What is wrong with spacetime?

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Abstract

Classical spacetime appears so directly from Einstein's two postulates that it appears to have a strong physical basis. However, it misses quantum mechanics, suggesting the 'physical basis' is somehow at odds with quantum mechanics. Despite this, classical spacetime has features in common with quantum mechanics and we investigate the connection by studying a model clock.

1. Introduction

Kronecker, has been credited with the saying that "God created the integers, all the rest is the work of man." From the perspective of measurement, he has a good point. The quantitative aspects of science at best use a very small subset of the rational numbers. Experiments extracting a dozen significant figures are rare and measuring hundreds of significant digits is beyond our capabilities in any direct physical context. In a universe with a finite number of objects, the representation of even a single real number with an infinite number of non-repeating decimal places is impossible. Ultimately, in the physical universe, Real numbers are *not* real. They are a mathematical convenience that allows us to avoid the tedious bookkeeping required by discrete models. To give them up, however, is unthinkable. It would rob us of the most powerful modelling language we have, the differential calculus.[1]

Like powerful machines, powerful languages are useful in many contexts, but can be a nuisance in situations where some delicacy is required. A bulldozer may be useful in clearing an archaeological site, but a trowel and paintbrush are more useful for finding, cleaning and classifying artifacts. Similarly the calculus is useful

[1]See however Kauffman's intriguing work on discrete forms of physical laws[1].

when all the relevant physics happens on scales above a certain threshold, but can be a liability if there are multiple characteristic lengths, or if Fractals are involved.

The concept of spacetime is a case in point. We can access it efficiently through Einstein's two postulates and it is convincing as a concept because the postulates themselves seem straightforward and unambiguous. It is so convincing in fact that we might wonder "What is wrong with the two postulates that their invocation fails to uncover quantum mechanics?" Here, the automatic assumption that Real numbers are real is a potential suspect. The requirement that particles move on smooth worldlines violates the uncertainty principle. As suggested by the path integral approach to quantum mechanics[2], to the extent that spacetime paths apply to quantum mechanics, they are Fractal with dimension two [3, 4, 5]. Since the calculus generally misses Fractals it is worth taking some care in noticing how the spacetime of special relativity appears if we start from a discrete perspective.

2. From Digital Clocks to Spacetime

Since continuous progression through time is suspect, we consider the idea that a particle is really a small digital clock.[2] Our model clock ticks periodically, once every unit of time (Fig. 1A). The clock mechanism is not specified between ticks, however notice that in a two dimensional spacetime, the forward and backwards light cones of each tick are nested and the intersection of these cones forms a chain of causal areas. We use the term causal in two ways. The clock mechanism that creates the ticks must be confined to these areas in order to be causal in the usual sense, with the mechanism being in the past of the subsequent tick and the future of the previous tick. The regions are also causal in the sense that their Euclidean areas are a Lorentz invariant[7]. Whatever the clock mechanism, its effect is to create periodic events by preserving the spacetime *area* between events, regardless of the inertial frame in which it is specified. If the events are to be considered intersections of locally smooth curves, the areas between events are oriented and their orientation alternates. We call these events 'audible' and the audible event sequence of our clock is the discrete analog of a particle's worldline. The corners in the chain of areas that are not audible we call 'inaudible'. They are not part of the event sequence that is the analog of the worldline but their positions in spacetime help determine the invariant areas between ticks.

So far, our clock of Fig. 1 ticks at the integers in the rest frame of the clock. We shall call this Compton's clock since if we think of it as a particle, the appropriate frequency of events is the Compton frequency.

