, **K**

We can give the last case as an example. Using antisymmetric wavefunctions, we find:

$$\begin{aligned}
&(\psi_1\,\psi_2 - \psi_2\psi_1) = -(\psi_2\psi_1 - \psi_1\psi_2) \\
&= (\pm\, \boldsymbol{ik}E_1 \pm \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1)\,(\pm\, \boldsymbol{ik}E_2 \pm \boldsymbol{i}\mathbf{p}_2 + \boldsymbol{j}m_2) \\
&- (\pm\, \boldsymbol{ik}E_2 \pm \boldsymbol{i}\mathbf{p}_2 + \boldsymbol{j}m_2)\,(\pm\, \boldsymbol{ik}E_1 \pm \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1) \\
&= 4\mathbf{p}_1\mathbf{p}_2 - 4\mathbf{p}_2\mathbf{p}_1 = 8\, i\, \mathbf{p}_1 \times \mathbf{p}_2 = -\,8\, i\, \mathbf{p}_2 \times \mathbf{p}_1
\end{aligned}$$

In real space, **p** must have a unique direction. At the same time, for a nilpotent wavefunction:

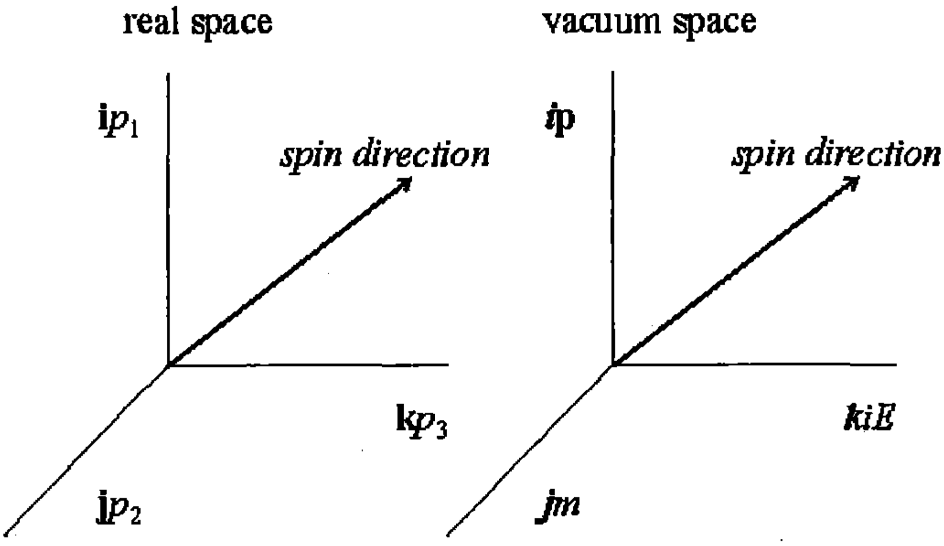
$$(\pm\, \boldsymbol{ik}E_1 \pm \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1)\,(\pm\, \boldsymbol{ik}E_1 \pm \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1) = 0$$

and so the 'space' created by the 'axes' of E, p, m ('antispace') must be unique. Both spaces record a unique instantaneous direction for the spin axis of any fermion.

Interestingly, the paired 'spaces' that are used in conventional theory – ordinary space and momentum space – are not commutative or truly independent as these two are, because momentum space is partly constructed from ordinary space. Heisenberg uncertainty is a consequence of this noncommutativity.

The nilpotent structure reveals the fundamental duality by creating a zero normed combination from the two spaces. Physically it manifests itself at a point, a region of zero space. In principle also, by gravity / gauge theory correspondence, and the vacuum annihilation of the total energy and momentum in the Dirac spinor, it creates a zeroing also in the dual space.

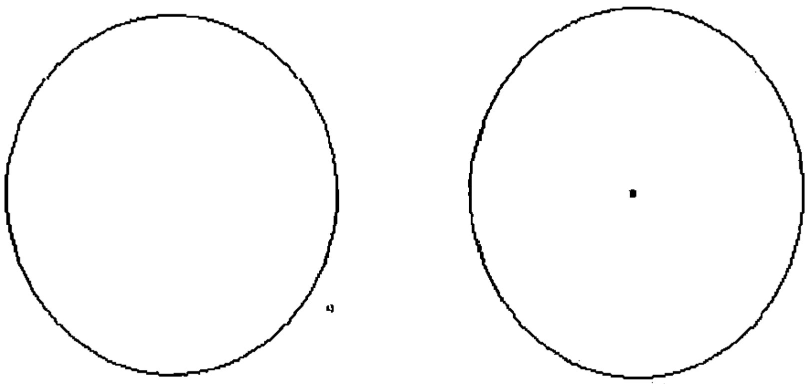
Pauli exclusion



antisymmetric wavefunctions nilpotency

Singularity and chirality

We can see that singularity is actually a result of duality. To understand this, we can use a reverse argument from topology. In topology, a singularity creates a multiply-connected space within a loop and a vector parallel transported around that loop needs a double circuit to arrive back at its starting position pointing in the same direction. We have to do a double circuit when we have a particle. A point is created by the simultaneous zeroing of two spaces.