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[2]The following is a development of a talk given at a previous ANPA meeting[6]

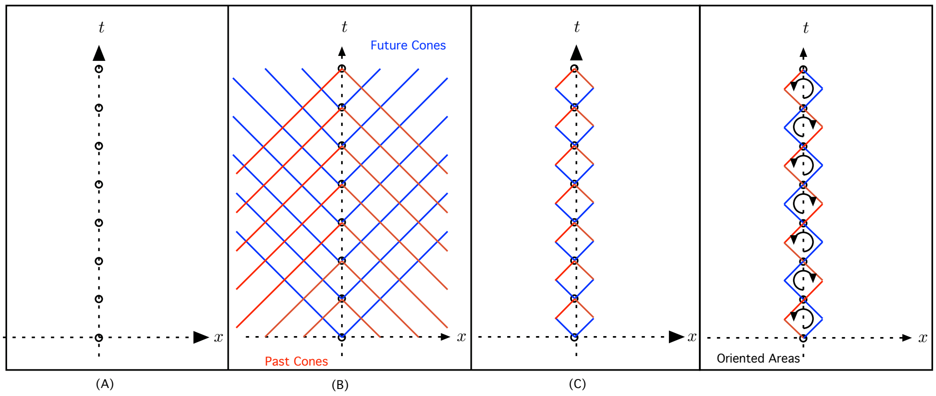


Figure 1: A periodic sequence of events has an associated sequence of forward and reverse light cones. The intersection of the events and the light cone area between the events sets up a chain of 'causal' areas. If the original events are the intersections of locally smooth curves, the areas are oriented.

If we are to make contact with conventional special relativity, we need clocks that can measure time to arbitrary precision, thus allowing us to interpolate between ticks. To this end we first use Compton's clock to count events using the right hand boundary of the chain of areas. To accommodate boosts we need a two component characteristic function. It needs to be two component because the clock sequence of ticks has projections on both the left and right light-cones. The tick frequencies on the cones is the same in the rest frame of the clock but different in inertial frames moving at constant velocity. Our time measurement must allow for this.

The mapping from path to characteristic function is shown in Fig.(2). The state variable s_k is a characteristic function for orientation and it takes on one of four values:

$$s_k \in S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} \tag{1}$$

in succession. If the Compton clock is in state s_k at time k then the subsequent state, after the next event is

$$s_{k+1} = T s_k \tag{2}$$

where T is the transfer matrix:

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

Since $s_k = T^k s_0$ the power of the transfer matrix corresponds to the (discrete) displacement in time.

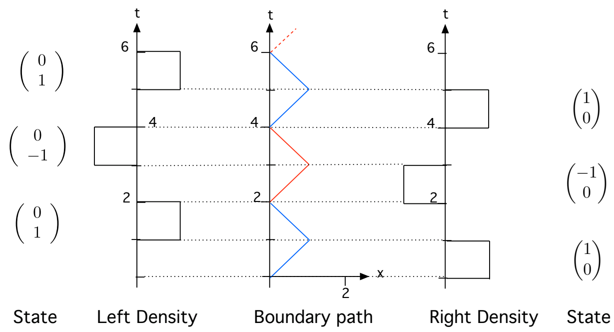


Figure 2: We use the right boundary of the chain of oriented areas to construct a two component state. The upper component registers a ± 1 or 0 for the signed projection onto the right light cone. The lower component registers the left light cone.

To obtain the continuous time of special relativity we need a clock of arbitrarily high frequency to allow real values of t. We can simply use a sequence of Compton clocks of ever higher frequency, Fig.3. If T_ν is the transfer matrix for the high-frequency N-clock, synchronized with T, we must have an equivalence between the N-th power of T_ν and T itself. Thus

$$T_\nu^N = T \tag{4}$$

giving T_ν as an Nth root of T or

$$T_\nu = \begin{pmatrix} \cos\left(\frac{\pi}{2N}\right) & -\sin\left(\frac{\pi}{2N}\right) \\ \sin\left(\frac{\pi}{2N}\right) & \cos\left(\frac{\pi}{2N}\right) \end{pmatrix} \tag{5}$$

Taking the limit as $N \to \infty$ via an eigenvalue expansion we get the transfer matrix

$$\begin{aligned} T_R(t) = \lim_{N\to\infty} T_\nu^{Nt} &= \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & -\sin\left(\frac{\pi t}{2}\right) \\ \sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix} \\ &= \mathrm{I}_2 \cos\left(\frac{\pi \mathrm{t}}{2}\right) + \mathrm{T}\sin\left(\frac{\pi \mathrm{t}}{2}\right) \end{aligned} \tag{6}$$