simply-connected space *multiply-connected space*

A classic example of this is the electron which exists in its own multiply connected space, and hence has spin ½. As we have seen, of course, the electron exists simultaneously in two spaces, only one of which is observable. Physically, this manifests itself in the electron having a rest mass – the mass being the result of the *zitterbewegung* that results from the continual switching between the real particle and its vacuum state. This is, in fact, a property of all fermions – all of which have the discrete rest mass.

Everywhere, we see the factor 2 or ½ in physics, it is ultimately because everything exists simultaneously in 2 spaces. Every electron exists in space, but has a vacuum partner which is an antielectron in 'vacuum space', and switches between them. This is why the electron has spin ½; the other ½ is in vacuum. It is why we normally only see matter, not antimatter. But, in fact, it also suggests that, to create a singularity, we need two spaces.

However, the fact that only ½ of the electron's spin is ever manifested in real space means that the electron's properties are chiral, as well as singular, as we know from the Dirac equation. *Zitterbewegung* and the Higgs mechanism are ultimately the same mass-generating process. To create a singularity from two spaces using a nilpotent (norm 0) structure means, as we have seen, that, from an observational point of view, the spaces are not equally symmetrical. Ultimately, the zeroing also means that the iE, **p** and m terms don't have an equal range of + and – values.

Further thoughts on the Higgs boson and fermion masses

The discovery of the Higgs boson was announced at CERN on 4 July 2012. The approximate mass of the particle appeared to be in the region 125 to 126 GeV, though some later measurements suggest that it might be slightly lower, perhaps closer to last year's prediction of around 123 GeV (Rowlands, 2012), based on the idea that the creation of Higgs requires approximately half of the vacuum expectation energy (246 GeV). Possibly the halving might refer more specifically to the mass realisation of the electroweak bosons W^+, W^-, Z^0 and γ, which total $2 \times 126 = 252$ GeV.

The coupling of the Higgs to the fermions to give them their observed masses is the least satisfactory aspect of the Standard Model, and there is no fully consistent theory. A significant problem appears to be in defining the *weak hypercharge*. The two quantum numbers associated with the $SU(2)$ of the weak interaction are weak hypercharge and weak isospin. If we take the electron neutrino and electron as an $SU(2)$ doublet, both particles have the same hypercharge, which is defined as 2 × the average charge of the doublet in units of e.

electron neutrino	charge	0	weak hypercharge	−1
electron	charge	−1	weak hypercharge	−1

The thing that distinguishes them is weak isospin, which is ½ for the neutrino and −½ for the electron.

Before it was established that neutrinos have masses, it was possible to find the mass of a member of an $SU(2)$ doublet by a Higgs coupling similar to that for the W and Z bosons. However, for two members of a doublet, we need two hypercharge numbers, 1 and −1. Standard theory says that electrons and neutrinos have the same hypercharge, but, if neutrinos are *Majorana particles* (i.e. their own antiparticles), then, in certain circumstances, they would have two possible hypercharge numbers, 1 and −1. The probability of this occurring would relate to that of CP violation and so be relatively low. Hence we would expect neutrinos to have small masses.

Quarks represent a different problem, because, according to conventional theory, up quarks have charge 2/3 and down quarks have −1/3, in units of e, so creating a hypercharge for *all* quarks of 2/3. However, if we suppose that the underlying structure reflects that of the colours, then the three options become (in units of e):

up	1	1	0
down	0	0	−1

So creating two hypercharge numbers of 1 and −1, exactly as required, and these can apply to either up or down quarks.

I have previously argued that this underlying structure for quark masses has many advantages, besides solving the Higgs problem: it avoids the arbitrariness of having *three* fundamental charge units: $2e/3$, $e/3$ and e; it creates Grand Unification at the Planck mass and $\sin^2\theta_W = 1$; it creates an exact parallel between the charge structures of quarks and leptons; and it suggests how the perfect gauge invariance of the strong interaction operates (Rowlands, 2007). We have an exact parallel of emergent fractional charges of this kind in the quantum Hall effect, where a single electron can simultaneously be associated with 3 magnetic flux lines, creating effective charges of $e/3$ in value.