The dependence of T_R on real values of t allows us in principle to have access to reference frame clocks in the rest frame of our original Compton clock. Notice that T_R reproduces the events of the Compton clock at the even integers but interpolates between the original discrete states *via a rotation*. To allow for moving frames we have to adjust the transfer matrix to allow for the two different projections onto

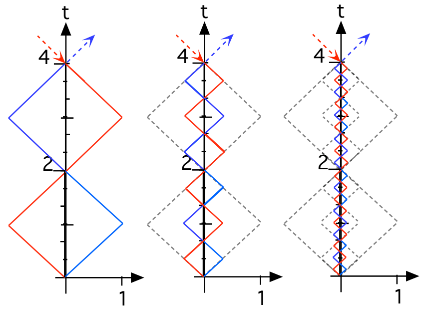


Figure 3: Contact with continuous time is made by allowing arbitrarily high frequency Compton clocks to fill in events on the t axis. Note how the area between events decreases as events are filled in. As the event frequency increases the causal area between events goes to zero.

the left and right light cones Fig.(3 B). The result is[8]

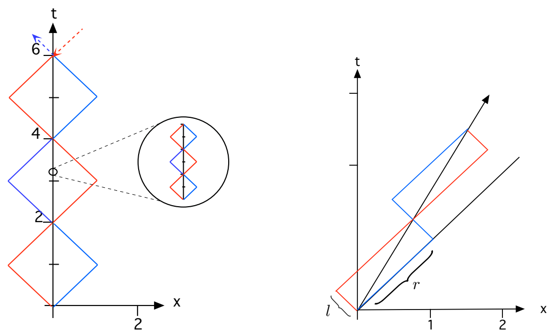
$$\begin{aligned} T_M(t) &= \begin{pmatrix} \cos\left(\frac{\pi t}{2\gamma}\right) - v\gamma \sin\left(\frac{\pi t}{2\gamma}\right) & -\gamma \sin\left(\frac{\pi t}{2\gamma}\right) \\ \gamma \sin\left(\frac{\pi t}{2\gamma}\right) & \cos\left(\frac{\pi t}{2\gamma}\right) + v\gamma \sin\left(\frac{\pi t}{2\gamma}\right) \end{pmatrix} \\ &= \mathrm{I}_2 \cos\left(\frac{\pi t}{2\gamma}\right) - \gamma(v\sigma_z + i\sigma_y) \sin\left(\frac{\pi t}{2\gamma}\right) \end{aligned} \tag{7}$$

Here v is the velocity of the moving frame and $\gamma = \frac{1}{\sqrt{1-v^2}}$ is the time dilation factor that just normalizes the 'vector' $M = \gamma(v\sigma_z + i\sigma_y)$.

It is useful to take stock at this point and compare eqn(6) with eqn(7). Newtonian mechanics allows an absolute time that is independent of the motion of the observer. In the case of absolute time, eqn(6) could be used as the reference clock for all frames and since T is a representation of the unit imaginary, our reference frame clock would be a translationally invariant unimodular complex number. There would then be little need to consider the clock aspect of particles since the argument of T_R would not depend on relative speeds and could be reasonably assumed to be 'available' to all particles. This would seem to justify Newon's absolute time and the segregation of time from space as a separate and independent coordinate. From the perspective of equation(7), the Newtonian approximation is achieved by setting $v = 0$.

In contrast, we see from (7) that with the relativity of simultaneity, space and time are intrinsically linked. The linkage here is not through the invocation of an ambient spacetime but simply through the geometry of a sequence of clocks.

In this view, absolute time is only an approximation and instead of a simple unimodular complex number appropriate for all frames we see the implication of geometric algebra[9] that allows for moving frames. From the fact that $M^2 = -I_2$ we see the origin of the odd signature of spacetime from the fact that spacetime



(a) Nested clocks.

(b) Viewed in a moving frame.

Figure 4: (a) A Compton clock at rest provides a chain of spacetime areas. To interpolate between events we construct higher frequency Compton clocks giving the continuous transfer matrix T_R. (b) In a moving reference frame the areas between events become rectangular, however the area is preserved and is calculated as the product of the lengths of the two sides of the rectangles. The effect on the transfer matrix is to weight the residence times of the two states differently.

areas in the plane defined by the t and v axes are invariant. The necessity to go beyond the single analog clock of Newton to require the Dirac algebra for a four dimensional spacetime is a direct consequence of the differing frequencies of clocks on the two lightcones in the reference frame specified by the relative velocity Fig.(4(b)). The full Dirac algebra is required in three dimensions of space since we need two more 'vectors' in the geometric algebra sense to embed our Compton clock in three space.

One more feature is worth noting at this point. To arrive at eqn(7) we had to take an infinite frequency limit of the Compton clock. It is this limit that establishes the Lorentz covariance of the spacetime frame that we are building from our digital clock. However, from the perspective of the clock, the reference frame is *an infinite mass idealization*. If classical 'particles' are simply small clocks that obey special relativity through geometry, the association of a smooth worldline with their passage through time will have the same effect. The arbitrarily high frequency of events implied by the association of 'event' with spacetime position on a smooth worldline is an infinite mass construction. This suggests that to get to the kinematics of conventional special relativity, we have 'frozen out' quantum propagation with an infinite mass limit. To introduce dynamics into relativity, mass has then to be reinserted as a background feature. What is wrong with spacetime is then that it hides a very relevant continuum limit. It is an infinite frequency construction that excludes quantum propagation by decoupling mass and frequency.

It seems clear from the above that we have taken an infinite frequency limit

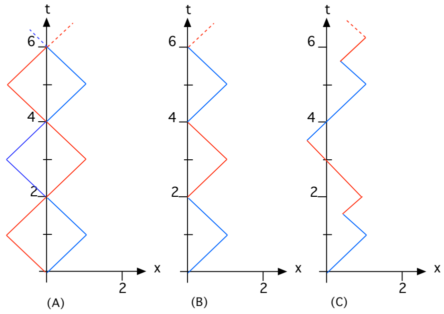


Figure 5: (A) Compton's clock. (B) The right hand boundary used to form the characteristic function for counting (Fig. 2). (C) A stochastic version of (B) used in the Feynman chessboard model. The orientation of the areas in (A) provide the 'corner rule' for Feynman's model.

that could be expected to eliminate quantum propagation via the quantum zeno effect. But is it really true that quantum propagation is implicated if we *forgo* the infinite frequency limit underlying (7)?

Consider Fig.(5). Compton's clock and the right hand path used for counting appear in the first two frames. The third frame is a sketch of a path in which the distance between the corners is stochastic. If the distribution governing the distance between corners is exponential and the matrix T provides the statistical weights of the paths, the resulting model is the Feynman chessboard model. This is Feynman's sum-over-paths formulation for the Dirac propagator.[10, 11, 12, 13, 7].

The appearance of Dirac propagation at this point may appear paradoxical. We have shown that *classical* spacetime appears as a high frequency limit of Compton's clock. The clock construction itself is a simple algorithm for counting and there has been no quantization procedure that would necessitate interpretation after the calculation. Mass does not appear as a characteristic frequency in Compton's clock since the high frequency limit leaves only the scale of the unit vectors of the spacetime frame. How then is wave propagation implicated in the chessboard model when there is none in the limit of Compton's clock?

From Fig.(5C)we see that the difference between Compton's clock and the chessboard model is the stochastic element built in to the causal areas. The Poisson process that governs the limit allows arbitrarily high frequency but gives a finite expected frequency at the Compton frequency. The remnant of Compton's clock is the 'zitterbewegung' that appears in the Dirac equation[14].