I have also previously represented the charge structures of quarks and leptons in 3 + 1 tables, where the fourth table included left-handed antileptons as an equivalent to the missing additional 'colours' (Rowlands, 2007). This suggests that the group symmetry has the right structure to incorporate Majorana behaviour in the neutrino sector. How the individual fermions couple to the Higgs boson to produce the particular masses they have has been a problem for a long time. The key to a solution probably lies in the idea that the Higgs only couples to a particle interacting with another field.

The electron, lightest neutrino and lightest quark masses are probably derivable from the respective electric, weak and strong coupling constants, α, α_2, α_3, via a version of the holographic principle. (These particles are asterisked in the table of particle masses in MEV, below.) (Rowlands, 2011) The increasing masses of quarks from d to s to b seem to be in proportion to the weak coupling constant, α_2, as might be expected. Also as expected, the masses of quarks s and b seem to relate to those of μ and τ in proportion to the running value of the strong coupling constant, α_3. The mass of the top (indicated by †) may represent the maximum possible coupling to the Higgs field ($f / \sqrt{2}$).

	u	1.5-3.3*			ν_e	0.00013*		
↑ α			d	3.5-6.0*			e	0.511*
	c	1,160 -1,340		↑ α_2	ν_μ	?		
↑ α			s	70-130	→ α_3		μ	105.7
	t	169,100 -173,300†		↑ α_2	ν_τ	?		
			b	4,130 -4370	→ α_3		τ	1,777

The relationships with the masses of the other quarks u, c, t requires further investigation, although there are some indications that the *generations* are related through the electric coupling constant, α. The masses of all fermions are a measure of their degree of right-handedness. Each increase in mass requires a new mechanism for an increase in the right-handed component of helicity. In leptons, for example, the 'down' state of weak isospin is more massive than the 'up' state, because there is now an electric charge, which responds to both left- and right-handed components. In quarks, except for the first generation, this is outweighed by other effects. One way of increasing the right-handed component is by a direct symmetry violation, and, between generations there is a 'step function' in right-handedness due to the successive introductions of parity violation (in the second generation) and time-reversal violation (in the third). In their physical effects, these two symmetry violations are difficult to distinguish, and the real distinction may be between one and two symmetry violations. It may be because the neutrino responds only to the weak interaction that there is maximal (~ 45°) mixing between the muon and tau neutrinos. Neutrino masses have not yet been measured beyond very wide limits, and the value given in the table for the electron neutrino is a prediction rather than a measurement. The neutrino sector remains the least understood aspect of particle physics.

REFERENCES

Diaz, B. and Rowlands, P., 2005, A computational path to the nilpotent Dirac equation, *International Journal of Computing Anticipatory Systems*, **16**, 203-18.

Hestenes, D. , 1966, *Space-time Algebras*, Gordon and Breach.

Hill, V. J. and Rowlands, P., 2008, Nature's code, *AIP Conference Proceedings*, **1051**, 117-126

Hill, V. J. and Rowlands, P., 2010, Nature's Fundamental Symmetry Breaking, *International Journal of Computing Anticipatory Systems*, **25**, 144-159

Hill, V. J. and Rowlands, P., 2010, The Numbers of Nature's Code, *International Journal of Computing Anticipatory Systems*, **25**, 160-175

Marcer, P. and Rowlands, P., 2010, The 'Logic' of Self-Organizing Systems, *AAAI Technical Reports* 2010-08-020

Marcer, P. and Rowlands, P., 2010, The Grammatical Universe and the Laws of Thermodynamics and Quantum Entanglement, *AIP Conference Proceedings*, **1303**, 161-167

Marcer, P. and Rowlands, P., 2010, Further Evidence in Support of the Universal Nilpotent Grammatical Computational Paradigm of Quantum Physics, *AIP Conference Proceedings*, **1316**, 90-101

Rowlands, P., 2007, *Zero to Infinity*, World Scientific, Singapore.

Rowlands, P., 2011, The Cosmological Implications of Nonlocal Gravity, presented in San Marino, in process of publication.

Rowlands, P., 2012, Nonlocality, in A. D. Ford (ed.) *Haecceity: Proceedings of ANPA 32*, 227-250 (also DVD recording by Anthony Blake, 4 August 2011).

Rowlands, P. and Diaz, B., 2002, A universal alphabet and rewrite system, arXiv:cs.OH/0209026.

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