Notice that the chessboard limit preserves the indeterminacy feature of the original digital clock. The rectangular area between two audible events in the Compton clock is causal in the sense that the clock mechanism is confined to that area. It is not however confined to the linear interpolant between the two events; that is the classical spacetime limit. Instead, the chessboard limit allows

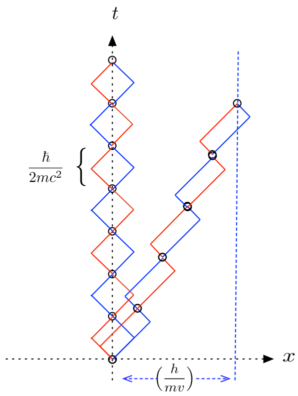


Figure 6: At fixed velocity, the length scale at which the moving clock has lost one tick is the deBroglie scale.

the clock to tick any integral number of times between the initial and final tick. The implementation of this allows the enumerative paths of a stochastic clock (the Chessboard paths) to cover the full causal area between two audible events.

The Dirac propagator implicated by the Feynman paths explicitly show that there are two characteristic lengths that arise from this picture of particles-as-clocks. The inner scale is determined by the mass of the particle, the Compton length. The outer scale is determined by the initial and final events, and presumably boundary conditions if the spatial domain is restricted. This is not however how conventional non-relativistic quantum mechanics sees quantum propagation. The Compton scale does not arise in the Schrödinger equation and we might question how the wave propagation we see here, so obviously relativistic in origin, could scale up to the deBroglie length.

The answer is remarkably simple. 'Moving clocks run slow' in special relativity and the length scale at which a moving clock resynchronizes with a stationary clock after missing a single tick is the deBroglie wavelength (Fig. 6). Using an analogy with sound, the deBroglie wavelength is simply the beat frequency arising from the fact that the Compton frequency of the moving clock is slightly lower than the frequency of the stationary clock due to time dilation. If the geometry of any confinement of the clock is closer in scale to the deBroglie length than the Compton length (as it is for example in the case of the electron orbitals of the Hydrogen atom) the result is that the deBroglie length and period become 'fundamental' to the clock. This suggests why the Bohr-Sommerfeld rules provided such a surprisingly accurate account of the spectral lines of Hydrogen prior to the advent of modern quantum theory. The quantized angular momentum rules are a generalization of the synchronization of stationary and moving clocks that lose an integral number of ticks in an orbit[15, 8].

3. Conclusions

There are many routes to spacetime and a frequent starting point is to note that

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = 0. \tag{8}$$

regardless of the inertial frame of reference. If we start with (8) we can quickly invoke spacetime algebra[9] and then bring in mass through dynamics.

From the perspective of Compton's clock, the convenience of assuming the continuum limit at the start is offset by the failure to see several features that illuminate the connection between relativity and quantum mechanics. To make contact with classical mechanics, note that the Compton clock has event intervals on the scale of the Compton time. This is well below the scale on which classical physics routinely holds and if we simply interpolate between ticks and allow very slight changes in velocity from one tick interval to another, we shall have a piecewise smooth approximation to a smooth classical worldline. The intrinsic mass associated with the clock can be linked to the mass of the particle through the kinetic energy term without having to resort to dynamics. We would not expect low energy collisions or chemical reactions to reveal the 'Compton machinery' of the clock for the same reason that Newtonian mechanics does not notice special relativity. The expansion of the relativistic energy:

$$E = mc^2 = m_0c^2 + \frac{1}{2}m_0v^2 + O((v/c)^4) \tag{9}$$

displays the reason clearly. In the absence of relativistic speeds, nature hides the first and last terms in this expansion in scales that are hard to measure. The second term by itself, being independent of c gives no hint of the relativistic time dilation that actually underlies its presence. Finally, our discrete clock has no need of a global spacetime to "tell matter how to move". Our clock moves simply by the implementation of a local rule requiring equal spacetime areas between events. Ultimately spacetime and its connection to mass emerges in a reasonably transparent way from considering Compton's clock.

The contact with Schrödinger's equation is more subtle. While the classical kinetic energy term on the right of eqn(9) can be directly transplanted into Newtonian mechanics with no other remnant of path dependent time, the transition to the Schrödinger equation retains a further remnant of time dilation through the phase dependence of Feynman paths. The formal similarity between the diffusion and Schrödinger equations becomes clearer in this case. The 'non-relativistic' limit of the telegraph equations give the diffusion equation with its relation to Wiener paths[16]. The Telegraph equations miss the Lorentz transformation because the analog of 'Compton's Clock' is not periodic but entropic; it is based on the decay of correlations. There is no analog of time dilation. However, a carefully placed

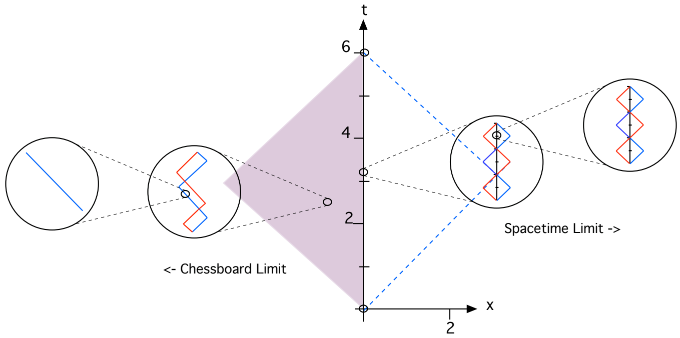


Figure 7: Two continuum limits. The limit indicated to the right of the figure corresponds to the limit taken for the Compton clock that provides a model for spacetime frames. The ever higher frequency reduces the spacetime area between the events at $t = 0$ and $t = 6$ to zero, ultimately filling in a linear worldline between the two events. The limit indicated to the left corresponds the Feynman chessboard model. There the continuum limit allows arbitrarily high frequencies through the Poisson distribution but the limit has a finite expected frequency that becomes the mass term in the Dirac propagator. The chessboard limit allows the 'clock mechanism' and hence the Chessboard paths to wander anywhere in the causal spacetime area between the first and last event. It does not scale down the causal area between observed events, it scales it up.

formal analytic continuation $t- > it$ imitates the rule for invariant spacetime areas and in the non-relativistic limit produces the Schrödinger equation[17]. The feature to note here is that the analytic continuation that restores Lorentz covariance also induces wave propagation! In the particle-as-clock picture, 'quantum propagation' is a relativistic effect that accommodates the path dependent time of special relativity in the Schrödinger regime. The physical constant "c" does not alert us to this fact for the same reason that it does not appear in the non-relativistic kinetic energy in eqn.(9).

This underlying structure, the fact that relativity and quantum mechanics share a common origin, is missed when we formulate our models directly at the level of differentials. The view that relativity and quantum mechanics are separate and need to be married appears here as an artifact of superimposing two different continuum limits. The assumption of smooth worldlines invokes an infinite mass/frequency limit that then has to be 'undone' by a quantization procedure.

Since we started this discussion with Kronecker, it seems appropriate to end with the philosophical descendants of Cantor[18] who embrace Fractals. From a Fractal perspective, the reason that modern quantum theory admits a differential formulation at all, relates to the dimensional 'accident' that random walk paths provide a bridge between one and two topological dimensions. This fact underlies both the Feynman and Wiener integrals, allowing sensible continuum limits from difference equations to differential equations (respectively the Schrödinger and diffusion equations).

We can see an analog of this in the above clock model *in reverse*. The classical limit of the Compton clock squeezes the chain of areas onto the t-axis making a smooth path by reducing the causal areas to zero(Fig.3). Having taken the limit we are left with a topologically one-dimensional object when in fact the mechanism for its maintainence is area-based. In contrast, the chessboard limit does not squeeze out the area between events or force the events onto a smooth worldline. As a result we are left with a form of wave propagation that leaves room for the uncertainty principle and wave-particle duality (Fig.7). Spacetime uses a continuum limit to force a dimensional collapse from area to length while the Chessboard limit does not. It exploits the fact that random paths can be area-filling.

The direct consideration of dimension as a conceptual and calculational tool goes back at least to Mandelbrot's work in the 1970's[19]. It is well known in non-linear dynamics and is becoming a regular, if sometimes unwelcome, visitor to considerations of quantum foundations [4, 5, 20, 21]. The deep philosophical divide between Kronecker and his pupil that shook the mathematical community in the nineteenth century, may yet show up as an echo in the foundations of physics in this century.

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